Proceedings of the Edinburgh Mathematical Society (2012) **55**, 403–427
DOI:10.1017/S0013091509001552

CONVERGENCE OF BROWNIAN MOTION WITH A SCALED DIRAC DELTA POTENTIAL

FLORIAN CONRAD[1,2], MARTIN GROTHAUS[1], JANNA LIERL[3] AND OLAF WITTICH[4]

[1] *Mathematics Department, University of Kaiserslautern, PO Box 3049, 67653 Kaiserslautern, Germany* (grothaus@mathematik.uni-kl.de)
[2] *Mathematics Department, Bielefeld University, PO Box 100131, 33501 Bielefeld, Germany* (fconrad@math.uni-bielefeld.de)
[3] *Department of Mathematics, Cornell University, 120 Malott Hall, Ithaca, NY 14853, USA* (jlierl@math.cornell.edu)
[4] *Lehrstuhl A für Mathematik, RWTH Aachen, 52056 Aachen, Germany* (olaf.wittich@matha.rwth-aachen.de)

(Received 3 November 2009)

Abstract The method of deriving scaling limits using Dirichlet-form techniques has already been successfully applied to a number of infinite-dimensional problems. However, extracting the key tools from these papers is a rather difficult task for non-experts. This paper meets the need for a simple presentation of the method by applying it to a basic example, namely the convergence of Brownian motions with potentials given by n multiplied by the Dirac delta at 0 to Brownian motion with absorption at 0.

Keywords: Dirichlet forms; limit theorems; strong Feller property

2010 *Mathematics subject classification:* Primary 47D07; 60J60; 31C25

1. Introduction

We study Brownian motion in $\mathbb{R}$ with a Dirac delta potential δ_0 at 0, scaled by $n \in \mathbb{N}$. Because of the singular nature of the potentials, the Dirichlet forms

$$\begin{aligned} \mathcal{E}^n(f,g) &= \frac{1}{2}\int_{\mathbb{R}} f'g'\,\mathrm{d}x + n\tilde{f}(0)\tilde{g}(0), \quad f,g \in D(\mathcal{E}^n), \\ D(\mathcal{E}^n) &= W^{1,2}(\mathbb{R}), \end{aligned}$$

provide a natural approach to constructing these stochastic processes (here $\tilde{f}$ and $\tilde{g}$ denote the continuous versions of the functions f and g, respectively, from the Sobolev space $W^{1,2}(\mathbb{R})$ of square-integrable functions on $\mathbb{R}$ having square-integrable weak derivatives). We use Dirichlet-form techniques not only to construct but also to characterize these processes, and finally to show convergence in law to Brownian motion with absorption at 0 in the limit $n \to \infty$ for all starting points $x \in \mathbb{R} \setminus \{0\}$. Our proof that the laws are weakly convergent is based on Mosco convergence of the corresponding Dirichlet forms and strong Feller properties of the associated semigroups $(T_t^n)_{t \geqslant 0}$, which are uniform in n in the following sense: for each bounded $f \in L^2(\mathbb{R})$ and each $t > 0$ the sequence $(T_t^n f)_{n \in \mathbb{N}}$ is equicontinuous. In characterizing the processes and for proving tightness, the family of martingales provided by the Dirichlet-form approach is of great importance.

We do *not* claim that the results in this paper cannot be shown by purely probabilistic methods (see § 5.5), or that the strategy for performing the limiting procedure is new. The method is used in a number of papers treating limits of finite- and infinite-dimensional systems of stochastic differential equations (see, for example, [**10–12**]). However, due to the advanced technical background used there, the key tools are not accessible to non-experts. (In this paper we even prove a result on convergence for any initial point, using sufficient regularity of the associated semigroups; even in infinite-dimensional settings, it is sometimes possible to obtain such regularity results [**5**].) In the elementary case treated here, the road map for identifying properties of the processes and for proving convergence is easy to follow. The technical problems that occur here can be solved in detail without greatly inconveniencing the reader. We consider the present paper to be a guideline for treating scaling limits of stochastic processes by Dirichlet-form techniques.

The paper is structured as follows: in § 2 we verify that $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, are local, regular Dirichlet forms, and derive that the corresponding infinitesimal generator is given by $L_n f = \frac{1}{2} f''$ with domain

$$D(L_n) = \{f \in W^{1,2}(\mathbb{R}) \cap W^{2,2}(\mathbb{R} \setminus \{0\}) \mid \widetilde{f'}_+(0) - \widetilde{f'}_-(0) = 2n\tilde{f}(0)\},$$

where the subscripts + and − denote the derivative from the right and left, respectively. Looking at the corresponding semigroups $(T_t^n)_{t>0}$, we show that each $T_t^n f$ has a (unique) continuous version, and that

$$\{\widetilde{T_t^n f} \mid n \in \mathbb{N},\ f \in L^2(\mathbb{R}),\ \|f\|_{L^2} \leqslant R\}$$

is equicontinuous for any $0 < t < \infty$ and all constants $0 \leqslant R < \infty$.

We then turn to the question of convergence. Mosco convergence of $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, is shown in § 3. We can describe the limit $(\mathcal{E}, D(\mathcal{E}))$ explicitly by

$$\mathcal{E}(f, f) = \sup_{n \in \mathbb{N}} \mathcal{E}^n(f, f), \quad f \in D(\mathcal{E}),$$
$$D(\mathcal{E}) = \Big\{f \in W^{1,2}(\mathbb{R}) \;\Big|\; \sup_{n \in \mathbb{N}} \mathcal{E}^n(f, f) < \infty\Big\} = \{f \in W^{1,2}(\mathbb{R}) \mid \tilde{f}(0) = 0\}.$$

Mosco convergence of $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, is equivalent to strong convergence of the semigroups $(T_t^n)_{t \geqslant 0}$, and convergence of the infinitesimal generators $(L_n, D(L_n))$ in both the strong resolvent and the strong graph sense.

The remaining sections are centred on the stochastic processes associated with the forms $(\mathcal{E}^n, D(\mathcal{E}^n))$, and weak convergence of their laws $\mathbb{P}_x^n$, pointwise for every starting point $x \in \mathbb{R} \setminus \{0\}$. There exists a Hunt diffusion process $\boldsymbol{M}^n$ properly associated with $(\mathcal{E}^n, D(\mathcal{E}^n))$. This is proved in § 4, using locality and regularity of the Dirichlet forms.

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

In Lemma 4.2 we prove that there is no non-empty set which is $\mathcal{E}^n$-exceptional and that every $\mathcal{E}^n$-quasi-continuous function is continuous. This also implies that, if $(P_t^n)_{t>0}$ denotes the transition semigroup of the associated process, $P_t^n f$ is the continuous version of $T_t^n f$ for all $f \in L^2(\mathbb{R})$, $t > 0$.

Section 5 is concerned with the properties of the associated Hunt diffusion process. First we prove by semigroup techniques that the process is not conservative. Then, by using the martingales provided by the associated martingale problem, it turns out that $\boldsymbol{M}^n$ is a Brownian motion with possible absorption at 0. Whenever the process approaches 0, it does not die immediately (as it would do if there was absorption at 0). Instead, it hits 0 almost surely. However, afterwards it might get killed instead of escaping from 0 again (see Lemmas 5.4 and 5.7 and § 5.3). As n increases, it becomes more likely that the process gets killed at 0 (see § 5.4). The proof follows from the Burkholder–Davis–Gundy inequality and the fact that the process $M^{[\mathrm{id}]}$ is a martingale, where

$$M_t^{[f]} = f(X_t) - f(X_0) - \frac{1}{2}\int_0^t f''(X_r)\,\mathrm{d}r.$$

The latter is proved by approximating the identity 'id' by functions $f_k \in C_{\mathrm{c}}^2(\mathbb{R})$ that vanish at 0, and showing that each $M^{[f_k]}$ is a martingale with quadratic variation process

$$\langle M^{[f_k]}\rangle_t = \int_0^t |f_k'(X_r)|^2\,\mathrm{d}r, \quad t \geqslant 0.$$

In § 5.5 we give an alternative description of the process in terms of Brownian motion.

In the last section we prove weak convergence of the laws $\mathbb{P}_x^n$ as $n \to \infty$, pointwise for all $x \in \mathbb{R} \setminus \{0\}$. Finally, we show that the limiting process is properly associated with the limiting Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ and is a Brownian motion with absorption at 0 (Theorem 6.6). The proof of convergence of the laws relies on convergence of the finite-dimensional distributions and tightness of the laws $\mathbb{P}_x^n$, $n \in \mathbb{N}$.

As mentioned above, our aim is to give a guideline on how to use Dirichlet-form techniques to treat scaling limits of stochastic processes. The essential techniques are Mosco convergence of the associated Dirichlet forms, strong Feller properties of the corresponding semigroups and the martingales provided by the associated martingale problem. The latter, in particular, are essential for proving tightness. In this context we note also that in the situation with an invariant measure, the Lyons–Zheng decomposition [**15**, **16**] for processes in the Dirichlet space is an additional, very powerful, tool for proving tightness, especially in very singular situations [**11**], at least when the initial distribution equals the invariant measure. Here we are interested in tightness of the processes for each initial point. (In addition, an invariant measure does not even exist.) Hence, for proving tightness we had to develop a different strategy.

1.1. Notation

We denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-field on $\mathbb{R}$, and by $\mathcal{B}_{\mathrm{b}}(\mathbb{R})$, $\mathcal{B}_+(\mathbb{R})$ the set of all $\mathcal{B}(\mathbb{R})$-measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ that are bounded or non-negative, respectively. $C^k(\mathbb{R})$, $k \in \mathbb{N}\cup\{0,\infty\}$, is the set of all k-times continuously differentiable real-valued functions on

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

$\mathbb{R}$. Subscripts 'c' and 'b' denote compact support and boundedness, respectively. Further, for $p \in [1, \infty]$ let $\mathcal{L}^p(\mathbb{R}) = \mathcal{L}^p(\mathbb{R}; \mathrm{d}x)$ be the set of all real-valued Borel functions f on $\mathbb{R}$ such that

$$\|f\|_{L^p} := \sqrt[p]{\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x} < \infty \quad \text{or} \quad \|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| < \infty.$$

The set of equivalence classes of $\mathcal{L}^p$-functions with respect to the Lebesgue measure $\mathrm{d}x$ is denoted by $L^p(\mathbb{R})$. If $f \in L^p(\mathbb{R})$ has a continuous representative, we denote it by $\tilde{f}$. For the Sobolev space of all k-times weakly differentiable L^2-functions such that their derivatives are L^2-integrable, we write $W^{k,2}(\mathbb{R})$. The corresponding norm is given by

$$\|f\|^2_{W^{k,2}} := \sum_{0 \leqslant m \leqslant k} \|\mathrm{d}^m f\|^2_{L^2}, \quad f \in W^{k,2}(\mathbb{R}),$$

where d^m, $m \in \mathbb{N}_0$, denotes the mth (weak) derivative. We sometimes also consider function spaces on open subsets of $\mathbb{R}$, which are then defined analogously. Moreover, we consider the space $C^{1/2}(K)$ of all functions on a bounded subset $K \subset \mathbb{R}$ that are Hölder continuous with parameter $\frac{1}{2}$. The usual norm on this space is given by

$$\|f\|_{C^{1/2}(K)} := \sup_{x \in K} |f(x)| + \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^{1/2}}, \quad f \in C^{1/2}(K).$$

For details on the considered function spaces, see, for example, [**1**].

2. Dirichlet form and generator with a scaled Dirac delta potential

We consider the sequence $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, of symmetric bilinear forms on $L^2(\mathbb{R})$:

$$\left.\begin{aligned} D(\mathcal{E}^n) &= W^{1,2}(\mathbb{R}), \\ \mathcal{E}^n(f, g) &= \frac{1}{2} \int_{\mathbb{R}} f' g' \, \mathrm{d}x + n \tilde{f}(0) \tilde{g}(0), \quad f, g \in D(\mathcal{E}^n). \end{aligned}\right\} \tag{2.1}$$

By Sobolev's Embedding Theorem [**1**, Theorem 5.4], $W^{1,2}((-k, k))$, $k \in \mathbb{N}$, can be continuously embedded to $C^{1/2}((-k, k))$. So $f, g \in W^{1,2}(\mathbb{R})$ have unique continuous versions $\tilde{f}$, $\tilde{g}$, and $(\mathcal{E}^n, D(\mathcal{E}^n))$ is well defined.

We need the following basic definitions.

Definition 2.1. Let $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ be a densely defined (i.e. $D(\hat{\mathcal{E}})$ is dense in $L^2(\mathbb{R})$) positive definite symmetric bilinear form on $L^2(\mathbb{R})$.

(i) $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ is called *regular* if $C^0_{\mathrm{c}}(\mathbb{R}) \cap D(\hat{\mathcal{E}})$ is dense in $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}}_1^{1/2})$ and in $(C^0_{\mathrm{c}}(\mathbb{R}), \|\cdot\|_\infty)$. Here $\hat{\mathcal{E}}_1^{1/2}$ denotes the norm

$$f \mapsto \hat{\mathcal{E}}_1(f, f)^{1/2} := \sqrt{\hat{\mathcal{E}}(f, f) + \|f\|^2_{L^2}}, \quad f \in D(\hat{\mathcal{E}}).$$

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

(ii) $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ is called *local* if, for any $u, v \in D(\hat{\mathcal{E}})$ with disjoint compact supports, it holds that $\hat{\mathcal{E}}(u, v) = 0$.

(iii) $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ is a *Dirichlet form* if it is closed (i.e. $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}}_1^{1/2})$ is complete) and for all $f \in D(\hat{\mathcal{E}})$ it holds that $f^+ \wedge 1 = \min(\max(f, 0), 1) \in D(\hat{\mathcal{E}})$ and

$$\hat{\mathcal{E}}(f^+ \wedge 1, f^+ \wedge 1) \leqslant \hat{\mathcal{E}}(f, f).$$

In the following, we often use definitions or facts concerning Dirichlet forms, in particular the fact that each $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, is uniquely associated with a generator $(L_n, D(L_n))$, a strongly continuous contraction semigroup $(T_t^n)_{t \geqslant 0}$ and a strongly continuous contraction resolvent $(G_\alpha^n)_{\alpha > 0}$. These are explained in [**9**, **17**].

Lemma 2.2. *$(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, are regular and local Dirichlet forms.*

Proof. In [**9**, Example 1.2.2, pp. 8–9] it is proved that, for any positive Radon measure κ on $\mathcal{B}(\mathbb{R})$, the bilinear form

$$\begin{aligned} \mathcal{E}^\kappa(f, g) &= \frac{1}{2} \int_{\mathbb{R}} f' g' \,\mathrm{d}x + \int_{\mathbb{R}} \tilde{f} \tilde{g} \,\mathrm{d}\kappa, \\ D(\mathcal{E}^\kappa) &= F^0, \end{aligned}$$

is a regular Dirichlet form on $L^2(\mathbb{R})$, where F^0 denotes the closure of $C_{\mathrm{c}}^\infty(\mathbb{R})$ with respect to the $(\mathcal{E}_1^\kappa)^{1/2}$-norm. Choosing $\kappa = n\delta$ with δ the Dirac delta, we see that $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, are special cases of $(\mathcal{E}^\kappa, D(\mathcal{E}^\kappa))$, since $W^{1,2}(\mathbb{R})$ is the closure of $C_{\mathrm{c}}^\infty(\mathbb{R})$ in $W^{1,2}$-norm, which is equivalent to the $(\mathcal{E}_1^n)^{1/2}$-norm:

$$\tfrac{1}{2} \|f\|_{W^{1,2}}^2 \leqslant \mathcal{E}_1^n(f, f) = \tfrac{1}{2} \|f\|_{W^{1,2}}^2 + n \tilde{f}(0)^2$$

for any $f \in C_{\mathrm{c}}^\infty(\mathbb{R})$, and there exists $C < \infty$ such that

$$\tilde{f}(0)^2 \leqslant \|\tilde{f}\|_{C^{1/2}((-1,1))}^2 \leqslant C \|f\|_{W^{1,2}((-1,1))}^2 \leqslant C \|f\|_{W^{1,2}}^2,$$

by Sobolev's Embedding Theorem.

Locality of $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, is immediate from the definition. □

Proposition 2.3. *For any $n \in \mathbb{N}$, the domain $D(L_n)$ of L_n is given by*

$$D(L_n) = D := \{f \in W^{1,2}(\mathbb{R}) \cap W^{2,2}(\mathbb{R} \setminus \{0\}) \mid \widetilde{f'}_+(0) - \widetilde{f'}_-(0) = 2n \tilde{f}(0)\}.$$

$L_n f = \frac{1}{2} f''$ holds for any $f \in D(L_n)$.

Here $\widetilde{f'}_+(0)$ and $\widetilde{f'}_-(0)$ denote the right and left limits of $\widetilde{f'}$ at 0, respectively. These limits exist, since $f \in W^{2,2}(\mathbb{R} \setminus \{0\})$ implies that $f' \in W^{1,2}(\mathbb{R} \setminus \{0\})$, and by Sobolev's Embedding Theorem it follows that $\widetilde{f'} \in C^{1/2}((-1, 0))$ and $\widetilde{f'} \in C^{1/2}((0, 1))$.

Proof. Fix $n \in \mathbb{N}$. Let $f \in D(L_n) \subset D(\mathcal{E}^n) = W^{1,2}(\mathbb{R})$. Then for all $g \in D(\mathcal{E}^n)$ it holds [**17**] that

$$\mathcal{E}^n(f,g) = -\int_{\mathbb{R}} L_n f g \, \mathrm{d}x.$$

In particular, we find for $g \in C_c^\infty(\mathbb{R} \setminus \{0\})$ that

$$-\int_{\mathbb{R}} L_n f g \, \mathrm{d}x = \mathcal{E}^n(f,g) = \frac{1}{2}\int_{\mathbb{R}} f'g' \, \mathrm{d}x + n\tilde{f}(0)g(0) = -\frac{1}{2}\int_{\mathbb{R}} fg'' \, \mathrm{d}x.$$

So $f \in W^{2,2}(\mathbb{R}\setminus\{0\})$ and $L_n f = \frac{1}{2}f''$ in $L^2(\mathbb{R}\setminus\{0\})$, and hence in $L^2(\mathbb{R})$. Let $g \in C_c^\infty(\mathbb{R})$, $g(0) \neq 0$. Using the fact that g is continuous at 0 and vanishes at ∞, we obtain

$$\begin{aligned}
\mathcal{E}^n(f,g) &= \frac{1}{2}\lim_{\varepsilon\to 0}\left(\int_{-1/\varepsilon}^{-\varepsilon} f'g' \, \mathrm{d}x + \int_{\varepsilon}^{1/\varepsilon} f'g' \, \mathrm{d}x\right) + n\tilde{f}(0)g(0) \\
&= -\frac{1}{2}\int_{\mathbb{R}} f''g \, \mathrm{d}x + n\tilde{f}(0)g(0) \\
&\quad + \frac{1}{2}\lim_{\varepsilon\to 0}\left(\widetilde{f}'g(-\varepsilon) - \widetilde{f}'g\left(-\frac{1}{\varepsilon}\right) + \widetilde{f}'g\left(\frac{1}{\varepsilon}\right) - \widetilde{f}'g(\varepsilon)\right) \\
&= -\int_{\mathbb{R}} L_n f g \, \mathrm{d}x + \tfrac{1}{2}g(0)(\widetilde{f}'_-(0) - \widetilde{f}'_+(0) + 2n\tilde{f}(0)).
\end{aligned}$$

Thus,

$$\widetilde{f}'_-(0) - \widetilde{f}'_+(0) + 2n\tilde{f}(0) = 0.$$

So we have shown that $D(L_n) \subset D$.

Conversely, for any $f \in D$, $g \in D(\mathcal{E}^n)$, the same calculation as above yields

$$\mathcal{E}^n(f,g) = -\frac{1}{2}\int_{\mathbb{R}} f''g \, \mathrm{d}x,$$

proving by [**17**, Proposition I.2.16] that $D \subset D(L_n)$, so the assertion is shown. □

Finally, we verify some (uniform) regularity properties of the semigroups $(T_t^n)_{t\geqslant 0}$, $n \in \mathbb{N}$.

Lemma 2.4. *Let $n \in \mathbb{N}$, $t > 0$ and $f \in L^2(\mathbb{R})$. Then $T_t^n f$ has a (unique) continuous version $\widetilde{T_t^n f}$. Moreover, for each bounded set $K \subset \mathbb{R}$, and any $t > 0$ and $R < \infty$, the set of restrictions*

$$\{\widetilde{T_t^n f}|_K \mid n \in \mathbb{N},\ f \in L^2(\mathbb{R}),\ \|f\|_{L^2} \leqslant R\}$$

is equicontinuous.

Proof. Let $t > 0$ and $n \in \mathbb{N}$. By [**9**, Lemma 1.3.3] and the fact that T_t^n is a contraction, it holds for any $f \in D(\mathcal{E}^n)$ that $T_t^n f \in D(\mathcal{E}^n) = W^{1,2}(\mathbb{R})$ and

$$\mathcal{E}_1^n(T_t^n f, T_t^n f) \leqslant \|T_t^n f\|_{L^2}^2 + \frac{1}{2t}(\|f\|_{L^2}^2 - \|T_t^n f\|_{L^2}^2) \leqslant \left(1 + \frac{1}{2t}\right)\|f\|_{L^2}^2.$$

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

Since $D(\mathcal{E}^n) \subset L^2(\mathbb{R})$ is dense, this extends to any $f \in L^2(\mathbb{R})$, so by Sobolev embedding we find that, for any such f, any bounded set $K \subset \mathbb{R}$ and any $n \in \mathbb{N}$ it holds that

$$\|\widetilde{T_t^n f}\|^2_{C^{1/2}(K)} \leqslant C\|T_t^n f\|^2_{W^{1,2}} \leqslant 2C\mathcal{E}_1^n(T_t^n f, T_t^n f) \leqslant 2C\biggl(1+\frac{1}{2t}\biggr)\|f\|^2_{L^2}$$

for some constant $C < \infty$ depending only on K, and the assertions follow. □

Remark 2.5.

(i) One can easily prove a similar property for the strongly continuous contraction resolvents associated with $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$. However, we omit the proof because we shall not use this result.

(ii) In fact, [**1**, Theorem 5.2] even implies global equicontinuity, i.e. the restriction to bounded sets K is not necessary. However, such a strong property of the semigroup is not needed for our further considerations.

3. Mosco convergence

In this section we prove Mosco convergence of the sequence $(\mathcal{E}^n, D(\mathcal{E}^n))$, $n \in \mathbb{N}$, introduced in § 2. We use the convention that any bilinear form $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ is defined on the entire underlying Hilbert space and takes the value $+\infty$ outside $D(\hat{\mathcal{E}})$. We restrict the following definition to the case of densely defined forms.

Definition 3.1. Let $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ and $(\hat{\mathcal{E}}^n, D(\hat{\mathcal{E}}^n))$, $n \in \mathbb{N}$, be densely defined positive definite symmetric bilinear forms on a separable real Hilbert space $(H, (\cdot,\cdot)_H)$. The sequence $(\hat{\mathcal{E}}^n, D(\hat{\mathcal{E}}^n))_{n\in\mathbb{N}}$ is said to be *Mosco convergent* to $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ if

(i) for any sequence $(u_n)_{n\in\mathbb{N}} \subset H$ weakly convergent to some $u \in H$, i.e. $(u_n, v)_H \to (u, v)_H$ for all $v \in H$, it holds that

$$\hat{\mathcal{E}}(u,u) \leqslant \liminf_{n\to\infty} \hat{\mathcal{E}}^n(u_n, u_n),$$

(ii) for every $u \in H$ there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset H$ such that $\lim_{n\to\infty} u_n = u$ in H and

$$\hat{\mathcal{E}}(u,u) = \lim_{n\to\infty} \hat{\mathcal{E}}^n(u_n, u_n).$$

The proof of convergence in the following theorem applies elementary facts from functional analysis to the particular situation under consideration. The convergence of an increasing sequence of closed symmetric forms can also be treated in a general way (see Remark 3.3).

Theorem 3.2. *The sequence* $(\mathcal{E}^n, D(\mathcal{E}^n))_{n\in\mathbb{N}}$ *Mosco converges to the form* $(\mathcal{E}, D(\mathcal{E}))$, *defined by*

$$D(\mathcal{E}) = \Big\{ f \in L^2(\mathbb{R}) \;\Big|\; \sup_{n\in\mathbb{N}} \mathcal{E}^n(f,f) < \infty \Big\} = \{ f \in W^{1,2}(\mathbb{R}) \mid \tilde{f}(0) = 0 \},$$

$$\mathcal{E}(f,f) = \sup_{n\in\mathbb{N}} \mathcal{E}^n(f,f),$$

$$\mathcal{E}(f,g) = \tfrac{1}{4}(\mathcal{E}(f+g,f+g) - \mathcal{E}(f-g,f-g)) = \frac{1}{2}\int_{\mathbb{R}} f'g' \,\mathrm{d}x,$$

where $f, g \in D(\mathcal{E})$. *Moreover,* $(\mathcal{E}, D(\mathcal{E}))$ *is a Dirichlet form and* $D(\mathcal{E})$ *is the closure of* $C_0^\infty(\mathbb{R}\setminus\{0\})$ *in* $W^{1,2}(\mathbb{R})$.

Proof. Condition (ii) in Definition 3.1 holds with $u_n := u$ for all $n \in \mathbb{N}$. To prove (i) let $(u_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R})$ be weakly convergent to some $u \in L^2(\mathbb{R})$ and assume that for some subsequence $(u_{n_k})_{k\in\mathbb{N}}$ it holds that $\lim_{k\to\infty} \mathcal{E}^{n_k}(u_{n_k}, u_{n_k}) < \infty$. We need to show that $u \in W^{1,2}(\mathbb{R})$ and $\tilde{u}(0) = 0$. Applying the Banach–Alaoglu Theorem we may assume without loss of generality that $(u_{n_k})_{k\in\mathbb{N}}$ converges weakly in $W^{1,2}(\mathbb{R})$ to some $\hat{u} \in W^{1,2}(\mathbb{R})$. The natural embedding $W^{1,2}(\mathbb{R}) \subset L^2(\mathbb{R})$ is continuous; hence, it is also continuous when considering both spaces together with their weak topologies, so $u = \hat{u} \in W^{1,2}(\mathbb{R})$. Moreover, since $\sup_{k\in\mathbb{N}} n_k \tilde{u}_{n_k}(0) < \infty$ and since $\lim_{k\to\infty} \tilde{u}_{n_k}(0) = \tilde{u}(0)$ due to the fact that evaluation at 0 is a continuous linear functional on $W^{1,2}(\mathbb{R})$, we obtain $\tilde{u}(0) = 0$. The estimate in part (i) of Definition 3.1 now follows from lower semicontinuity of the norm with respect to weak convergence in $W^{1,2}(\mathbb{R})$.

Closedness of $(\mathcal{E}, D(\mathcal{E}))$ is immediate and the Dirichlet property follows as in [**9**, Example 1.2.2]. The last assertion follows by standard approximation arguments. □

The limit Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is associated with a Brownian motion with absorption at 0 (see Theorem 6.6).

Let $(L, D(L))$, $(G_\alpha)_{\alpha>0}$ and $(T_t)_{t\geqslant 0}$ be the generator, resolvent and semigroup in $L^2(\mathbb{R})$, respectively, corresponding to $(\mathcal{E}, D(\mathcal{E}))$. There are three more formulations of convergence of non-negative symmetric closed bilinear forms that are equivalent to Mosco convergence (cf. Corollary 2.6.1 and Theorem 2.4.1 of [**18**] and (the proof of) Theorem VIII.26 in [**19**]):

(i) strong convergence of the corresponding semigroups, i.e.

$$\lim_{n\to\infty} T_t^n f = T_t f \quad \text{for all } f \in L^2(\mathbb{R}), \tag{3.1}$$

uniformly on every compact interval $t \in [0, T]$;

(ii) convergence of the corresponding generators in the strong resolvent sense, i.e.

$$\lim_{n\to\infty} G_\alpha^n f = G_\alpha f \quad \text{for all } f \in L^2(\mathbb{R}),\ \alpha > 0;$$

(iii) convergence of L_n to L in the strong graph sense, i.e.

$$D(L) = \Big\{ f \in L^2(\mathbb{R}) \;\Big|\; \exists f_n \in D(L_n),\ n \in \mathbb{N}\colon \lim_{n\to\infty} f_n = f \text{ and } \exists \lim_{n\to\infty} L_n f_n \Big\},$$
$$Lf = \lim_{n\to\infty} L_n f_n, \quad f \in D(L) \text{ with } (f_n)_{n\in\mathbb{N}} \text{ as above},$$

is the generator of the limiting semigroup $(T_t)_{t\geqslant 0}$.

Remark 3.3. As mentioned above, Mosco convergence can also be shown by more general arguments: in [**20**] (see also [**13**, § VIII.3.4]) it is shown that, for any increasing sequence of closed positive definite symmetric forms dominated by a densely defined positive definite symmetric form, the convergence in the above sense holds. It is also possible to derive this in the more general framework of epiconvergence or Γ-convergence (see, for example, [**3**, **4**]).

4. The associated diffusion process

In the following two sections, we keep $n \in \mathbb{N}$ fixed. We need the following notions from the potential theory for Dirichlet forms.

Definition 4.1. Let $(\hat{\mathcal{E}}, D(\hat{\mathcal{E}}))$ be a Dirichlet form on $L^2(E;\mu)$, where E is a Hausdorff topological space and μ is a σ-finite measure on the Borel σ-field of E.

(i) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an $\hat{\mathcal{E}}$-nest if $\bigcup_{k\geqslant 1} D(\hat{\mathcal{E}})_{F_k}$ is dense in $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}}_1^{1/2})$, where $F_k^{\mathrm{c}} = E \setminus F_k$ and

$$D(\hat{\mathcal{E}})_{F_k} = \{ f \in D(\hat{\mathcal{E}}) \mid f = 0 \text{ on } F_k^{\mathrm{c}} \}.$$

(ii) A subset $N \subset E$ is called $\hat{\mathcal{E}}$-exceptional if $N \subset \bigcap_{k\geqslant 1} F_k^{\mathrm{c}}$ for some $\hat{\mathcal{E}}$-nest $(F_k)_{k\in\mathbb{N}}$. '$\hat{\mathcal{E}}$-quasi-everywhere' ($\hat{\mathcal{E}}$-q.e.) means 'outside an $\hat{\mathcal{E}}$-exceptional set'.

(iii) An $\hat{\mathcal{E}}$-q.e. defined function $f\colon A \to \mathbb{R}$ is called $\hat{\mathcal{E}}$-quasi-continuous if there exists an $\hat{\mathcal{E}}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $\bigcup_{k\geqslant 1} F_k \subset A \subset E$ and $f|_{F_k}$ is continuous for every $k \in \mathbb{N}$.

In the situation considered here we have $E = \mathbb{R}$ and μ equal to the Lebesgue measure.

For the reader unfamiliar with these notions we note that an $\mathcal{E}$-exceptional set as in the above definition always has measure 0 with respect to the underlying measure μ. An exceptional set is, roughly speaking, a set which is not only almost surely not 'hit' by μ, but also almost surely not hit by a special standard process that is properly associated (see below) with $\mathcal{E}$ when this process starts, e.g. in a probability distribution that is absolutely continuous with respect to μ. ($\mathcal{E}$-nests can also be described in this way: the process almost surely dies after leaving all F_k, $k \in \mathbb{N}$.) For instance, in the case of a two-dimensional Brownian motion, a line segment is not exceptional, but a singleton is, whereas one-dimensional Brownian motion hits every singleton. Since the Dirichlet form $\mathcal{E}^n$ differs from that associated with Brownian motion only by the killing term $n\tilde{f}(0)\tilde{g}(0)$, the situation for $\mathcal{E}^n$ should be the same as for one-dimensional Brownian motion; we give a proof in Lemma 4.2. For more details on the notions from Definition 4.1 see, for example, [**17**, Chapter III and § IV.5].

Lemma 4.2. *Only the empty set is $\mathcal{E}^n$-exceptional. Moreover, $\mathcal{E}^n$-quasi-continuity reduces to ordinary continuity.*

Proof. Let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}^n$-nest. We first show that any compact interval $[a,b]\subset\mathbb{R}$ is contained in some F_k. Let $f\in D(\mathcal{E}^n)$ be such that $\tilde{f}(x)\neq 0$ for all $x\in[a,b]$. By definition of an $\mathcal{E}^n$-nest, there exists a sequence $(f_l)_{l\in\mathbb{N}}\subset\bigcup_{k\in\mathbb{N}}D(\mathcal{E}^n)_{F_k}$ converging to f in $D(\mathcal{E}^n)$ in the $(\mathcal{E}^n_1)^{1/2}$-norm, and hence in the $W^{1,2}$-norm. Suppose $[a,b]$ is not contained in any of the F_k, $k\in\mathbb{N}$. Then there exists for every $l\in\mathbb{N}$ some $y_l\in[a,b]$ such that $\tilde{f}_l(y_l)=0$. We may without loss of generality assume that $(y_l)_{l\in\mathbb{N}}$ converges to some $y\in[a,b]$. By Sobolev's Embedding Theorem, $\tilde{f}_l\to\tilde{f}$ uniformly on $[a,b]$ as $l\to\infty$; hence, $\tilde{f}(y)=0$: a contradiction.

The first assertion follows by setting $a=b$, showing that no singleton is $\mathcal{E}^n$-exceptional. The second assertion follows without additional effort: the above considerations imply that every $\mathcal{E}^n$-quasi-continuous function f is necessarily continuous on any bounded interval $[a,b]\subset\mathbb{R}$, and hence on the whole of $\mathbb{R}$. □

Let $\mathbb{R}^{\Delta}=\mathbb{R}\cup\{\Delta\}$ be the one-point compactification of $\mathbb{R}$. The Borel σ-algebra on $\mathbb{R}^{\Delta}$ is given by $\mathcal{B}(\mathbb{R}^{\Delta})=\mathcal{B}(\mathbb{R})\cup\{B\cup\{\Delta\}\mid B\in\mathcal{B}(\mathbb{R})\}$.

Definition 4.3 (cf. [17, Chapter IV.1]). Let

$$\boldsymbol{M}=(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},(X_t)_{t\geqslant 0},(\mathbb{P}_x)_{x\in\mathbb{R}^{\Delta}})$$

be a right process, i.e. a strong Markov process that has right-continuous paths and the normal property, with state space $\mathbb{R}$, cemetery Δ and lifetime ζ. Here $\mathcal{F}=\sigma(\bigcup_{t\geqslant 0}\mathcal{F}_t)$, where $(\mathcal{F}_t)_{t\geqslant 0}$ is the natural filtration (or minimum completed admissible filtration) with respect to $X=(X_t)_{t\geqslant 0}$.

(i) $\boldsymbol{M}$ is called a *special standard process* if it has the following properties for any probability measure μ on $(\mathbb{R}^{\Delta},\mathcal{B}(\mathbb{R}^{\Delta}))$.

(a) (*Left limits up to ζ.*) $X_{t_-}=\lim_{s\uparrow t}X_s$ exists in $\mathbb{R}$ for all $t\in(0,\zeta)$ $\mathbb{P}_\mu$-almost surely (a.s.). Here $\mathbb{P}_\mu:=\int\mathbb{P}_x\,\mathrm{d}\mu(x)$.

(b) (*Quasi-left continuity up to ζ.*) Let τ and τ_k, $k\in\mathbb{N}$, be stopping times with respect to $(\mathcal{F}^\mu_t)_{t\geqslant 0}$, the completion of the natural filtration with respect to $\mathbb{P}_\mu$. If $\tau_k\uparrow\tau$, then $X_{\tau_k}\to X_\tau$ as $k\to\infty$ $\mathbb{P}_\mu$-a.s. on $\{\tau<\zeta\}$.

(c) (*Special.*) If τ and τ_k, $k\in\mathbb{N}$, are as in (b) and $\tau_k\uparrow\tau$, then X_τ is measurable with respect to $\bigvee_{k\in\mathbb{N}}\mathcal{F}^\mu_{\tau_k}$, the smallest σ-algebra that contains all $\mathcal{F}^\mu_{\tau_k}$, $k\in\mathbb{N}$.

(ii) $\boldsymbol{M}$ is called a *Hunt process* if (i) holds with ζ replaced by ∞ and $\mathbb{R}$ replaced by $\mathbb{R}^{\Delta}$.

(iii) $\boldsymbol{M}$ is called a *diffusion process* if almost all paths are continuous up to ζ, i.e.

$$\mathbb{P}_x(t\mapsto X_t \text{ is continuous on } [0,\zeta))=1\quad\text{for all } x\in\mathbb{R}.$$

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

Theorem 4.4. *There exists a Hunt diffusion process*

$$\boldsymbol{M}^n = (\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_{t\geqslant 0}, (X^n_t)_{t\geqslant 0}, (\mathbb{P}^n_x)_{x\in\mathbb{R}^\Delta})$$

that is properly associated with $(\mathcal{E}^n, D(\mathcal{E}^n))$*, i.e. for its transition semigroup* $(P^n_t)_{t\geqslant 0}$ *and any* $f \in L^2(\mathbb{R})$*,* $P^n_t f$ *is an* $\mathcal{E}^n$*-quasi-continuous version of* $T^n_t f$*.*

The transition semigroup is defined by $P^n_t f(x) := \mathbb{E}^n_x f(X_t)$, $x \in \mathbb{R}$, $f \in \mathcal{B}(\mathbb{R})$ such that the integration makes sense. Here one always defines $f(\Delta) = 0$.

Proof. By [**17**, Proposition V.2.12(ii)], the regularity of $(\mathcal{E}^n, D(\mathcal{E}^n))$ implies strict quasi-regularity. By [**17**, Theorem V.2.13, Remark V.2.8], this is sufficient for $(\mathcal{E}^n, D(\mathcal{E}^n))$ to be (strictly) properly associated with a Hunt process $\boldsymbol{M}^n$. Furthermore, the locality of $(\mathcal{E}^n, D(\mathcal{E}^n))$ implies by [**17**, Theorem V.1.5] that

$$\mathbb{P}^n_x(t \mapsto X_t \text{ is continuous on } [0, \zeta)) = 1$$

holds for $\mathcal{E}^n$-quasi-every initial point $x \in \mathbb{R}$, i.e. for all x outside some $\mathcal{E}^n$-exceptional set N. But, by Lemma 4.2, only the empty set is $\mathcal{E}^n$-exceptional, proving the assertion. □

Remark 4.5.

(i) Note that Lemma 4.2 implies that $P^n_t f$ is continuous for any $f \in L^2(\mathbb{R})$, so $P^n_t f = \widetilde{T^n_t f}$ (cf. Lemma 2.4).

(ii) We may, and will, assume in the following that $\Omega^n = D([0,\infty), \mathbb{R}^\Delta)$, the space of all càdlàg (right continuous with left limits) paths in $\mathbb{R}^\Delta$.

5. Properties of the associated process

In this section, we show how we may obtain a description of the Hunt diffusion process

$$\boldsymbol{M}^n = (D([0,\infty), \mathbb{R}^\Delta), \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (X_t)_{t\geqslant 0}, (\mathbb{P}^n_x)_{x\in\mathbb{R}^\Delta})$$

from Theorem 4.4 by considerations based on the associated semigroup. After proving that this semigroup is not conservative, which implies a possibly finite lifetime of the process, we derive more precise results by using the fact that $\boldsymbol{M}^n$ solves the martingale problem for the generator $(L_n, D(L_n))$. In particular, we find that the process describes Brownian motion with possible absorption at 0 and obtain a quantitative result on the rate of absorption. Finally, we give a complementary result showing how the process can be described using methods from stochastic calculus, namely as a Brownian motion which is killed when its local time at 0 exceeds an independent exponentially distributed random variable with expectation value $1/n$.

5.1. Non-conservativity of the semigroup

By the Beurling–Deny Theorem [**14**, Proposition 1.8] there exists, for any $p \in [1, \infty)$, a strongly continuous contraction semigroup $(T^n_{t,p})_{t \geqslant 0}$ on $L^p(\mathbb{R})$ such that $T^n_{t,p}$ is an extension of $T^n_t|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})}$ for each $t \geqslant 0$. We denote by $(L_{n,p}, D(L_{n,p}))$ the corresponding generator for $p \in [1, \infty)$. For $p = \infty$, $t \geqslant 0$, we define $T^n_{t,\infty}$ to be the adjoint operator of $T^n_{t,1}$.

Proposition 5.1. *The semigroup $(T^n_{t,\infty})_{t \geqslant 0}$ is not conservative, i.e. there exists $t > 0$ and a set $A \subset \mathbb{R}$ of positive Lebesgue measure such that $T^n_{t,\infty} 1_{\mathbb{R}}(x) < 1$ for (almost every (a.e.)) $x \in A$.*

Proof. Pick $f \in D(L_{n,2})$ such that f has compact support, $f(0) \neq 0$ and $L_{n,2} f = \frac{1}{2} f'' \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then

$$\left\| \frac{T^n_t f - f}{t} - L_{n,2} f \right\|_{L^1} = \left\| \frac{1}{t} \int_0^t T^n_s L_{n,2} f - L_{n,2} f \,\mathrm{d}s \right\|_{L^1} \leqslant \sup_{0 \leqslant s \leqslant t} \| T^n_s L_{n,2} f - L_{n,2} f \|_{L^1}, \tag{5.1}$$

which converges to 0 as $t \downarrow 0$ by the strong continuity of $(T^n_{t,1})_{t \geqslant 0}$. (Note that by the assumptions on f the integral in (5.1) exists in the L^2-sense as well as in the L^1-sense.) So $f \in D(L_{n,1})$ and $L_{n,1} f = L_{n,2} f$. Suppose $(T^n_{t,\infty})_{t \geqslant 0}$ is conservative. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} 1_{\mathbb{R}}(x)(T^n_{t,1} f)(x) \,\mathrm{d}x \bigg|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} (T^n_{t,\infty} 1_{\mathbb{R}})(x) f(x) \,\mathrm{d}x \bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} f(x) \,\mathrm{d}x \bigg|_{t=0} \\ &= 0, \end{aligned}$$

hence we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} T^n_{t,1} f(x) \,\mathrm{d}x \bigg|_{t=0} \\ &= \int_{-\infty}^{\infty} L_n f(x) \,\mathrm{d}x \\ &= \frac{1}{2} \int_{-\infty}^{\infty} f''(x) \,\mathrm{d}x \\ &= \frac{1}{2} \left(\lim_{\varepsilon \to 0} f'(-\varepsilon) - f'\left(-\frac{1}{\varepsilon}\right) + f'\left(\frac{1}{\varepsilon}\right) - f'(\varepsilon) \right) \\ &= -n f(0). \end{aligned}$$

This contradicts $f(0) \neq 0$. □

Non-conservativity of the semigroup implies that the associated process $\boldsymbol{M}^n$ may have finite lifetime: by the symmetry of P^n_t with respect to Lebesgue measure, one finds that $P^n_t f = T^n_{t,\infty} f$ $\mathrm{d}x$-almost everywhere for any $f \in \mathcal{B}_{\mathrm{b}}(\mathbb{R})$. Thus,

$$\int 1_{\mathbb{R}}(X_t) \,\mathrm{d}\mathbb{P}^n_x = P^n_t 1_{\mathbb{R}}(x) = T^n_{t,\infty} 1_{\mathbb{R}}(x) < 1 \quad \text{for a.e. } x \in A.$$

So for any initial point $x \in \tilde{A}$, where $\tilde{A} \subset A$ is a suitably chosen measurable subset of positive Lebesgue measure, the process $\boldsymbol{M}^n$ leaves $\mathbb{R}$ with positive probability.

One way to obtain additional information about the process is potential theoretic considerations. For a proof of the following proposition see [**9**, Example 4.5.1, p. 166].

Proposition 5.2. *Let $\sigma_y = \inf\{t \geqslant 0\colon X_t = y\}$ be the first hitting time of y. Then*

$$\mathbb{P}^n_x(\sigma_y < \infty) > 0$$

for all $x, y \in \mathbb{R}$.

We know from Proposition 5.1 that for every initial point $x \in A$, where A is a set of positive Lebesgue measure, there is a positive probability that $\boldsymbol{M}^n$ has finite lifetime. Proposition 5.2 now implies that this extends to *any* initial point $x \in \mathbb{R}$. Moreover, we can conclude from Proposition 5.2 that there is no point in $\mathbb{R}$ such that the process gets killed almost surely when approaching this point.

However, the probability of getting killed when approaching 0 increases as $n \to \infty$. In fact, as $n \to \infty$, the processes $\boldsymbol{M}^n$ asymptotically behave as if being immediately killed at 0. These facts are shown in § 5.4.

5.2. The martingale problem

As mentioned before, any real-valued function g on $\mathbb{R}$ is extended to $\mathbb{R}^{\Delta}$ by defining $g(\Delta) = 0$. Let us introduce the process

$$M_t^{[f]} = f(X_t) - f(X_0) - \frac{1}{2}\int_0^t f''(X_r)\,\mathrm{d}r$$

for any $f \in C^2(\mathbb{R})$ with $f(0) = 0$ and for $t \geqslant 0$ such that the integral exists, and

$$M_t^{[f]} = \tilde{f}(X_t) - \tilde{f}(X_0) - \int_0^t L_n f(X_r)\,\mathrm{d}r, \quad t \geqslant 0, \tag{5.2}$$

for any $f \in D(L_n)$ such that $L_n f$ is bounded. By Remark 4.5 (i) for any $g \in \mathcal{L}^2(\mathbb{R})$ which equals 0 almost everywhere it holds that $P_t^n g(x) = 0$ for all $x \in \mathbb{R}$, $t > 0$. This implies that for $x \in \mathbb{R}$ the integral in (5.2) $\mathbb{P}^n_x$-a.s. does not depend on the version of $L_n f$ we choose. By Proposition 2.3, the two definitions of $M^{[f]}$ are consistent. Clearly, $(M_t^{[f]})_{t \geqslant 0}$ is $(\mathcal{F}_t)_{t \geqslant 0}$-adapted.

Lemma 5.3. *Let $x \in \mathbb{R}$ and $f \in D(L_n)$ such that f and $L_n f$ are bounded. Then, under $\mathbb{P}^n_x$,*

$$M_t^{[f]} = \tilde{f}(X_t) - \tilde{f}(x) - \int_0^t L_n f(X_r)\,\mathrm{d}r, \quad t \geqslant 0, \tag{5.3}$$

is an $(\mathcal{F}_t)_{t \geqslant 0}$-martingale starting at 0.

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

Proof. Using Lebesgue's Dominated Convergence Theorem, the boundedness of $L_n f$ and Remark 4.5 (i), we find that the function

$$\mathbb{R} \ni x \mapsto \int_0^t P_r^n L_n f(x)\,\mathrm{d}r$$

is continuous. It is therefore a continuous version of the L^2-element

$$\int_0^t T_r L_n f\,\mathrm{d}r = T_t f - f.$$

It follows that for all $x \in \mathbb{R}$ it holds that

$$P_t^n \tilde{f}(x) - \tilde{f}(x) = \int_0^t P_r^n L_n f(x)\,\mathrm{d}r.$$

(See the proof of [**2**, Lemma 5.1] for a similar argument which works in cases where $L_n f$ is not essentially bounded.) The rest is now standard using the Markov property. Let $0 \leqslant s \leqslant t$. For any bounded $\mathcal{F}_s$-measurable function $G\colon D([0,\infty), \mathbb{R}^{\Delta}) \to \mathbb{R}$ we find that

$$\begin{aligned}\mathbb{E}_x^n[(M_t^{[f]} - M_s^{[f]})G(X)] &= \mathbb{E}_x^n\biggl[\mathbb{E}_{X_s}^n\biggl[\tilde{f}(X_{t-s}) - \tilde{f}(X_0) - \int_0^{t-s} L_n f(X_r)\,\mathrm{d}r\biggr]G(X)\biggr]\\ &= \mathbb{E}_x^n\biggl[\biggl(P_{t-s}^n \tilde{f}(X_s) - \tilde{f}(X_s) - \int_0^{t-s} P_r^n L_n f(X_s)\,\mathrm{d}r\biggr)G(X)\biggr]\\ &= 0.\end{aligned}$$

This proves the assertion. □

We now prove several lemmas that lead to a generalization of Lemma 5.3 to $f = \mathrm{id}$; see Theorem 5.9. The proof of the latter theorem requires that $\boldsymbol{M}^n$ does not explode, but we can derive an even more general statement: the only state where $\boldsymbol{M}^n$ possibly gets killed is 0 (see Lemma 5.7). Note that, by the existence of left limits up to ∞, on $\{\zeta < \infty\}$ one can fix some point $X_{\zeta-} \in \mathbb{R}^{\Delta}$ at which the process is killed.

Lemma 5.4. *The process $\boldsymbol{M}^n$ cannot be killed at any non-zero state $y \in \mathbb{R} \setminus \{0\}$.*

Proof. Fix any initial point $x > 0$ and define

$$\tau_m = \inf\{t \geqslant 0\colon X_t \notin [1/m, m] \cup \{\Delta\}\}$$

for $m \in \mathbb{N}$ sufficiently large such that $1/m < x < m$. Here we use the convention $\inf(\emptyset) = \infty$. τ_m is a stopping time with respect to $(\mathcal{F}_t)_{t \geqslant 0}$, because $t \mapsto X_t$ is right continuous. We choose functions $f_k \in C_c^2(\mathbb{R})$, $k \in \mathbb{N}$, such that $f_k(0) = 0$ and $f_k = -a$ on $[1/k, k]$ for some $a > 0$. For any $k \in \mathbb{N}$, by Lemma 5.3 $M^{[f_k]}$ is a martingale under $\mathbb{P}_x^n$ starting at 0. For $k > m$ we have that $f_k''(X_r) = 0$ on $\{r \leqslant \tau_m\}$; hence, the stopped process

$$M_{t \wedge \tau_m}^{[f_k]} = f_k(X_{t \wedge \tau_m}) + a, \quad t \geqslant 0,$$

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

is a non-negative martingale starting at 0. Hence, $M^{[f_k]}_{t\wedge\tau_m} \equiv 0$ $\mathbb{P}^n_x$-a.s. for all $t \geqslant 0$. Therefore, we obtain, for all $t \geqslant 0$,

$$\begin{aligned}\mathbb{P}^n_x(X_{r\wedge\tau_m} = \Delta \text{ for some } r \leqslant t) &= \mathbb{P}^n_x(X_{t\wedge\tau_m} = \Delta)\\ &= \mathbb{P}^n_x(M^{[f_k]}_{t\wedge\tau_m} = a)\\ &= 0.\end{aligned}$$

$\mathbb{P}^n_x$-almost surely, the diffusion process $\boldsymbol{M}^n$ cannot enter the cemetery before leaving $[1/m, m]$. Letting m tend to ∞, we find that $\boldsymbol{M}^n$ does not enter the cemetery before it either explodes or approaches 0. Similarly, one can prove this result for any initial point $x < 0$. Using the strong Markov property (and the existence of left limits), it can now be verified that one can exclude the case that $\boldsymbol{M}^n$ starts at some $x \in \mathbb{R}$, hits 0 without being absorbed and is killed later at some $y \neq 0$. So $\boldsymbol{M}^n$ cannot be killed at any state $y \neq 0$. □

Lemma 5.5. *Let $x \in \mathbb{R}$ and $f \in C^2_{\mathrm{c}}(\mathbb{R})$ with $f(0) = 0$. Then, under $\mathbb{P}^n_x$,*

$$K^{[f]}_t = (M^{[f]}_t)^2 - \int_0^t |f'(X_r)|^2 \,\mathrm{d}r, \quad t \geqslant 0,$$

defines an $(\mathcal{F}_t)_{t\geqslant 0}$-martingale. In particular,

$$\langle M^{[f]}\rangle_t = \int_0^t |f'(X_r)|^2 \,\mathrm{d}r, \quad t \geqslant 0,$$

is the quadratic variation of $M^{[f]}$.

Proof. $K^{[f]}_t$ is $\mathcal{F}_t$-measurable and bounded (hence, integrable) for any $t \geqslant 0$, since $f \in C^2_{\mathrm{c}}(\mathbb{R})$. Let $s \geqslant 0$ and g be some bounded $\mathcal{F}_s$-measurable function. We consider the map $t \mapsto \mathbb{E}^n_x(gK^{[f]}_t)$, $t \geqslant s$. It is continuous by Lebesgue's Dominated Convergence Theorem, because $\boldsymbol{M}^n$ is continuous up to ζ and $(f(X_t))_{t\geqslant 0}$ continuously approaches 0 at time ζ (recall that $f(\Delta) = 0$ and that $\boldsymbol{M}^n$ can at most be killed at 0 or ∞, where f vanishes). It is shown in the proof of [**6**, Theorem 4.6] that the right derivatives of $t \mapsto \mathbb{E}^n_x(gK^{[f]}_t)$, $t \geqslant s$, exist and equal 0. Thus, this function is constant and the assertion follows. □

Lemma 5.6. *For any $0 < p < \infty$, $x \in \mathbb{R}$, $T > 0$, there exists a constant $C(p, T)$ such that*

$$\mathbb{E}^n_x\Big[\sup_{t\leqslant T} |M^{[\mathrm{id}]}_t|^p\Big] \leqslant C(p, T) \quad \text{for all } T > 0. \tag{5.4}$$

$C(p, T)$ is independent of n and goes to 0 as $T \to 0$. Here $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ denotes the identity function.

Proof. Let $0 < p < \infty$, $x \in \mathbb{R}$ and $T > 0$. We approximate the identity map by functions $f_k \in C^2_{\mathrm{c}}(\mathbb{R})$, $k \in \mathbb{N}$, with $f_k = \mathrm{id}$ on $[-k, k]$ and $|f'_k|, |f''_k| \leqslant 1$. By Lemma 5.3, $M^{[f_k]}$ are martingales. Define

$$\tau_m = \inf\{t \geqslant 0\colon |M^{[\mathrm{id}]}_t| \geqslant m\}, \quad m \in \mathbb{N},$$

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

which are stopping times due to the right continuity of $t \mapsto M_t^{[\mathrm{id}]}$ on $[0,\infty)$. For any $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that the stopped process $(M_{t\wedge\tau_m}^{[\mathrm{id}]})_{t\geqslant 0}$ coincides with $(M_{t\wedge\tau_m}^{[f_k]})_{t\geqslant 0}$. In particular, $(M_{t\wedge\tau_m}^{[\mathrm{id}]})_{t\geqslant 0}$ is a martingale. Using the Burkholder–Davis–Gundy inequality and Lemma 5.5, we obtain

$$\begin{aligned}\mathbb{E}_x^n\Big[\sup_{t\leqslant T\wedge\tau_m}|M_t^{[\mathrm{id}]}|^p\Big] &= \mathbb{E}_x^n\Big[\sup_{t\leqslant T\wedge\tau_m}|M_t^{[f_k]}|^p\Big]\\ &\leqslant C\mathbb{E}_x^n[\langle M^{[f_k]}\rangle_{T\wedge\tau_m}^{p/2}]\\ &\leqslant CT^{p/2}\end{aligned}$$

for some constant $C<\infty$. Letting m tend to ∞, the result follows from the Monotone Convergence Theorem. □

Lemma 5.7. *For any initial point $x \in \mathbb{R}$, the process $\boldsymbol{M}^n$ does not explode. In particular, it can only be killed at* 0.

Proof. By Lemma 5.4, it suffices to show that $\boldsymbol{M}^n$ does not explode. Estimate (5.4) implies that, for any $T>0$, $k \in \mathbb{N}$,

$$\mathbb{P}_x^n\Big[\sup_{t\leqslant T}|M_t^{[\mathrm{id}]}| \geqslant k\Big] \leqslant \frac{1}{k}\mathbb{E}_x^n\Big[\sup_{t\leqslant T}|M_t^{[\mathrm{id}]}|\Big] \leqslant \frac{1}{k}C(1,T) \xrightarrow[k\to\infty]{} 0.$$

Thus, the probability that $M^{[\mathrm{id}]}$ explodes before time T is 0 for all $T>0$. □

Remark 5.8.

(i) Note that Lemma 5.7 implies continuity of $M_t^{[\mathrm{id}]} = \mathrm{id}(X_t) - \mathrm{id}(X_0)$ at time ζ, because $\mathrm{id}(\Delta) = 0$ by definition. In this respect, $M^{[\mathrm{id}]}$ is obviously different from the actual process $(X_t)_{t\geqslant 0}$, which enters the cemetery discontinuously, though $X_{\zeta-} = 0 \in \mathbb{R}$. This continuity property of $M^{[\mathrm{id}]}$ will become useful in § 6.

(ii) In § 6 we shall consider Δ to be isolated instead of compactifying $\mathbb{R}^{\Delta}$. This does not change the σ-algebra $\mathcal{B}(\mathbb{R}^{\Delta})$. Moreover, right continuity of the paths is not lost. Hence, with respect to the new topology, $\boldsymbol{M}^n$ is still a diffusion process on $D([0,\infty),\mathbb{R}^{\Delta})$. The fact that the process does not explode shows that quasi-left continuity and existence of left limits in $\mathbb{R}^{\Delta}$ up to ∞ are preserved, so $\boldsymbol{M}^n$ is also still a Hunt process in the modified setting.

We now prove the martingale property for $M^{[\mathrm{id}]}$, which is clearly not covered by Lemma 5.3.

Theorem 5.9. *For any $x \in \mathbb{R}$, $M^{[\mathrm{id}]}$ is a continuous $(\mathcal{F}_t)_{t\geqslant 0}$-martingale starting at* 0 *under $\mathbb{P}_x^n$. The quadratic variation of $M^{[\mathrm{id}]}$ is given by*

$$\langle M^{[\mathrm{id}]}\rangle_t = t\wedge\zeta, \quad t \geqslant 0.$$

Proof. As in the proof of Lemma 5.6, pick approximation functions f_k of the identity and stopping times τ_m, $k, m \in \mathbb{N}$. Note that $(\tau_m)_{m\in\mathbb{N}}$ goes almost surely to ∞ as $m \to \infty$, because $\boldsymbol{M}^n$ does not explode by Lemma 5.7. Since $(M^{[\mathrm{id}]}_{t\wedge\tau_m})_{t\geqslant 0}$ are martingales, $M^{[\mathrm{id}]}$ is a local martingale. By (5.4), we have for any $T > 0$ that

$$\mathbb{E}^n_x\Big[\sup_{t\leqslant T}|M^{[\mathrm{id}]}_t|\Big] \leqslant C(1,T) < \infty, \tag{5.5}$$

so $M^{[\mathrm{id}]}$ is a martingale by [**7**, Theorem 2.2.5]. The quadratic variation of the stopped process $(M^{[f_k]}_{\cdot\wedge\tau_m})_{t\geqslant 0}$ is (by Lemma 5.5) given by

$$\langle M^{[\mathrm{id}]}_{\cdot\wedge\tau_m}\rangle_t = \langle M^{[f_k]}_{\cdot\wedge\tau_m}\rangle_t = t\wedge\tau_m\wedge\zeta, \quad t\geqslant 0.$$

So the quadratic variation of $M^{[\mathrm{id}]}$ is $\langle M^{[\mathrm{id}]}\rangle_t = t\wedge\zeta$, $t \geqslant 0$. □

5.3. Brownian paths

We conclude from Theorem 5.9 that $M^{[\mathrm{id}]}$ is a time change of Brownian motion (or rather a stopped Brownian motion); in particular, it coincides with a Brownian motion as long as the process $\boldsymbol{M}^n$ is alive. To be precise, let $(B_t)_{t\geqslant 0}$ be an $\mathbb{R}$-valued Brownian motion starting at 0 defined on some probability space $(\Omega^{\mathrm{B}}, \mathcal{F}^{\mathrm{B}}, \mathbb{P}^{\mathrm{B}})$. On the set $\{\zeta < \infty\}$,

$$M^{[\mathrm{id}]}_\infty := \lim_{t\to\infty} M^{[\mathrm{id}]}_t$$

exists $\mathbb{P}^n_x$-almost surely for $x \in \mathbb{R}$. Hence, we can define the process

$$W_u(\omega_1,\omega_2) = \begin{cases} M^{[\mathrm{id}]}_u(\omega_1), & u<\zeta, \\ M^{[\mathrm{id}]}_\infty(\omega_1) + B_{u-\langle M^{[\mathrm{id}]}\rangle_\infty}(\omega_2), & u \geqslant \zeta, \end{cases} \tag{5.6}$$

for $(\omega_1,\omega_2) \in \Omega\times\Omega^{\mathrm{B}}$. Then $(W_t)_{t\geqslant 0}$ is a Brownian motion on the product space $(D([0,\infty),\mathbb{R})\times\Omega^{\mathrm{B}}, \mathcal{F}\otimes\mathcal{F}^{\mathrm{B}}, \mathbb{P}^n_x\otimes\mathbb{P}^{\mathrm{B}})$ by [**7**, Theorem 3.4.8].

5.4. A quantitative result on the killing of the process

For $x \in \mathbb{R}$ we know from (5.6) that $\boldsymbol{M}^n$ behaves $\mathbb{P}^n_x$-almost surely like a Brownian motion up to ζ, and killing can only take place at 0. But there is also the possibility that the process passes through 0 without being killed immediately. Let us examine how likely this is. Let $x \in \mathbb{R}$, $k \geqslant |x|$ and $f_n \in D(L_n)$ be a function with $f_n(y) = 1 + n|y|$ for $y \in [-k,k]$, going smoothly to 0 outside $[-k,k]$ such that f_n and $L_n f_n$ are bounded. Then $M^{[f_n]} + f_n(x)$ is a martingale under $\mathbb{P}^n_x$ by Lemma 5.3, and so is the stopped process $(M^{[f_n]}_{t\wedge\tau} + f_n(x))_{t\geqslant 0} = (f_n(X_{t\wedge\tau}))_{t\geqslant 0}$, where τ is the following (a.s. finite; see § 5.3) stopping time

$$\tau = \inf\{t\geqslant 0\colon |X_t| = k \text{ or } X_t = \Delta\}.$$

Therefore,

$$1 + n|x| = f_n(x) = \mathbb{E}^n_x(f_n(X_\tau)) = (1+nk)\mathbb{P}^n_x(|X_\tau| = k).$$

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

Thus,

$$\mathbb{P}_x^n(|X_\tau| = k) = \frac{1 + n|x|}{1 + nk}. \tag{5.7}$$

Hence, $\mathbb{P}_x^n$-a.s. the process remains bounded (or reaches Δ in finite time). Due to the almost sure unboundedness of Brownian motion we conclude in view of § 5.3 that $\boldsymbol{M}^n$ has a.s. finite lifetime with respect to all $\mathbb{P}_x^n$, $x \in \mathbb{R}$.

Moreover,

$$\mathbb{P}_x^n(X_\tau = \Delta) = \frac{n(k - |x|)}{nk + 1} < \frac{k - |x|}{k}. \tag{5.8}$$

As $\mathbb{P}_x^n(X_\tau = \Delta)$ is strictly increasing in n and converges to $(k - |x|)/k$, it becomes more likely that the process gets killed when approaching 0 as n increases. By [**7**, Theorem 3.1.4], $(k - |x|)/k$ is the probability that $(X_t)_{t \geqslant 0}$ approaches 0 before hitting k. Since (5.8) is a strict inequality, this is another proof that there is a positive probability that $(X_t)_{t \geqslant 0}$ passes through 0 without being killed immediately (cf. Proposition 5.2).

In fact, the probability that $(X_t)_{t \geqslant 0}$ gets instantaneously killed when approaching 0 is null: Let (without loss of generality) $0 < x < k$ and define the stopping time $\tau_0 = \inf\{t \geqslant 0 \colon X_t \in \{0, \Delta, k\}\}$. Let $p = \mathbb{P}_x^n(X_{\tau_0} = \Delta \mid X_{\tau_0} \in \{0, \Delta\})$. Then we have by (5.8) and the strong Markov property that

$$\begin{aligned}
\frac{n(k - |x|)}{nk + 1} &= \mathbb{P}_x^n(X_{\tau_0} = \Delta) + \mathbb{P}_x^n(X_{\tau_0} = 0)\mathbb{P}_0^n(X_\tau = \Delta) \\
&= \mathbb{P}_x^n(X_{\tau_0} \in \{0, \Delta\})p + \mathbb{P}_x^n(X_{\tau_0} \in \{0, \Delta\})(1 - p)\mathbb{P}_0^n(X_\tau = \Delta) \\
&= \frac{k - |x|}{k} p + \frac{nk}{nk + 1} \frac{k - |x|}{k} (1 - p);
\end{aligned}$$

hence,

$$0 = p \frac{k - |x|}{k} \left(1 - \frac{nk}{nk + 1}\right),$$

so $p = 0$.

Remark 5.10. The last result can also be seen by the following argument. Without loss of generality let $x > 0$. Consider the sequence of stopping times $\tau_n = \inf\{t \geqslant 0 \colon X_t \leqslant 1/n\}$, $n \in \mathbb{N}$. We set $\tau = \lim_{n \to \infty} \tau_n$; then $\mathbb{P}_x^n$-a.s. it holds that $\tau = \inf\{t > 0 \mid X_{t-} = 0\}$. Since $\{\tau < \infty\}$ is an almost sure event, quasi-left continuity of $\boldsymbol{M}^n$ up to ∞ implies that $X_{\tau_n} = 1/n$ converges to X_τ as $n \to \infty$, $\mathbb{P}_x^n$-almost surely. Hence, $X_\tau = 0$ with probability 1, i.e. $(X_t)_{t \geqslant 0}$ almost surely survives the first time it approaches 0. However, note that the quantitative result (5.8) is rather unlikely to be seen from the abstract properties of $\boldsymbol{M}^n$.

5.5. Description of the process via stochastic calculus

In many applications (in particular for infinite particle systems; see, for example, [**11**]) the description of the process via the martingale problem is the only information one may obtain about its behaviour. However, for the present process it is not difficult to derive

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

a nice representation using the Meyer–Tanaka formula, independently of the previous considerations in § 5.

For $x \in \mathbb{R}$ we denote by $\mathbb{P}_x^{\mathrm{B}}$ the law of a Brownian motion starting in x with paths denoted by $(B_t)_{t \geqslant 0}$. Let L_t^y, $t \geqslant 0$, be the local time of $(B_t)_{t \geqslant 0}$ in $y \in \mathbb{R}$. Without loss of generality we may assume that L_t^y is jointly continuous in y and t (see, for example, [**7**, Theorem 2.11.8]).

The following lemma is the Feynman–Kac formula for a potential given by $n\delta_0$.

Lemma 5.11. *For $(P_t^n)_{t \geqslant 0}$ as in Theorem 4.4 and any bounded measurable $f : \mathbb{R} \to \mathbb{R}$ the following holds:*

$$P_t^n f(x) = \mathbb{E}_x^{\mathrm{B}}(\mathrm{e}^{-nL_t^0} f(B_t)). \tag{5.9}$$

Proof. Denoting the right-hand side of the above equation by $\hat{P}_t^n f(x)$, we find by the Markov property of Brownian motion that the $\hat{P}_t^n$, $t \geqslant 0$, have the semigroup property and map at least bounded continuous functions to continuous functions. To see the latter, note that $\hat{P}_t^n f(x) = \mathbb{E}_0^{\mathrm{B}}(\mathrm{e}^{-nL_t^{-x}} f(B_t + x))$, $x \in \mathbb{R}$, f bounded and continuous, and apply Lebesgue's Dominated Convergence Theorem. Moreover, as $(\hat{P}_t^n)_{t \geqslant 0}$ is dominated by the classical heat semigroup, we find that it maps $L^2(\mathbb{R})$ into itself, respects L^2-classes and consists of L^2-contractions.

Denote by D the set of real-valued functions of the form $f = g + ng(0)|\cdot|\chi$ with $g \in C_0^\infty(\mathbb{R})$ and $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighbourhood of 0. It is readily seen that D is dense in the L_n-graph norm in $D(L_n)$.

Let $f \in D$ be as above and let $h(x) = g''(x) + 2ng(0)\chi'(x)\,\mathrm{sgn}(x) + ng(0)\chi''(x)|x|$, $x \in \mathbb{R}$. Using the Meyer–Tanaka formula, Itô's formula and [**7**, Theorem 2.11.7] we obtain

$$\begin{aligned} \mathrm{e}^{-nL_t^0} f(B_t) - f(B_0) &= \int_0^t \mathrm{e}^{-nL_s^0} f'_-(B_s)\,\mathrm{d}B_s - n\int_0^t \mathrm{e}^{-nL_s^0} f(B_s)\,\mathrm{d}L_s^0 \\ &\quad + \frac{1}{2}\int_0^t \mathrm{e}^{-nL_s^0} h(B_s)\,\mathrm{d}s + n\int_0^t \mathrm{e}^{-nL_s^0} f(0)\,\mathrm{d}L_s^0 \\ &= \int_0^t \mathrm{e}^{-nL_s^0} f'_-(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \mathrm{e}^{-nL_s^0} h(B_s)\,\mathrm{d}s. \end{aligned}$$

Taking the expectation with respect to $\mathbb{P}_x^{\mathrm{B}}$, $x \in \mathbb{R}$, subtracting $\frac{1}{2}th(x)$ (which equals $tL_n f(x)$ almost everywhere) and dividing by t we arrive at

$$\frac{1}{t}(\hat{P}_t^n f(x) - f(x)) - \tfrac{1}{2}h(x) = \frac{1}{2t}\mathbb{E}_x^{\mathrm{B}} \int_0^t (\mathrm{e}^{-nL_s^0} h(B_s) - h(x))\,\mathrm{d}s, \tag{5.10}$$

$x \in \mathbb{R}$. Note that

$$\sup_{t \in [0,1]} \frac{1}{t}\biggl(\mathbb{E}_{(\cdot)}^{\mathrm{B}} \int_0^t |h(B_s)|\,\mathrm{d}s\biggr) = \sup_{t \in [0,1]} \frac{1}{t}\int_0^t \frac{1}{\sqrt{2\pi s}}\int_{\mathbb{R}} \exp\biggl\{-\frac{|\cdot - y|^2}{2s}\biggr\} h(y)\,\mathrm{d}y\,\mathrm{d}s \in L^2(\mathbb{R})$$

since this function is bounded and decays exponentially at ∞ due to the compact support of h. Thus, by Lebesgue's Dominated Convergence Theorem the right-hand side of (5.10) tends to 0 in $L^2(\mathbb{R})$ as $t \to 0$. Since $D \subset L^2(\mathbb{R})$ is dense, it follows that $(\hat{P}_t^n)_{t \geqslant 0}$ is strongly

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

continuous in $L^2(\mathbb{R})$. Moreover, we see that it is generated by an extension of (L_n, D). Since D is a core for $(L_n, D(L_n))$, this shows that $(\hat{P}_t^n)_{t\geqslant 0}$ equals $(P_t^n)_{t\geqslant 0}$ as a semigroup on $L^2(\mathbb{R})$ and the assertion follows for bounded continuous $f \in L^2(\mathbb{R})$ from the continuity of $\hat{P}_t^n f$ and $P_t^n f$, and then for general f, e.g. using monotone convergence. □

Let T be an exponentially distributed random variable (the survival time at 0) with expectation value $1/n$ which is independent of $(B_t)_{t\geqslant 0}$. We define, for $t \geqslant 0$,

$$\hat{X}_t := \begin{cases} B_t & \text{if } L_t^0 < T, \\ \Delta & \text{if } L_t^0 \geqslant T. \end{cases}$$

Using the above representation of $(P_t^n)_{t\geqslant 0}$ and the Markov property of Brownian motion, it is readily verified that the finite-dimensional distributions of $(\hat{X}_t)_{t\geqslant 0}$ under $\mathbb{P}_x^{\mathrm{B}}$ coincide with the finite-dimensional distributions of $(X_t)_{t\geqslant 0}$ under $\mathbb{P}_x^n$, proving that both coincide as laws on $D([0,\infty);\mathbb{R}^\Delta)$. Thus, we obtained a complete description of the processes corresponding to the $\mathcal{E}^n$ in terms of Brownian motion.

6. Tightness and convergence of the laws

6.1. Tightness

We now apply the following well-known tightness criterion to the families $(\mathbb{P}_n^x)_{n\in\mathbb{N}}$ of laws on $D([0,\infty),\mathbb{R}^\Delta)$, $x \in \mathbb{R}\setminus\{0\}$.

Proposition 6.1. *Let (E, r) be a complete separable metric space and let $(\mathbb{P}^n)_{n\in\mathbb{N}}$ be a sequence of probability laws on $D([0,\infty), E)$. Assume that the following conditions are satisfied:*

(a) *for all $\eta > 0$ and all $T > 0$ there exists a compact set $\Gamma_{\eta,T} \subset E$ such that*

$$\inf_{n\in\mathbb{N}} \mathbb{P}^n(X_T \in \Gamma_{\eta,T}) \geqslant 1-\eta;$$

(b) *for all $T > 0$ there exist $\beta > 0$, $C < \infty$ and $\theta > 1$ such that for $t \in [0,T]$ and $h \leqslant t$ it holds that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n[r(X_t, X_{t-h})^{\beta/2} r(X_{t+h}, X_t)^{\beta/2}] \leqslant Ch^\theta$$

and

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \mathbb{E}^n[r(X_\delta, X_0)^\beta] = 0.$$

Proof. See [**8**, Theorems 3.7.2, 3.8.6 and 3.8.8]. □

In this section, we consider Δ to be adjoined to $\mathbb{R}$ as an isolated point, as we may do due to Remark 5.8 (ii). We define a metric r on $\mathbb{R}^\Delta$ by setting $r(x,y) := |x-y| \wedge 1$ for $x, y \in \mathbb{R}$ and $r(x,\Delta) := 1$ for $x \in \mathbb{R}$. Note that this metric is complete and generates the correct topology on $\mathbb{R}^\Delta$.

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

Theorem 6.2. *For every $x \in \mathbb{R} \setminus \{0\}$, the sequence $(\mathbb{P}_x^n)_{n\in\mathbb{N}}$ of laws on $D([0,\infty),\mathbb{R}^{\varDelta})$ is tight.*

Proof. Before applying Proposition 6.1 we make two estimates. First, note that by the Burkholder–Davis–Gundy inequality and Theorem 5.9, for any $t, h \geqslant 0$ and $n \in \mathbb{N}$ the following holds:

$$\mathbb{E}_x^n[|\operatorname{id}(X_{t+h}) - \operatorname{id}(X_t)|^4] = \mathbb{E}_x^n[|M_{t+h}^{[\mathrm{id}]} - M_t^{[\mathrm{id}]}|^4] \leqslant C\mathbb{E}_x^n[\langle M_{t+\cdot}^{[\mathrm{id}]} - M_t^{[\mathrm{id}]}\rangle_h^2] \leqslant Ch^2, \quad (6.1)$$

where $C < \infty$ is independent of n and t.

The second observation concerns the probability that the process is killed after a short time. Let $(W_t)_{t\geqslant 0}$ be as in (5.6). By Lemma 5.7 we know that

$$\zeta \geqslant \inf\{t > 0 \colon W_t = -x\} =: \tau_0$$

holds $\mathbb{P}_x^n \otimes \mathbb{P}^{\mathrm{B}}$-a.s. and that $(W_t)_{t\geqslant 0}$ is a Brownian motion starting at 0 with respect to this probability distribution. Suppose that $x > 0$ (for $x < 0$ one makes similar considerations). By the reflection principle (see [**7**, Example 1.3.3]) and since W_δ is normally distributed with mean 0 and variance δ we obtain, for any $n \in \mathbb{N}$,

$$\mathbb{P}_x^n(\zeta \leqslant \delta) \leqslant \mathbb{P}_x^n \otimes \mathbb{P}^{\mathrm{B}}(\tau_0 \leqslant \delta) = 2\mathbb{P}_x^n \otimes \mathbb{P}^{\mathrm{B}}(W_\delta < -x) = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{-x/\sqrt{\delta}} \mathrm{e}^{-z^2/2}\,\mathrm{d}z, \quad (6.2)$$

which is independent of $n \in \mathbb{N}$ and converges to 0 as $\delta \to 0$.

For verifying assumption (a) in Proposition 6.1 we observe that, for $k \in \mathbb{N}$, $T > 0$ and $n \in \mathbb{N}$, it holds by (6.1) that

$$\begin{aligned} \mathbb{P}_x^n(X_T \notin [-k,k] \cup \{\varDelta\}) &= \mathbb{P}_x^n(|\operatorname{id}(X_T)| > k) \\ &\leqslant \frac{1}{k^4}\mathbb{E}_x^n[|\operatorname{id}(X_T)|^4] \\ &\leqslant \frac{8}{k^4}(x^4 + \mathbb{E}_x^n[|\operatorname{id}(X_T) - \operatorname{id}(X_0)|^4]) \\ &\leqslant \frac{8}{k^4}(x^4 + CT^2) \to 0 \end{aligned}$$

as $k \to \infty$. Thus, (a) holds with $\varGamma_{\eta,T} := [-k,k] \cup \{\varDelta\}$, if k is chosen large enough.

We now verify assumption (b). Let $T > 0$, $0 \leqslant h \leqslant t$ and $n \in \mathbb{N}$. Observe that $\mathbb{P}_x^n$-a.s. the following holds:

$$r(X_t, X_{t-h})^4 r(X_{t+h}, X_t)^4 \leqslant r(X_t, X_{t-h})^4 1_{\{X_t \neq \varDelta\}},$$

i.e. the expression on the left-hand side equals 0 if the process dies before time t and can be estimated by $r(X_t, X_{t-h})^4$ otherwise. This implies that

$$\begin{aligned} \sup_{n\in\mathbb{N}} \mathbb{E}_x^n[r(X_t, X_{t-h})^4 r(X_{t+h}, X_t)^4] &\leqslant \sup_{n\in\mathbb{N}} \mathbb{E}_x^n[r(X_t, X_{t-h})^4 1_{\{X_t \neq \varDelta\}}] \\ &\leqslant \sup_{n\in\mathbb{N}} \mathbb{E}_x^n[|\operatorname{id}(X_t) - \operatorname{id}(X_{t-h})|^4] \\ &\leqslant Ch^2, \end{aligned}$$

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

where we have used (6.1). Moreover, for $\delta > 0$ the following holds:

$$\sup_{n\in\mathbb{N}} \mathbb{E}_x^n[r(X_\delta, X_0)^4] \leqslant \sup_{n\in\mathbb{N}} \mathbb{E}_x^n[|\operatorname{id}(X_\delta) - \operatorname{id}(X_0)|^4] + \sup_{n\in\mathbb{N}} \mathbb{P}_x^n(\zeta \leqslant \delta).$$

Using (6.1) and (6.2) we find that this converges to 0 as $\delta \to 0$. This completes the proof. □

Proposition 6.3. *The sequence $(\mathbb{P}_0^n)_{n\in\mathbb{N}}$ is not tight; hence, it does not converge.*

Proof. By [**8**, Theorems 3.7.2 and 3.8.6] a necessary condition for tightness of $(\mathbb{P}_0^n)_n$ is that there exists $\beta > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}_0^n[r^\beta(X_\delta, X_0)] \to 0 \quad \text{as } \delta \to 0. \tag{6.3}$$

For $0 < \varepsilon$, define the stopping time $\tau = \inf\{t \geqslant 0 \colon |X_t| = \varepsilon \text{ or } X_t = \Delta\}$. Then we have, by (5.6) and (5.7), for any $\delta > 0$,

$$\begin{aligned}
\mathbb{E}_0^n[r^\beta(X_\delta, X_0)] &\geqslant \mathbb{P}_0^n(X_\delta = \Delta) \\
&\geqslant \mathbb{P}_0^n \otimes \mathbb{P}^{\mathrm{B}}(|W_\delta| \geqslant \varepsilon) - \mathbb{P}_0^n(|X_\tau| = \varepsilon) \\
&= 2\int_\varepsilon^\infty \frac{1}{\sqrt{2\pi\delta}} \mathrm{e}^{-y^2/2\delta}\,\mathrm{d}y - \frac{1}{1+n\varepsilon}
\end{aligned}$$

Thus,

$$\sup_{n\in\mathbb{N}} \mathbb{E}_0^n[r^\beta(X_\delta, X_0)] \geqslant 2\int_\varepsilon^\infty \frac{1}{\sqrt{2\pi\delta}} \mathrm{e}^{-y^2/2\delta}\,\mathrm{d}y \to 1 \quad \text{as } \varepsilon \to 0.$$

Since this holds for any $\delta > 0$, it follows that $\lim_{\delta\downarrow 0} \sup_{n\in\mathbb{N}} \mathbb{E}_0^n[r^\beta(X_\delta, X_0)] = 1$, contradicting (6.3). □

6.2. Convergence of the laws

In order to prove weak convergence of the sequence $(\mathbb{P}_x^n)_{n\in\mathbb{N}}$ for any $x \in \mathbb{R}\backslash\{0\}$, it is now sufficient to prove that it has at most one accumulation point. By [**8**, Proposition 3.7.1] this reduces to proving that the finite-dimensional distributions of any two accumulation points $\mathbb{P}_x, \hat{\mathbb{P}}_x$ coincide on a set D dense in $[0,\infty)$, i.e. for $f_1,\dots,f_k \in C_{\mathrm{b}}^0(\mathbb{R}^\Delta)$ and $0 \leqslant t_1 < \cdots < t_k$, $t_1,\dots,t_k \in D$, the expectation of $f_1(X_{t_1})\cdots f_k(X_{t_k})$ is the same with respect to $\mathbb{P}_x$ and $\hat{\mathbb{P}}_x$. Here $f_k(\Delta) \neq 0$ would be allowed. However, since $f_i = 1_{\mathbb{R}} f_i + (1 - 1_{\mathbb{R}}) f_i(\Delta)$, $1 \leqslant i \leqslant k$, and by a monotone convergence argument, we find that we only have to consider $f_1,\dots,f_k \in C_{\mathrm{c}}^0(\mathbb{R})$ (with $f_i(\Delta) = 0$ for all $1 \leqslant i \leqslant k$).

Lemma 6.4. *For any $x \in \mathbb{R}\setminus\{0\}$ the sequence $(\mathbb{P}_x^n)_{n\in\mathbb{N}}$ converges weakly to some law $\mathbb{P}_x$ on $D([0,\infty),\mathbb{R}^\Delta) =: \Omega$.*

Proof. Let $\mathbb{P}_x$ be an accumulation point of $(\mathbb{P}_x^n)_{n\in\mathbb{N}}$, $\mathbb{P}_x = \lim_{k\to\infty} \mathbb{P}_x^{n_k}$. By [**8**, Theorem 3.7.8.], the finite-dimensional distributions of $\mathbb{P}_x^{n_k}$ converge to those of $\mathbb{P}_x$ as $k \to \infty$

on a set $D_{\mathbb{P}_x} \subset [0, \infty)$ having an at most countable complement in $[0, \infty)$. At first, we consider one-dimensional distributions. Let $t > 0$, $t \in D_{\mathbb{P}_x}$ and $f \in C_{\mathrm{c}}^0(\mathbb{R})$. Then

$$\int_\Omega f(X_t)\,\mathrm{d}\mathbb{P}_x^n = P_t^n f(x) \quad \text{for all } x \in \mathbb{R} \setminus \{0\}, \tag{6.4}$$

since $(P_t^n)_{t \geqslant 0}$ is the transition semigroup of $\boldsymbol{M}^n$. Let $K \subset \mathbb{R}$ be compact. By Lemma 2.4, $P_t^n f|_K$, $n \in \mathbb{N}$, are equicontinuous. So there is a subsequence of $P_t^n f$ which converges uniformly on K to some function g. Since $P_t^n f$ is a continuous version of $T_t^n f$ for any $n \in \mathbb{N}$ and $T_t^n f$ converges to $T_t f$ in $L^2(\mathbb{R})$ by Theorem 3.2 and (3.1), we find that the limit g is a continuous version of $T_t f|_K$; in particular, the limit is uniquely determined. Therefore, $P_t^n f$ converges locally uniformly to the continuous version $P_t f$ of $T_t f$ as $n \to \infty$ and

$$P_t f(x) = \int_\Omega f(X_t)\,\mathrm{d}\mathbb{P}_x.$$

We consider the step from one-dimensional to two-dimensional distributions. Let $f, g \in C_{\mathrm{c}}^0(\mathbb{R})$ and $0 < s \leqslant t < \infty$, $s, t \in D_{\mathbb{P}_x}$. By the Markov property,

$$\begin{aligned}
\int_\Omega f(X_s)g(X_t)\,\mathrm{d}\mathbb{P}_x^n &= \mathbb{E}_x^n(f(X_s)g(X_t)) \\
&= \mathbb{E}_x^n[f(X_s)\mathbb{E}_x^n(g(X_t)|\mathcal{F}_s)] \\
&= \mathbb{E}_x^n[f(X_s)\mathbb{E}_{X_s}^n(g(X_{t-s}))] \\
&= \int_\Omega f(X_s) \int_\Omega g(X_{t-s})\,\mathrm{d}\mathbb{P}_{X_s}^n\,\mathrm{d}\mathbb{P}_x^n \\
&= P_s^n(fP_{t-s}^n g)(x).
\end{aligned} \tag{6.5}$$

Furthermore, L^2-convergence of $(P_r^n h)_{n \in \mathbb{N}}$ for all $h \in L^2(\mathbb{R})$ and $r \geqslant 0$ implies that $(P_s^n(fP_{t-s}^n g))_{n \in \mathbb{N}}$ converges in $L^2(\mathbb{R})$:

$$\begin{aligned}
\|P_s^n f P_{t-s}^n g - P_s f P_{t-s} g\| &\leqslant \|P_s^n(fP_{t-s}^n g - fP_{t-s}g)\| + \|(P_s^n - P_s)fP_{t-s}g\| \\
&\leqslant \|f\|_\infty \|(P_{t-s}^n - P_{t-s})g\| + \|(P_s^n - P_s)fP_{t-s}g\| \to 0
\end{aligned}$$

as $n \to \infty$. Moreover, since $\|fP_{t-s}^n g\|_{L^2} \leqslant \|f\|_\infty \|g\|_{L^2}$, using Lemma 2.4 we obtain equicontinuity of $(P_s^n(fP_{t-s}^n g))_{n \in \mathbb{N}}$ on compact sets. So we can conclude as above and obtain

$$P_s(fP_{t-s}g)(x) = \int_\Omega f(X_s)g(X_t)\,\mathrm{d}\mathbb{P}_x.$$

A similar calculation can be done for all finite-dimensional distributions. Therefore, and by the considerations preceding this lemma, the finite-dimensional distributions of any two accumulation points $\mathbb{P}_x$ and $\hat{\mathbb{P}}_x$ of $(\mathbb{P}_x^n)_{n \in \mathbb{N}}$ coincide on a dense set $D_{\mathbb{P}_x} \cap D_{\hat{\mathbb{P}}_x}$, proving the assertion. □

Remark 6.5. There are two further important consequences of the above proof:

(i) for any $f \in L^2(\mathbb{R})$ there exist continuous versions $P_t f$ of $T_t f$, $t > 0$;

(ii) the finite-dimensional distributions of the limiting law $\mathbb{P}_x$, $x \in \mathbb{R} \setminus \{0\}$, are given in terms of $(P_t)_{t > 0}$, similarly to (6.4) and (6.5), at least on a dense subset of $[0, \infty)$.

Set $\mathbb{P}_\Delta := \delta_{\{X_t=\Delta \text{ for all } t\}}$, $\mathbb{P}_0 := \delta_{\{X_t=0 \text{ for all } t\}}$, i.e. these two laws put full measure on constant paths at Δ and 0, respectively. Define

$$\boldsymbol{M} := (D([0,\infty), \mathbb{R}^\Delta), \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (X_t)_{t\geqslant 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^\Delta}).$$

Theorem 6.6. *The limiting diffusion process* $\boldsymbol{M}$ *defined above is properly associated with the Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *from Theorem 3.2. Moreover, each* $\mathbb{P}_x$, $x \in \mathbb{R}\setminus\{0\}$, *is the law of a Brownian motion starting in* x *with absorption at* 0.

Proof. When considered on $L^2(\mathbb{R}\setminus\{0\})$ instead of $L^2(\mathbb{R})$, the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is regular and local. (Both spaces are isomorphic; the difference lies in considering 0 not to be an element of the state space.) As in Theorem 4.4 we find that $(\mathcal{E}, D(\mathcal{E}))$ is properly associated with a Hunt diffusion process

$$\bar{\boldsymbol{M}} = (\bar{\Omega}, (\bar{\mathcal{F}}_t)_{t\geqslant 0}, (\bar{X}_t)_{t\geqslant 0}, (\bar{\mathbb{P}}_x)_{x\in(\mathbb{R}\setminus\{0\})^\Delta}).$$

Δ is adjoined to $\mathbb{R}\setminus\{0\}$ as the one-point compactification. This clearly implies that the process is killed as soon as it 'approaches' 0. Using the same arguments as in the proof of Lemma 4.2 we find that any $\mathcal{E}$-quasi-continuous function is continuous on $\mathbb{R}\setminus\{0\}$. Therefore, the arguments from § 5 are also valid for $\bar{\mathbb{P}}_x$, $x \in \mathbb{R}\setminus\{0\}$, and $\bar{\mathbb{P}}_x$ describes a Brownian motion with absorption at 0. Considering Δ again as an isolated point and extending $\bar{\boldsymbol{M}}$ trivially to $\mathbb{R}^\Delta$ (adding 0 to the state space and setting $\bar{\mathbb{P}}_0 := \delta_{\{X_t=0 \text{ for all } t\}}$ [**17**, p. 118]), we obtain a special standard diffusion process, which we also denote by

$$\bar{\boldsymbol{M}} = (\bar{\Omega}, (\bar{\mathcal{F}}_t)_{t\geqslant 0}, (\bar{X}_t)_{t\geqslant 0}, (\bar{\mathbb{P}}_x)_{x\in(\mathbb{R})^\Delta}).$$

This process is properly associated with $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathbb{R})$, since it follows from the last assertion in Theorem 3.2 and Definition 4.1 that 0 is an $\mathcal{E}$-exceptional set and the notion of $\mathcal{E}$-quasi-continuity is not changed by this extension. We may without loss of generality assume that $\bar{\Omega} = D([0,\infty); \mathbb{R}^\Delta)$.

By definition, $\mathbb{P}_0 = \bar{\mathbb{P}}_0$ and $\mathbb{P}_\Delta = \bar{\mathbb{P}}_\Delta$. Let us look at the initial points $x \in \mathbb{R}\setminus\{0\}$. Denote by $(\bar{P}_t)_{t>0}$ the transition semigroup corresponding to $\bar{\boldsymbol{M}}$. For $f \in L^2(\mathbb{R})$ we find by continuity that $\bar{P}_t f$ coincides with the continuous version $P_t f$ of $T_t f$ everywhere on $\mathbb{R}\setminus\{0\}$. Therefore, by Remark 6.5 and the considerations preceding Lemma 6.4, we obtain $\mathbb{P}_x = \bar{\mathbb{P}}_x$. □

Acknowledgements. The authors are grateful to Heinrich von Weizsäcker for communicating the problem at the 'Probability and Functional Analysis' Seminar at TU Kaiserslautern.

References

1. R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Volume 65 (Academic Press, New York, 1975).
2. S. Albeverio, Yu. G. Kondratiev and M. Röckner, Strong Feller properties for distorted Brownian motion and applications to finite particle systems with singular interactions, in *Finite and infinite dimensional analysis in honor of Leonard Gross*, Contemporary Mathematics, Volume 317 (American Mathematical Society, Providence, RI, 2003).

https://doi.org/10.1017/S0013091509001552 Published online by Cambridge University Press

3. H. Attouch, *Variational convergence for functions and operators*, Applicable Mathematics Series (Pitman Advanced Publishing Program, Boston, MA, 1984).
4. G. Dal Maso, *An introduction to gamma-convergence* (Birkhäuser, Boston, MA, 1993).
5. G. Da Prato and M. Röckner, Singular dissipative stochastic equations in Hilbert spaces, *Prob. Theory Relat. Fields* **124**(2) (2002), 261–303.
6. A. de Masi, N. Ianiro, A. Pellegrinotti and E. Presutti, A survey of the hydrodynamical behavior of many-particle systems, in *Nonequilibrium phenomena II: from stochastics to hydrodynamics*, Studies in Statistical Mechanics, Volume 11, pp. 193–294 (North-Holland, Amsterdam, 1984).
7. R. Durrett, *Stochastic calculus: a practical introduction* (CRC Press, New York, 1996).
8. S. Ethier and T. Kurtz, *Markov processes: characterization and convergence* (Wiley, 1986).
9. M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes* (Walter de Gruyter, Berlin, 1994).
10. M. Grothaus, Scaling limit of fluctuations for the equilibrium Glauber dynamics in continuum, *J. Funct. Analysis* **239** (2006), 414–445.
11. M. Grothaus, Yu. G. Kondratiev, E. Lytvynov and M. Röckner, Scaling limit of stochastic dynamics in classical continuous systems, *Annals Probab.* **31** (2003), 1494–1532.
12. M. Grothaus, Yu. G. Kondratiev and M. Röckner, N/V-limit for stochastic dynamics in continuous particle systems, *Prob. Theory Relat. Fields* **137** (2007), 121–160.
13. T. Kato, *Perturbation theory for linear operators* (Springer, 1966).
14. V. Liskevich and Y. A. Semenov, *Some problems on Markov semigroups*, Mathematical Topics, Volume 11 (Akademie, Berlin, 1996).
15. T. J. Lyons and T. S. Zhang, Decomposition of Dirichlet processes and its applications, *Annals Probab.* **22** (1994), 494–524.
16. T. J. Lyons and W. A. Zheng, A crossing estimate for the canonical process on a Dirichlet space and a tightness result, *Astérisque* **157/158** (1988), pp. 249–272.
17. Z.-M. Ma and M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms* (Springer, 1992).
18. U. Mosco, Composite media and asymptotic Dirichlet forms, *J. Funct. Analysis* **123** (1994), 368–421.
19. M. Reed and B. Simon, *Methods of modern mathematical physics, I, Functional analysis* (Academic Press, New York, 1980).
20. B. Simon, A canonical decomposition for quadratic forms with applications to monotone convergence theorems, *J. Funct. Analysis* **28** (1978), 377–385.