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LARGE $\mathcal{F}$-FREE SUBGRAPHS IN r-CHROMATIC GRAPHS

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Abstract

For a graph G and a family of graphs $\mathcal{F}$, the Turán number $\mathrm{ex}(G, \mathcal{F})$ is the maximum number of edges an $\mathcal{F}$-free subgraph of G can have. We prove that $\mathrm{ex}(G, \mathcal{F}) \geq \mathrm{ex}(K_r, \mathcal{F})$ if the chromatic number of G is r and $\mathcal{F}$ is a family of connected graphs. This result answers a question raised by Briggs and Cox ['Inverting the Turán problem', *Discrete Math.* **342**(7) (2019), 1865–1884] about the inverse Turán number for all connected graphs.

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1. Introduction

Let G be a graph. We denote the set of vertices of G by $V(G)$, the number of edges of G by $e(G)$ and its chromatic number by $\chi(G)$. We say that a graph G is k-critical if $\chi(G) = k$ and $\chi(H) < \chi(G)$ for every proper subgraph H of G. Let $\mathcal{F}$ be a family of graphs. The Turán number of $\mathcal{F}$ is defined by $\mathrm{ex}(G, \mathcal{F}) = \max\{e(H) : H \subseteq G \text{ and } H \text{ is } \mathcal{F}\text{-free}\}$. When $\mathcal{F} = \{F\}$, we simply write $\mathrm{ex}(G, F)$. Turán [11] determined the exact value of $\mathrm{ex}(K_n, K_t)$, which can be thought of as the first major result in extremal graph theory. Since then, the Turán problem has attracted much attention. A well-known result is the Erdős–Stone–Simonovits theorem [5, 6], which gives the asymptotic Turán number of all nonbipartite graphs. It states that

$$\mathrm{ex}(K_n, F) = \left(\frac{\chi(F) - 2}{\chi(F) - 1}\right)\frac{n^2}{2} + o(n^2).$$

When F is a bipartite graph, it is natural to consider $\mathrm{ex}(K_{m,n}, F)$, replacing the host graph K_n with a complete bipartite graph $K_{m,n}$ (the so-called Zarankiewicz problem). More generally, the host graph can also be replaced by other graphs. Frankl and Rödl [9] studied $\mathrm{ex}(G, F)$ in the case when G is a random graph. Dowden [3] considered the Turán problem when the host graph G is a planar graph. Erdős [4] studied the problem $\mathrm{ex}(Q_n, F)$, where Q_n is the hypercube graph.

Inspired by these problems, Briggs and Cox [2] introduced a dual problem called the inverse Turán problem. That is, for a given set $\mathcal{F}$, if its Turán number is bounded, then what is the behaviour of the set of host graphs

$$\mathcal{G}(k,\mathcal{F}) = \{G : \mathrm{ex}(G,\mathcal{F}) < k\}.$$

The most interesting problem is to study the maximum size of the graphs in $\mathcal{G}(k,\mathcal{F})$. Let $\varepsilon(k,\mathcal{F}) = \sup\{e(G) : \mathrm{ex}(G,\mathcal{F}) < k\}$. In [2], Briggs and Cox obtained the asymptotic value of $\varepsilon(k,F)$ for all nonbipartite graphs and determined the exact value of $\varepsilon(k,\mathcal{F})$ when $\mathcal{F} = \{C_3, C_4, C_5, \ldots\}$ or $\mathcal{F} = \{C_4, C_6, C_8, \ldots\}$. For a path or an even cycle, Győri *et al.* [10] obtained results about the order of magnitude of $\varepsilon(k,F)$, in several cases. Before this problem was formally raised, there were several papers dealing with the function $\varepsilon(k,\mathcal{F})$ in a different form, for example, $\mathcal{F} = \{C_3, C_5, C_7, \ldots\}$ by Alon [1], $\mathcal{F} = \{C_3, C_4, \ldots, C_{2r+1}\}$ and $\mathcal{F} = \{C_4, C_6, \ldots, C_{2r}\}$ by Foucaud *et al.* [8] and $\mathcal{F} = \{K_3, P_4\}$ by Ferneyhough *et al.* [7].

At the end of [2], Briggs and Cox also suggested investigating the maximum chromatic number of the graphs in $\mathcal{G}(k,\mathcal{F})$:

$$\varphi(k,\mathcal{F}) = \sup\{\chi(G) : \mathrm{ex}(G,\mathcal{F}) < k\}.$$

The third author and Chen [12] determined the value of $\varphi(k,\mathcal{F})$ for families of graphs of diameter at most 2.

THEOREM 1.1 (Zhu and Chen [12]). *Let $\mathcal{F}$ be a family of graphs, each member of which has diameter at most* 2. *Then,*

$$\varphi(k,\mathcal{F}) = \max\{r : \mathrm{ex}(K_r,\mathcal{F}) < k\}.$$

At the same time, they proposed the following conjecture.

CONJECTURE 1.2 (Zhu and Chen [12]). For any $F \neq 2K_2$,

$$\varphi(k,F) = \max\{r : \mathrm{ex}(K_r,F) < k\}.$$

In this note, we continue this work and prove the following theorem which implies that Conjecture 1.2 is true for all connected graphs.

THEOREM 1.3. *Let $\mathcal{F}$ be a family of connected graphs and r an integer. For any graph G with $\chi(G) = r$,*

$$\mathrm{ex}(G,\mathcal{F}) \geq \mathrm{ex}(K_r,\mathcal{F}).$$

If some r-chromatic graph G is contained in $\{G : \mathrm{ex}(G,\mathcal{F}) < k\}$, then by Theorem 1.3, K_r is also in $\{G : \mathrm{ex}(G,\mathcal{F}) < k\}$. Hence we obtain the following corollary.

COROLLARY 1.4. *If F is a connected graph, then*

$$\varphi(k,F) = \max\{r : \mathrm{ex}(K_r,F) < k\}.$$

2. Proof of Theorem 1.3

Let G be a graph with chromatic number r and let $c : V(G) \to [r]$ be a proper vertex-colouring of G. We use C_i to denote the set of vertices assigned colour i and let $c_i = |C_i|$. Without loss of generality, we may assume $c_1 \le c_2 \le \cdots \le c_r$. We call $(c_1, c_2, \ldots, c_r)$ the *colour-sequence* of c. Let c and c' be two different proper colourings of G. We say $(c_1, c_2, \ldots, c_r) < (c'_1, c'_2, \ldots, c'_r)$ if $c_j < c'_j$ for $j = \max\{i : c_i \ne c'_i\}$. For convenience, we simply write $c < c'$. We first prove a lemma which will be used in the proof of Theorem 1.3.

LEMMA 2.1. *Let G be any graph with $\chi(G) = r$ and let c be a proper vertex colouring of G such that its colour-sequence $(c_1, c_2, \ldots, c_r)$ is minimal. Then $V(G)$ has a partition $\{V_1, V_2, \ldots, V_s\}$ such that:*

(i) $|V_i| \le r$;
(ii) $s \le c_r$;
(iii) $\sum_{i=1}^{s} e(G[V_i]) \ge \binom{r}{2}$.

PROOF. We prove this lemma by induction on r. If $r = 2$, the result is trivially true. Let $r \ge 3$ and suppose that the lemma holds for all $r' < r$. Let G be an r-chromatic graph on n vertices and c be a vertex-colouring of G such that the colour-sequence $(c_1, c_2, \ldots, c_r)$ is minimal. It is easy to see that $c_r \ge \lceil n/r \rceil$. Now let G' be an r-critical subgraph of G and c' be the minimum vertex-colouring of G' with colour-sequence $(c'_1, c'_2, \ldots, c'_r)$ and colour classes $\{C'_1, \ldots, C'_r\}$. If c' is restricted to the subgraph $G'[C'_1 \cup \cdots \cup C'_{r-1}]$, it is still a minimal proper vertex colouring of $G' - C'_r$ and the colour-sequence is $(c'_1, \ldots, c'_{r-1})$. Thus, by the induction hypothesis, $G' - C'_r$ has a partition $V'_1, \ldots, V'_{s'}$ such that (i) $|V'_i| \le r - 1$, (ii) $s' \le c'_{r-1}$ and (iii) $\sum_{i=1}^{s'} e(G'[V'_i]) \ge \binom{r-1}{2}$.

For each $v \in C'_r$ and $i \le s'$, let $e_{G'}(v, V'_i)$ denote the number of edges in G' having v as one endvertex and having the other endvertex in V'_i. Observe that

$$s' \le c'_{r-1} \le c'_r.$$

We choose s' vertices $\{v_1, \ldots, v_{s'}\}$ from C'_r step by step by the following greedy algorithm. Suppose we have found $\{v_1, \ldots, v_{i-1}\}$ from C'_r, $i \le s'$. Then we choose v_i from $C'_r \setminus \{v_1, \ldots, v_{i-1}\}$ such that $e_{G'}(v_i, V'_i)$ is the maximum among all remaining vertices. Consider the last vertex $v_{s'}$ we choose. Since in each step $e_{G'}(v_i, V'_i)$ is maximal and C'_r is independent in G', we have

$$d_{G'}(v_{s'}) = \sum_{i=1}^{s'} e_{G'}(v_{s'}, V'_i) \le \sum_{i=1}^{s'} e_{G'}(v_i, V'_i). \tag{2.1}$$

Suppose $C'_r \setminus \{v_1, v_2, \ldots, v_{s'}\} = \{v_{s'+1}, \ldots, v_{c'_r}\}$ and let $V_i = V'_i \cup \{v_i\}$ for $1 \le i \le s'$ and $V_j = \{v_j\}$ for $s' + 1 \le j \le c'_r$. Then $V_1, V_2, \ldots, V_{s'}, V_{s'+1}, \ldots, V_{c'_r}$ is a partition of

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$V(G')$ with $|V_i| \le r$. Furthermore, by (2.1),

$$\sum_{i=1}^{c'_r} e(G'[V_i]) = \sum_{i=1}^{s'} e(G'[V'_i]) + \sum_{i=1}^{s'} e_{G'}(v_i, V'_i)$$
$$\ge \binom{r-1}{2} + \sum_{i=1}^{s'} e_{G'}(v_s, V'_i) \ge \binom{r-1}{2} + \delta(G') \ge \binom{r}{2}.$$

The last inequality holds because G' is r-critical and so $\delta(G') \ge r-1$.

It is possible that $V_1, V_2, \ldots, V_{c'_r}$ is not a partition of $V(G)$. In this case, we take the additional sets $V_{c'_r+1}, \ldots, V_{c_r}$ if $c_r > c'_r$ and put the remaining vertices $V(G) - V(G')$ into $V_1, V_2, \ldots, V_{c_r}$ in such a way that $|V_i| \le r$ holds for each i. This can be done because $r \cdot c_r \ge r\lceil n/r \rceil \ge n$. Finally, $V(G) = \bigcup_{i=1}^{c_r} V_i$ is a partition satisfying conditions (i)–(iii). □

Proof of Theorem 1.3. Now, let G be an r-chromatic graph and suppose $V_1, \ldots, V_s$ is a partition of $V(G)$ satisfying the properties in Lemma 2.1. For any $i \in [s]$, let K be the complete graph on the vertex set V_i and let T be a randomly chosen copy of an extremal graph for $\mathrm{ex}(K, \mathcal{F})$. Let H be the subgraph of $G[V_i]$ with the edge set $\{uv \in E(G[V_i]) : uv \in E(T)\}$. Then,

$$\mathrm{ex}(G[V_i], \mathcal{F}) \ge \mathbb{E}(e(H)) = e(G[V_i]) \frac{\mathrm{ex}(K_r, \mathcal{F})}{\binom{r}{2}},$$

where $\mathbb{E}(e(H))$ denotes the expectation of $e(H)$. Because all graphs in $\mathcal{F}$ are connected,

$$\mathrm{ex}(G, \mathcal{F}) \ge \sum_{i=1}^{s} \mathrm{ex}(G[V_i], \mathcal{F}) \ge \sum_{i=1}^{s} e(G[V_i]) \frac{\mathrm{ex}(K_r, \mathcal{F})}{\binom{r}{2}} \ge \mathrm{ex}(K_r, \mathcal{F}).$$

This completes the proof of Theorem 1.3. □

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