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CONTRIBUTIONS TO A GENERAL THEORY OF VIEW-OBSTRUCTION PROBLEMS

V. C. DUMIR, R. J. HANS-GILL AND J. B. WILKER

ABSTRACT In the original view-obstruction problem congruent closed, centrally symmetric convex bodies centred at the points of the set $(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}) + \mathbb{N}^n$ in $\mathbb{R}^n$ are expanded uniformly until they block all rays from the origin into the open positive cone The central problem is to determine the minimal blocking size and this value is known for balls in dimensions $n = 2, 3$ and for symmetrically placed cubes in dimensions $n = 2, 3, 4$

In order to explain fully the distinction between rational and irrational rays in the original problem, we extend consideration to the blocking of subspaces of all dimensions In order to appreciate the special properties of balls and cubes, we give a discussion of the convex body with respect to reflection symmetry, lower dimensional sections, and duality We introduce topological considerations to help understand when the critical parameter of the theory is an attained maximum and we add substantially to the list of known values of this parameter In particular, when the dimension is $n = 2$ our dual body considerations furnish a complete solution to the view-obstruction problem

1. **Introduction.** In 1973, T. W. Cusick [6] introduced a problem in geometry of numbers which he called the *view-obstruction problem*. This problem in its original form has attracted sustained interest over the years ([3]–[12]) but we introduce it here in a slightly generalized form to accomodate the results we intend to present.

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space; $\mathbb{Z}^n$, the integer lattice; $\frac{1}{2}$, the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and Λ, the shifted lattice $\frac{1}{2} + \mathbb{Z}^n$. Let $\mathbf{C}$ be a closed, centrally symmetric convex body centred at the origin in $\mathbb{R}^n$ and let $\mathbf{L}$ be a vector subspace of the real n-space underlying $\mathbb{R}^n$. We define the function

$$\nu(\mathbf{C}, \mathbf{L}) = \inf\{\alpha > 0 : (\alpha\mathbf{C} + \Lambda) \cap \mathbf{L} \neq \emptyset\}$$

and seek to understand its behaviour, especially when $\mathbf{C}$ is fixed and $\mathbf{L}$ varies. In terms of the metric on $\mathbb{R}^n$ defined by

$$d_c(\mathbf{x}, \mathbf{y}) = \inf\{\alpha > 0 : \mathbf{y} \in \alpha\mathbf{C} + \mathbf{x}\}$$

the function $\nu(\mathbf{C}, \mathbf{L})$ is just the distance between the two sets Λ and $\mathbf{L}$, that is,

$$\nu(\mathbf{C}, \mathbf{L}) = d_c(\Lambda, \mathbf{L}).$$

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Another useful observation to make at this stage is that if $\mathbf{C}' \supset \mathbf{C}$ and $\mathbf{L}' \supset \mathbf{L}$ then $\nu(\mathbf{C}', \mathbf{L}') \leq \nu(\mathbf{C}, \mathbf{L})$

In the original problem, attention is restricted to the open positive cone $\mathbf{P} = (\mathbb{R}^{+})^{n}$ and to rays that lie in $\mathbf{P}$ The central problem is to determine

$$\nu(\mathbf{C}) = \nu(\mathbf{C}, 1) = \sup\{\nu(\mathbf{C}, \mathbf{L}) \quad \mathbf{L} \cap \mathbf{P} \neq \emptyset, \dim \mathbf{L} = 1\}$$

where the supremum is known to exist because the set on the right is bounded above by the number $d_{c}(\frac{1}{2}, \mathbf{0})$ This statement of the problem in terms of 1-dimensional subspaces can be shown to be equivalent to the original formulation in terms of rays

In number theoretic problems involving extreme values of certain functions it is natural to study successive extreme values and find out whether a Markoff type chain of isolated extreme values exists In the present context these investigations in $(n-1)$ dimensions can sometimes be used to determine $\nu(\mathbf{C}, 1)$ in n dimensions (see [10]–[12]) In a later paper we shall show that these investigations for special bodies can lead to results about other convex bodies in the same dimension These remarks direct interest to the full spectrum

$$\{\nu(\mathbf{C}, \mathbf{L}) \quad \mathbf{L} \text{ 1-dimension}\}$$

In the same spirit we shall now explain how a thorough treatment of irrational rays in the original problem leads very naturally to consideration of higher dimensional subspaces

A 1-dimensional subspace $\mathbf{L}$ of $\mathbb{R}^{n}$ is called *rational* if it can be written $\mathbf{L} = \langle \mathbf{a} \rangle$ with $\mathbf{a} \in \mathbb{Q}^{n}$, otherwise it is called *irrational* If $\mathbf{L}$ is rational, it can be written in the form $\mathbf{L} = \langle \mathbf{a} \rangle$ with $\mathbf{a} \in \mathbb{Z}^{n}$ and the fact that Λ and $\mathbf{L}$ are both invariant under the translation $\mathbf{x} \longrightarrow \mathbf{x} + \mathbf{a}$ can be used to simplify the calculation of $\nu(\mathbf{C}, \mathbf{L})$ and to improve its status from an infimum to an attained minimum On the other hand, if $\mathbf{L}'$ is an irrational 1-dimensional subspace then for any $\varepsilon > 0$ there is, by Kronecker's Theorem, a rational 1-dimensional subspace $\mathbf{L} = \mathbf{L}(\varepsilon)$ such that $\nu(\mathbf{C}, \mathbf{L}') < \nu(\mathbf{C}, \mathbf{L}) + \varepsilon$ A number of authors ([5], [6], [10]) have used this type of reasoning in discussions of $\nu(\mathbf{C}, 1)$ to show that is sufficient to take the supremum over rational subspaces, that is

$$\nu(\mathbf{C}, 1) = \sup\{\nu(\mathbf{C}, \mathbf{L}) \quad \mathbf{L} \cap \mathbf{P} \neq \emptyset, \dim \mathbf{L} = 1, \mathbf{L} \text{ is rational}\}$$

While the argument given above is quite correct, it leaves a false impression about the behaviour of irrational subspaces In dimension $n = 2$, the full truth is that if $\mathbf{L}$ is irrational then $\nu(\mathbf{C}, \mathbf{L}) = 0$ In higher dimensions there are "degrees of irrationality" and what we can say is that an irrational 1-dimensional subspace $\mathbf{L}$ lies in a uniquely determined rational m-dimensional subspace $\mathbf{M} = \mathbf{M}(\mathbf{L})$, $1 < m \leq n$, and satisfies $\nu(\mathbf{C}, \mathbf{L}) = \nu(\mathbf{C}, \mathbf{M})$ More generally, an irrational d-space $\mathbf{L}$ (see §2 for definitions) lies in a uniquely determined rational m-space $\mathbf{M} = \mathbf{M}(\mathbf{L})$, $d < m \leq n$, and satisfies $\nu(\mathbf{C}, \mathbf{L}) = \nu(\mathbf{C}, \mathbf{M})$ This general result gives good understanding about links in the full spectrum

$$\{\nu(\mathbf{C}, \mathbf{L}) \quad \mathbf{L} \text{ a subspace of } \mathbb{R}^{n}\}$$

and can be used to eliminate irrational subspaces from the calculation of quantities such as

$$\nu(\mathbf{C}, d) = \sup\{\nu(\mathbf{C}, \mathbf{L}) : \mathbf{L} \cap \mathbf{P} \neq \emptyset, \dim \mathbf{L} = d\}.$$

A tool used in our proofs about irrationality is the fractional parts map $\varphi \colon \mathbb{R}^n \longrightarrow [0, 1)^n$ given by $\varphi\big((x_1, x_2, \ldots, x_n)\big) = (\{x_1\}, \{x_2\}, \ldots, \{x_n\})$. By extension of an earlier remark we have

$$\nu(\mathbf{C}, \mathbf{L}) = d_c(\Lambda, \mathbf{L}) = d_c\big(\varphi(\Lambda), \varphi(\mathbf{L})\big) = d_c\Big(\frac{1}{2}, \varphi(\mathbf{L})\Big)$$

but this formula must be used with caution because the d_c-distance on $[0, 1)^n$ is to be taken in the sense of the torus $[0, 1)^n = \mathbb{R}^n/\mathbb{Z}^n$ rather than in the sense of the metric subspace $[0, 1)^n \subset \mathbb{R}^n$. However, when the metric d_c on $\mathbb{R}^n$ is such that the cells of the partition $\mathbb{R}^n = [0, 1)^n + \mathbb{Z}^n$ are Dirichlet cells for Λ, this distinction can be dropped, at least for distances measured from $\frac{1}{2} \in [0, 1)^n$. We show that this situation prevails whenever the symmetry group of $\mathbf{C}$ includes reflections in all the coordinate hyperplanes $x_i = 0$, $i = 1, 2, \ldots, n$, and it brings with it a substantial simplification in the calculation of $\nu(\mathbf{C}, \mathbf{L})$. In other considerations related to the symmetries of $\mathbf{C}$, we mention cases where the values of $\nu(\mathbf{C}, \mathbf{L})$ for subspaces $\mathbf{L}$ lying in a coordinate hyperplane $\mathbf{H}$ can and cannot be related to the lower dimensional problem $\nu(\mathbf{C} \cap \mathbf{H}, \mathbf{L})$.

A consideration which has not appeared previously in view-obstruction problems is the relation of a convex body $\mathbf{C}$ to its dual body $\mathbf{C}^*$. We exploit this relation in n dimensions to give an explicit formula for $\nu(\mathbf{C}, \mathbf{L})$ when $\mathbf{L}$ has dimension $n - 1$.

We have mentioned an instance of the result that when $\mathbf{L}$ is rational,

$$\nu(\mathbf{C}, \mathbf{L}) = d_c(\Lambda, \mathbf{L}) = \inf\{d_c(\mathbf{z}, \mathbf{x}) : \mathbf{z} \in \Lambda, \mathbf{x} \in \mathbf{L}\}$$

is an attained minimum. We prove this result in full generality and it prompts us to ask whether

$$\nu(\mathbf{C}, d) = \sup\{\nu(\mathbf{C}, \mathbf{L}) : \mathbf{L} \cap \mathbf{P} \neq \emptyset, \dim \mathbf{L} = d\}$$

is an attained maximum. In pursuit of the answer to this question we place a metric on the space of subspaces of $\mathbb{R}^n$ which renders it compact and show that $\nu(\mathbf{C}, \mathbf{L})$ is upper semi-continuous in $\mathbf{L}$ with respect to this metric. This machinery reduces the problem of showing that $\nu(\mathbf{C}, d)$ is an attained maximum to the problem of showing that the set of subspaces $\mathbf{L}$ which needs to be considered, can be taken to be closed.

Finally, by bringing all of our techniques to bear on the view-obstruction problem in dimension $n = 2$ we can give a rather complete analysis. For any 2-dimensional closed, centrally symmetric convex body $\mathbf{C}$, the value $\nu(\mathbf{C}, 1)$ is always an attained maximum but there are two quite different extremes of behaviour. On the one hand we find a large class of bodies $\mathbf{C}$ including the square $\max\{|x_1|, |x_2|\} \leq \frac{1}{2}$ and the disk $x_1^2 + x_2^2 \leq \frac{1}{4}$ for which it is true that

$$\nu(\mathbf{C}, 1) = \max\Big\{\nu\big(\mathbf{C}, \langle(2, 1)\rangle\big), \nu\big(\mathbf{C}, \langle(1, 2)\rangle\big)\Big\}$$

and for which the only centres in Λ that are actually needed for blocking arbitrary subspaces are those with $x_1 = \frac{1}{2}$ or $x_2 = \frac{1}{2}$ In contrast we identify bodies $\mathbf{C}$ for which the critical subspace $\mathbf{L}$ that determines $\nu(\mathbf{C}, 1)$ is blocked only at centres in Λ that are far from the axes This last class of examples provides a nice justification for the original idea of using all the centres in Λ for view-obstruction

2 **Irrational subspaces.** A d-dimensional subspace $\mathbf{L}$ of $\mathbb{R}^n$ is called *rational* if it has a basis consisting of vectors in $\mathbb{Q}^n$

$$\mathbf{L} = \langle \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d \rangle, \quad \mathbf{a}_i \in \mathbb{Q}^n, \; i = 1, 2, \ldots, d,$$

or, equivalently, if it is the intersection of $n - d$ independent hyperplanes with normals in $\mathbb{Q}^n$

$$\mathbf{L} = \langle \mathbf{c}_1, \mathbf{c}_2, \ldots \mathbf{c}_{n-d} \rangle^{\perp} = \bigcap_{j=1}^{n-d} \mathbf{c}_j^{\perp}, \quad \mathbf{c}_j \in Q^n, \; j = 1, 2, \ldots, n - d$$

A subspace $\mathbf{L}$ of $\mathbb{R}^n$ which is not rational is called *irrational* Any subspace $\mathbf{L}$ of $\mathbb{R}^n$ is contained in a unique minimal rational subspace $\mathbf{M} = \mathbf{M}(\mathbf{L})$ which can be characterized as

$$\mathbf{M}(\mathbf{L}) = \bigcap \{ \mathbf{c}^{\perp} : \mathbf{c} \in \mathbb{Q}^n, \mathbf{c}^{\perp} \supset \mathbf{L} \}$$

Of course, if $\mathbf{L}$ is rational then $\mathbf{M}(\mathbf{L}) = \mathbf{L}$

If $\mathbf{L}$ is a d-dimensional subspace and $\mathbf{L} \cap \mathbf{P} \neq \emptyset$ then it is possible to translate each vector in an arbitrary basis for $\mathbf{L}$ by a vector in $\mathbf{L} \cap \mathbf{P}$ with sufficiently large norm that the resulting vectors form a basis for $\mathbf{L}$ and lie in $\mathbf{P}$ It follows that $\mathbf{L} \cap \mathbf{P}$ has topological dimension d On the other hand, the rational hyperplanes $\mathbf{c}^{\perp}$ which do not contain $\mathbf{L}$ satisfy $\dim(\mathbf{c}^{\perp} \cap \mathbf{L}) < d$ and there are only countably many of them These two observations show the existence of a vector

$$\mathbf{a}_0 \in (\mathbf{L} \cap \mathbf{P}) \setminus \bigcup \{ \mathbf{c}^{\perp} : \mathbf{c} \in \mathbb{Q}^n, \mathbf{c}^{\perp} \not\supset \mathbf{L} \}$$

Then $\mathbf{L}_0 = \langle \mathbf{a}_0 \rangle$ is a 1-dimensional subspace of $\mathbf{L}$, and

$$\mathbf{M}(\mathbf{L}_0) = \bigcap_{\mathbf{c}^{\perp} \supset \mathbf{L}} \mathbf{c}^{\perp} = \bigcap_{\mathbf{c}^{\perp} \supset \mathbf{L}} \mathbf{c}^{\perp} = \mathbf{M}(\mathbf{L})$$

Moreover, since $\mathbf{L}_0 \subset \mathbf{L} \subset \mathbf{M}$,

$$\nu(\mathbf{C}, \mathbf{L}_0) \geq \nu(\mathbf{C}, \mathbf{L}) \geq \nu(\mathbf{C}, \mathbf{M})$$

In a series of lemmas we shall show that under the fractional parts map $\varphi(\mathbf{L}_0)$ is dense in $\varphi(\mathbf{M})$ and therefore

$$\nu(\mathbf{C}, \mathbf{L}_0) = d_c\Big(\frac{1}{2}, \varphi(\mathbf{L}_0)\Big) = d_c\Big(\frac{1}{2}, \varphi(\mathbf{M})\Big) = \nu(\mathbf{C}, \mathbf{M})$$

This will show that $\nu(\mathbf{C}, \mathbf{L}) = \nu(\mathbf{C}, \mathbf{M})$ These results are summarized as

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THEOREM 1. *Every irrational d-space* $\mathbf{L} \subset \mathbb{R}^n$ *is contained in a unique rational m-space* $\mathbf{M} = \mathbf{M}(\mathbf{L})$ *of minimum dimension* $m(\mathbf{L}) > d$ *and this space satisfies* $\nu(\mathbf{C}, \mathbf{L}) = \nu(\mathbf{C}, \mathbf{M})$.

COROLLARY. *The constant* $\nu(\mathbf{C}, d)$ *defined earlier can be computed as*

$$\nu(\mathbf{C}, d) = \sup\{\nu(\mathbf{C}, \mathbf{L}) : \mathbf{L} \cap \mathbf{P} \neq \emptyset, \dim \mathbf{L} = d, \mathbf{L} \textit{ is rational}\}.$$

PROOF OF COROLLARY. Let $\mathbf{L}'$ be an irrational subspace of dimension d which meets $\mathbf{P}$. Let $\mathbf{L}$ be any rational subspace of $\mathbf{M}(\mathbf{L}') = \mathbf{M}$ of dimension d which meets $\mathbf{P}$. Then

$$\nu(\mathbf{C}, \mathbf{L}') = \nu(\mathbf{C}, \mathbf{M}) \leq \nu(\mathbf{C}, \mathbf{L}).$$

The rest of the section is devoted to the discussion required for the proof of the fact that $\varphi(\mathbf{L}_0)$ is dense in $\varphi(\mathbf{M})$.

Suppose $\mathbf{L}$ is an irrational d-space given in the form

$$\mathbf{L} = \langle \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d \rangle.$$

By writing the vectors $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ as rows we can regard $\mathbf{L}$ as the row space of the $d \times n$ matrix $A = (a_{ij})$. By elementary row operations and possible reordering of coordinates we can assume that

1) for $j \leq d$, $a_{ij} = \delta_{ij}$, and
2) there is an integer m with $d < m \leq n$ such that the first m columns of A are linearly independent over $\mathbb{Q}$ and the remaining $n - m$ columns are $\mathbb{Q}$-linear combinations of the first m.

The first condition, which is mainly intended to help fix ideas, says that the points of $\mathbf{L}$ are of the form

$$\mathbf{L} = \Big\{\big(x_1, x_2, \ldots, x_d, (x_1, x_2, \ldots, x_d)L\big) : x_1, x_2, \ldots, x_d \in \mathbb{R}\Big\}$$

where L is a *real* $d \times (n - d)$ matrix equal to the last $n - d$ columns of the matrix A. The second condition invites us to consider the m-dimensional space

$$\mathbf{M} = \Big\{\big(x_1, x_2, \ldots, x_m, (x_1, x_2, \ldots, x_m)M\big) : x_1, x_2, \ldots, x_m \in \mathbb{R}\Big\}$$

where M is the *rational* $m \times (n - m)$ matrix whose j-th column gives the coefficients of the expression for the $m + j$-th column of A in terms of its first m columns.

LEMMA 1. *The subspace* $\mathbf{M}$ *defined above is equal to the minimal rational subspace* $\mathbf{M}(\mathbf{L})$.

PROOF. Let I be the $(n - m) \times (n - m)$ identity matrix. Then the $n - m$ columns of $\binom{M}{-I}$ are independent rational vectors perpendicular to the rows of A. No rational vector outside their span could be perpendicular to the rows of A because it together with these columns would generate a non-trival $\mathbb{Q}$-linear relation among the first m columns of A. This completes the proof that $\mathbf{M} = \mathbf{M}(\mathbf{L})$.

LEMMA 2. *If* $\mathbf{L} \cap \mathbf{P} \neq \emptyset$ *there is a vector* $\mathbf{a}_0 = \big(1, a_2, \ldots, a_m, (1, a_2, \ldots, a_m)M\big) \in \mathbf{L} \cap \mathbf{P}$ *with* $\{1, a_2, \ldots, a_m\}$ *linearly independent over* $\mathbb{Q}$.

PROOF. The existence of $\mathbf{a}_0 \in \mathbf{L} \cap \mathbf{P}$ was argued in the preamble to Theorem 1. The normalization to $a_1 = 1$ is possible because the eligible vectors form a positive cone. The form of $\mathbf{a}_0$ as regards the matrix M applies to all vectors in $\mathbf{L}$ because $\mathbf{L} \subset \mathbf{M}$. The $\mathbb{Q}$-dependence of $\{1, a_2, \ldots, a_m\}$ would imply the existence of $\mathbf{c} \in \mathbb{Q}^n$ with $\mathbf{a}_0 \in \mathbf{c}^{\perp}$ and $\mathbf{L} \not\subset \mathbf{c}^{\perp}$ contrary to the definition of $\mathbf{a}_0$.

It is convenient to write

$$\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} = \{(\mathbf{u}, \mathbf{v}) : \mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^{n-m}\}.$$

Then the subspace $\mathbf{M}$ is equal to the graph of the linear mapping $\mathbf{v} = \mathbf{u}M$. We let q be the least common multiple of the denominators occurring in the rational matrix M and for $1 \leq i \leq m$ set $\mathbf{e}_i = (0, 0, \ldots, 0, q, 0, \ldots, 0) \in \mathbb{R}^m$ with q in the i-th place. The vectors $(\mathbf{e}_i, \mathbf{e}_i M)$ lie in $\mathbb{Z}^n$ and generate a translation group G consisting of symmetries for both Λ and $\mathbf{M}$. As a first step in applying the fractional parts map $\varphi: \mathbb{R}^n \longrightarrow [0, 1)^n$ we define $\varphi_1: \mathbb{R}^n \longrightarrow [0, q)^m \times \mathbb{R}^{n-m}$ by noting that $[0, q)^m \times \mathbb{R}^{n-m}$ is a fundamental domain for G and letting φ_1 be the quotient map. Since the elements of G are symmetries of Λ and $\mathbf{M}$, $\varphi_1(\Lambda)$ and $\varphi_1(\mathbf{M})$ are just the sections of Λ and $\mathbf{M}$ by the cylinder $[0, q)^m \times \mathbb{R}^{n-m}$. The subspaces $\mathbf{L}_0 = \langle \mathbf{a}_0 \rangle$ and $\mathbf{L}$ are nested subspaces of $\mathbf{M}$ and so their images under φ_1 satisfy $\varphi_1(\mathbf{L}_0) \subset \varphi_1(\mathbf{L}) \subset \varphi_1(\mathbf{M})$. The sets $\varphi_1(\mathbf{L}_0)$ and $\varphi_1(\mathbf{L})$ are obtained by restricting $\mathbf{v} = \mathbf{u}M$ to suitable domains in $[0, q)^m$ and we can show that both these sets are dense in $\varphi_1(\mathbf{M})$ by showing that the domain of $\varphi_1(\mathbf{L}_0)$ is dense in $[0, q)^m$. This will complete the proof of Theorem 1 because φ is equal to $\varphi_2 \circ \varphi_1$ where φ_2 maps the cells of $[0, 1)^n + \mathbb{Z}^n$ that lie in $[0, q)^m \times \mathbb{R}^{n-m}$ isometrically to $[0, 1)^n$.

LEMMA 3. *The domain of* $\varphi_1(\mathbf{L}_0)$ *is dense in* $[0, q)^m$.

PROOF. With $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$, the subspace $\mathbf{L}_0$ is given by restricting the linear mapping $\mathbf{v} = \mathbf{u}M$ to the domain $\mathbf{u} = (t, a_2 t, \ldots, a_m t)$ with $t \in \mathbb{R}$. The domain of $\varphi_1(\mathbf{L}_0)$ is given by reducing these coordinates modulo q to

$$\left\{\left(q\left\{\frac{t}{q}\right\}, q\left\{\frac{a_2 t}{q}\right\}, \ldots, q\left\{\frac{a_m t}{q}\right\}\right) : t \in \mathbb{R}\right\} = \left\{q(\{t'\}, \{a_2 t'\}, \ldots, \{a_m t'\}) : t' = \frac{t}{q} \in \mathbb{R}\right\}.$$

This set is dense in $[0, q)^m$ because a direct application of Kronecker's Theorem ([14], p. 382) to the rationally independent numbers $\{1, a_2, \ldots, a_m\}$ assures us that the set

$$\{(\{t\}, \{a_2 t\}, \ldots, \{a_m t\}) : t \in \mathbb{R}\}$$

is dense in $[0, 1)^m$.

3. **Rational subspaces.** We have seen that $\nu(\mathbf{C},d) = \sup\{\nu(\mathbf{C},\mathbf{L}) : \mathbf{L}\cap\mathbf{P}\neq\emptyset, \dim\mathbf{L}=d\}$ can be computed by considering only rational subspaces $\mathbf{L}$. The object of this section is to investigate the calculation of $\nu(\mathbf{C},\mathbf{L})$ when $\mathbf{L}$ is a rational subspace.

Since rational subspaces $\mathbf{L}$ satisfy $\mathbf{M}(\mathbf{L})=\mathbf{L}$, the analysis of §2 shows, in effect, that rational d-spaces admit a representation in

$$\mathbb{R}^n = \mathbb{R}^d\times\mathbb{R}^{n-d} = \{(\mathbf{u},\mathbf{v}) : \mathbf{u}\in\mathbb{R}^d, \mathbf{v}\in\mathbb{R}^{n-d}\}$$

of the form $\mathbf{v}=\mathbf{u}L$ where L is a rational matrix. Moreover, if q is the least common multiple of the denominators occurring in L then every point on $\mathbf{L}$ is equivalent by an integral translation to at least one point on the section of $\mathbf{L}$ by the *closed* cylinder $[0,q]^d\times\mathbb{R}^{n-d}$ and therefore $d_c(\Lambda,\mathbf{L}) = d_c\big(\Lambda,\mathbf{L}\cap([0,q]^d\times\mathbb{R}^{n-d})\big)$. It is an important fact that this section is compact because this allows us to prove

THEOREM 2. *If $\mathbf{L}$ is a rational d-space then*

$$\nu(\mathbf{C},\mathbf{L}) = d_c(\Lambda,\mathbf{L}) = \inf\{d_c(\mathbf{z},\mathbf{x}) : \mathbf{z}\in\Lambda, \mathbf{x}\in\mathbf{L}\}$$

is an attained minimum.

PROOF. For any subset $\mathbf{S}\subset\mathbb{R}^n$ the related function $f\colon\mathbb{R}^n\to\mathbb{R}$ defined by $f(\mathbf{x}) = d_c(\mathbf{S},\mathbf{x})$ is Lipschitz in $\mathbf{x}$ and hence continuous. In particular the function $f(\mathbf{x}) = d_c(\Lambda,\mathbf{x})$ is continuous and therefore attains a minimum, at $\mathbf{x}_0$, say, on the compact set $\mathbf{L}\cap([0,q]^d\times\mathbb{R}^{n-d})$. Since Λ is discrete there must be a point $\mathbf{z}_0$ of Λ such that $f(\mathbf{x}_0) = d_c(\mathbf{z}_0,\mathbf{x}_0)$. It follows that $d_c(\Lambda,\mathbf{L}) = d_c(\mathbf{z}_0,\mathbf{x}_0)$ as required.

Theorem 2 suggests an idea for computing $\nu(\mathbf{C},\mathbf{L})$. First, by considering those $\mathbf{z}\in\Lambda$ that are "near" $\mathbf{L}\cap([0,q]^d\times\mathbb{R}^{n-d})$ we can produce a finite list of $\mathbf{z}_i\in\Lambda$ which must include a closest point $\mathbf{z}_0$. Next we can solve the "calculus"problem of minimizing $f_i(\mathbf{x}) = d_c(\mathbf{z}_i,\mathbf{x})$ with $\mathbf{x}\in\mathbf{L}$ for each of the centres $\mathbf{z}_i$ on our list. Finally, we can obtain $\nu(\mathbf{C},\mathbf{L})$ as the smallest of these finitely many minima. One situation in which this algorithm works particularly well is when the cells of $[0,1)^n+\mathbb{Z}^n$ are Dirichlet cells for Λ under the metric d_c on $\mathbb{R}^n$.

Let Γ be a shifted lattice in $\mathbb{R}^n$. For $\mathbf{z}_0\in\Gamma$, $D(\mathbf{z}_0)$ is called a *Dirichlet cell* if it contains

$$\{\mathbf{x}\in\mathbb{R}^n : d_c(\mathbf{z}_0,\mathbf{x}) < d_c(\mathbf{z},\mathbf{x})\ \forall\mathbf{z}\in\Gamma \text{ with } \mathbf{z}\neq\mathbf{z}_0\}$$

and is contained in

$$\{\mathbf{x}\in\mathbb{R}^n : d_c(\mathbf{z}_0,\mathbf{x}) \leq d_c(\mathbf{z},\mathbf{x})\ \forall\mathbf{z}\in\Gamma \text{ with } \mathbf{z}\neq\mathbf{z}_0\}.$$

If $\Gamma' = \Gamma+\mathbf{d}$ where $\mathbf{d}\in\mathbb{R}^n$ is fixed, then $D(\mathbf{z}_0)+\mathbf{d}$ is a Dirichlet cell for $\mathbf{z}_0+\mathbf{d}\in\Gamma'$. In particular $[0,1)^n+\mathbb{Z}^n$ is a family of Dirichlet cells for Λ if and only if $[-\frac{1}{2},\frac{1}{2})^n+\mathbb{Z}^n$ is a family of Dirichlet cells for $\mathbb{Z}^n$. We proceed to give an example in $\mathbb{R}^2$ which reveals some surprising possibilities and helps to clarify the definition of Dirichlet cell.

Let $\mathbf{C}$ be the rectangular convex body

$$\mathbf{C}\{(x_1,x_2)\in\mathbb{R}^2 : -3\leq x_1\leq 3 \text{ and } -2\leq x_2\leq 2\}.$$

The inequality $d_c(\mathbf{0},\mathbf{x}) < d_c(\mathbf{z},\mathbf{x})$ for all $\mathbf{z} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ defines the open convex octagon with vertices $(\frac{1}{2},\frac{1}{3})$, $(\frac{1}{4},\frac{1}{2})$, and others related to these by reflections in the coordinate axes On the other hand, the inequality $d_c(\mathbf{0},\mathbf{x}) \le d_c(\mathbf{z},\mathbf{x})$ for all $\mathbf{z} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ defines the closed, non-convex (but star-shaped) octagon with vertices $(\frac{1}{2},\frac{1}{3})$, $(\frac{3}{4},\frac{1}{2})$, and others related to these by reflections in the coordinate axes (See Figure 1 and note that the points of the closed triangle with vertices $(\frac{1}{2},\frac{1}{3})$, $(\frac{3}{4},\frac{1}{2})$, $(\frac{1}{4},\frac{1}{2})$ are all equidistant from $\mathbf{0}$ and $\mathbf{z} = (1,0)$ in the metric d_c) One might have expected the second set to be the closure of the first set, but this is not the case One might also have expected the translates of one or the other of these sets to the centres of $\mathbb{Z}^2$ to give a partition of $\mathbb{R}^2$ (modulo boundary points) but this too is false. Nevertheless, the set $[-\frac{1}{2},\frac{1}{2})^2$ does lie between these two sets and its translates do give the desired partition As a generalization of the essential features of this example we shall prove

THEOREM 3 *The cells of* $[0,1)^n + \mathbb{Z}^n$ *are Dirichlet cells for* $\Lambda = \frac{1}{2} + \mathbb{Z}^n$ *under the metric* d_c *whenever* $\mathbf{C}$ *is symmetric by reflection in all n coordinate hyperplanes*

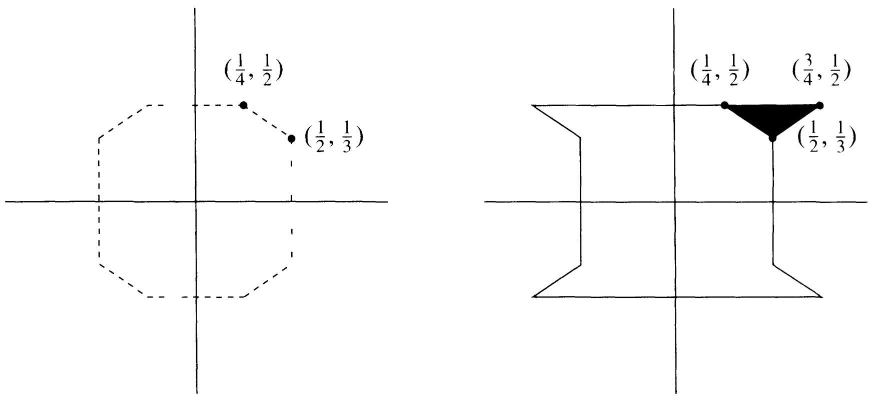

FIGURE 1 The sets defined by $d_c(0,x) < d_c(z\ x)$ and by $d_c(0,x) \le d_c(z,x)$ with $C = [-3,3] \times [-2,2]$

This theorem has a strong bearing on our algorithm for computing $\nu(\mathbf{C},\mathbf{L})$ because of the following

COROLLARY. *If* $\mathbf{C}$ *is symmetric by reflections in all n coordinate hyperplanes then at least one of the points of* Λ *closest to the rational d-space* $\mathbf{L}$ *is among the centres of the cells of* $[0,1)^n + \mathbb{Z}^n$ *which meet* $\mathbf{L} \cap ([0,q]^d \times \mathbb{R}^{n-d})$.

PROOF For each point $\mathbf{x}$ of $\mathbf{L}$, no point of Λ is closer than the centre of the cell to which $\mathbf{x}$ itself belongs. Accordingly we need only examine those centres $\mathbf{z}_i \in \Lambda$ with the property that the cell centred at $\mathbf{z}_i$ meets $\mathbf{L} \cap ([0,q]^d \times \mathbb{R}^{n-d})$ and indeed contains one of the points of $\mathbf{L}$ closest to $\mathbf{z}_i$ The second condition is informative but impossible to phrase as a selection criterion so we include only the first condition in the statement of the corollary

As an application of the corollary we take up the example of a 1-dimensional rational subspace $\mathbf{L} = \langle \mathbf{a} \rangle$ where $\mathbf{a} \in (\mathbb{Z}^+)^n$ and g. c. d. $\{a_1, a_2, \ldots, a_n\} = 1$. The greatest integer function allows us to write the finite list of centres $\mathbf{z}_i \in \Lambda$ that need to be considered, in the compact form

$$\mathbf{z} = \frac{1}{2} + ([a_1, t], [a_2 t], \ldots, [a_n t]), \quad 0 \le t \le \frac{1}{2}.$$

The extra reduction from $0 \le t < 1$ to $0 \le t \le \frac{1}{2}$ comes about because we can avail ourselves of the symmetry $\mathbf{x} \longrightarrow -\mathbf{x}$ as well as the symmetry $\mathbf{x} \longrightarrow \mathbf{x} + \mathbf{a}$. The number of these trial centres is at most $1 + \sum^{*n}_{i=1} [\frac{1}{2} a_i]$ where the $*$ indicates summation over distinct a_i's. This is because the line $\mathbf{L}$ crosses $[\frac{1}{2} a_i]$ cell walls parallel to the i-th coordinate hyperplane as t runs through the interval $0 \le t \le \frac{1}{2}$ and each new crossing, counting simultaneous crossings as one, generates one new centre. Earlier papers have suggested $\prod_{i=1}^n a_i$ trial centres in the special case when $\mathbf{C}$ is a ball or a box, so our reduction to fewer than $1 + \sum^{*n}_{i=1} [\frac{1}{2} a_i]$ under more general conditions represents a substantial improvement.

Before turning to the proof of Theorem 3 we should see an example where the algorithm fails in the absence of the symmetry condition. Consider the 1-dimensional subspace $\mathbf{L}$ given by $(t, t, 2t)$, $t \in \mathbb{R}$, in $\mathbb{R}^3$. The fact that it is rational implies that there are points $\mathbf{z} \in \Lambda$ and $\mathbf{x} \in \mathbf{L}$ which realize $d_c(\Lambda, \mathbf{L})$ and moreover the point $\mathbf{x}$ can be taken to lie in $\mathbf{L} \cap ([0, 1] \times \mathbb{R}^2)$. However if we take $\mathbf{C}$ to be a long narrow ellipsoid of revolution with its major axis in the direction $(-1, -1, 8)$ the possible values of $\mathbf{z}$ related to these values of $\mathbf{x}$ are $\mathbf{z}_1 = (\frac{1}{2}, \frac{1}{2}, -\frac{3}{2})$ which pairs with $\mathbf{x} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and $\mathbf{z}_2 = (\frac{1}{2}, \frac{1}{2}, \frac{7}{2})$ which pairs with $\mathbf{x} = (\frac{3}{4}, \frac{3}{4}, \frac{3}{2})$. Neither of the corresponding cells meets the line $\mathbf{L}$.

Now towards the proof of Theorem 3 we state and prove

LEMMA 4. *Suppose that* $\mathbf{C}$ *is invariant under reflection in the coordinate hyperplane parallel to the hyperplane* $\mathbf{H}$ *and* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{x}$ *are points of* $\mathbb{R}^n$ *with* $\mathbf{a}$ *and* $\mathbf{b} = \mathbf{a}^{\mathbf{H}}$ *mirror images in* $\mathbf{H}$. *Then*

(i) if $\mathbf{x} \in \mathbf{H}$, $d_c(\mathbf{a}, \mathbf{x}) = d_c(\mathbf{b}, \mathbf{x})$, *and*

(ii) if $\mathbf{x}$ *lies in the same half space as* $\mathbf{a}$ *relative to* $\mathbf{H}$, $d_c(\mathbf{a}, \mathbf{x}) \le d_c(\mathbf{b}, \mathbf{x})$.

The inequality in (ii) is strict if $\mathbf{C}$ *is strictly convex.*

PROOF. (i) Suppose $d_c(\mathbf{a}, \mathbf{x}) = \alpha$ so that $\mathbf{x} \in \mathrm{bdy}(\alpha\mathbf{C} + \mathbf{a})$. Then $\mathbf{x} = \mathbf{x}^{\mathbf{H}} \in \mathrm{bdy}(\alpha\mathbf{C} + \mathbf{a})^{\mathbf{H}}$. If $\mathbf{H}_1$ is the hyperplane parallel to $\mathbf{H}$ through $\mathbf{a}$, the product of reflections $\mathbf{H}_1\mathbf{H}$ is equal to the translation $\mathbf{x} \longrightarrow \mathbf{x} + (\mathbf{b} - \mathbf{a})$. The reflection symmetry of $\mathbf{C}$ shows that

$$(\alpha\mathbf{C} + \mathbf{a})^{\mathbf{H}} = (\alpha\mathbf{C} + \mathbf{a})^{\mathbf{H}_1\mathbf{H}} = (\alpha\mathbf{C} + \mathbf{a}) + (\mathbf{b} - \mathbf{a}) = \alpha\mathbf{C} + \mathbf{b}.$$

Thus $\mathbf{x} \in \mathrm{bdy}(\alpha\mathbf{C} + \mathbf{b})$ and $d_c(\mathbf{b}, \mathbf{x}) = \alpha = d_c(\mathbf{a}, \mathbf{x})$.

(ii) Let $\mathbf{xb}$ meet $\mathbf{H}$ at $\mathbf{y}$. Then

$$d_c(\mathbf{a}, \mathbf{x}) \le d_c(\mathbf{a}, \mathbf{y}) + d_c(\mathbf{y}, \mathbf{x}) = d_c(\mathbf{b}, \mathbf{y}) + d_c(\mathbf{y}, \mathbf{x}) = d_c(\mathbf{b}, \mathbf{x}).$$

The inequality comes from the triangle inequality applies to a proper triangle. It is known that this leads to strict inequality whenever $\mathbf{C}$ is strictly convex.

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LEMMA 5. *If* $\mathbf{C}$ *is invariant under reflection in each of the coordinate hyperplanes and* $\mathbf{x} \in [-\frac{1}{2}, \frac{1}{2})^n$ *then the inequality* $d_c(\mathbf{0}, \mathbf{x}) \leq d_c(\mathbf{z}, \mathbf{x})$ *holds for all* $\mathbf{z} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$.

PROOF. The proof is by induction on $\delta(\mathbf{z}) = \sum_{i=1}^n |z_i|$. If $\delta(\mathbf{z}) = 1$ and the non-zero coordinate of $\mathbf{z}$ is $z_j = \pm 1$ we apply Lemma 4(ii) to the hyperplane $\mathbf{H}$ given by $x_j = \pm\frac{1}{2}$.

Now suppose the result is true for δ_0. Consider $\mathbf{x} \in [-\frac{1}{2}, \frac{1}{2})^n$ and $\mathbf{z} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $\delta(\mathbf{z}) = \delta_0 + 1$. Suppose that the segment $\mathbf{zx}$ meets a face $\mathbf{H}$ of $[-\frac{1}{2}, \frac{1}{2}]^n$ at $\mathbf{y}$. If $\mathbf{H}$ is the hyperplane $x_j = \pm\frac{1}{2}$ then $\mathbf{z}^{\mathbf{H}}$ differs from $\mathbf{z}$ only in its j-th component and this is reduced by 1 in absolute value so $\delta(\mathbf{z}^{\mathbf{H}}) = \delta_0$ and $d_c(\mathbf{z}^{\mathbf{H}}, \mathbf{x}) \geq d_c(\mathbf{0}, \mathbf{x})$ by induction. This proves that

$$\begin{aligned} d_c(\mathbf{z}, \mathbf{x}) &= d_c(\mathbf{z}, \mathbf{y}) + d_c(\mathbf{y}, \mathbf{x}) \\ &= d_c(\mathbf{z}^{\mathbf{H}}, \mathbf{y}) + d_c(\mathbf{y}, \mathbf{x}) \\ &\geq d_c(\mathbf{z}^{\mathbf{H}}, \mathbf{x}) \geq d_c(\mathbf{0}, \mathbf{x}) \end{aligned}$$

as required.

Lemma 5 implies that $[-\frac{1}{2}, \frac{1}{2})^n + \mathbb{Z}^n$ is a Dirichlet partition for $\mathbb{Z}^n$ under the metric d_c and hence $[0, 1)^n + \mathbb{Z}^n$ is a Dirichlet partition for Λ. This completes the proof of Theorem 3.

4. **Sections of C.** If a subspace $\mathbf{L} \subset \mathbb{R}^n$ does not lie in any of the coordinate hyperplanes then it meets one of the open cones which comprise the complement of these hyperplanes and there is no serious loss of generality in assuming, as we have done, that it meets the positive cone. For by applying reflections in suitably chosen coordinate hyperplanes we can move $\mathbf{L}$ to a subspace $\mathbf{L}'$ that does meet $\mathbf{P}$ and either these reflections belong to the symmetry group of $\mathbf{C}$, in which case $\nu(\mathbf{C}, \mathbf{L}) = \nu(\mathbf{C}, \mathbf{L}')$ or else they transform $\mathbf{C}$ to a related body $\mathbf{C}'$, in which case interest shifts to the related problem $\nu(\mathbf{C}, \mathbf{L}) = \nu(\mathbf{C}', \mathbf{L}')$. We now give some consideration to subspaces $\mathbf{L}$ that lie in a coordinate hyperplane $\mathbf{H}$; for reasons similar to the ones given above, we can assume that $\mathbf{L}$ meets the part of $\mathbf{H}$ that constitutes a wall of $\mathbf{P}$.

We are interested in cases where consideration of these subspaces reduces to an $(n-1)$-dimensional problem in $\mathbf{H} = \mathbb{R}^{n-1}$ and $\nu(\mathbf{C}, \mathbf{L})$ is nicely related to $\nu(\mathbf{C} \cap \mathbf{H}, \mathbf{L})$. This will happen if, for suitable values of α, there is a function $\beta = \beta(\alpha)$ such that

$$\left(\alpha\mathbf{C} + \frac{1}{2} + \mathbb{Z}^n\right) \cap \mathbf{H} = \beta(\mathbf{C} \cap \mathbf{H}) + \frac{1}{2} + \mathbb{Z}^{n-1}.$$

We shall see that this condition holds for many bodies including the generalized ℓ_p-balls

$$\mathbf{C}_{p,b} : \sum_{i=1}^n \left(\frac{2|x_i|}{b_i}\right)^p \leq 1; \quad b_i > 0,\ 1 \leq p \leq \infty.$$

On the other hand the property obviously fails whenever

(1) the sections of $\mathbf{C}$ by hyperplanes parallel to $\mathbf{H}$ are not mutually similar, or

(2) the collection of the bodies $(\alpha\mathbf{C} + \frac{1}{2} + \mathbb{Z}^n) \cap \mathbf{H}$ are not properly situated in $\mathbf{H}$.

Bodies $\mathbf{C}$ symmetric by reflection in the coordinate hyperplane $\mathbf{H}$ were an important feature of the last section. They reappear here whenever pairs of bodies from $\alpha\mathbf{C} + \Lambda$ which lie closest to $\mathbf{H}$ but on opposite sides of $\mathbf{H}$, have the same intersection with $\mathbf{H}$. This is a very natural precursor of our desired condition, thought not the only one possible.

LEMMA 6. *If* $\mathbf{H}$ *is a coordinate hyperplane,* $\mathbf{a} \notin \mathbf{H}$*, and* $(\alpha\mathbf{C}+\mathbf{a})\cap\mathbf{H} = (\alpha\mathbf{C}+\mathbf{a}^{\mathbf{H}})\cap\mathbf{H}$ *for all* $\alpha > 0$ *then* $\mathbf{C}$ *is symmetric by reflection in* $\mathbf{H}$.

PROOF. Since $\mathbf{H}$ is invariant under the dilatation $\mathbf{x} \longrightarrow \alpha\mathbf{x}$, the condition can be rewritten

$$\alpha[(\mathbf{C} + \alpha^{-1}\mathbf{a}) \cap \mathbf{H}] = \alpha[(\mathbf{C} + \alpha^{-1}\mathbf{a}^{\mathbf{H}}) \cap \mathbf{H}]$$

and then

$$\mathbf{C} \cap (\mathbf{H} - \alpha^{-1}\mathbf{a}) + \alpha^{-1}\mathbf{a} = \mathbf{C} \cap (\mathbf{H} - \alpha^{-1}\mathbf{a}^{\mathbf{H}}) + \alpha^{-1}\mathbf{a}^{\mathbf{H}}.$$

If the component of $\alpha^{-1}\mathbf{a}$ that is perpendicular to $\mathbf{H}$ is denoted $\mathbf{n} = \mathbf{n}(\alpha)$, this last equation says that the symmetrically placed sets $\mathbf{C} \cap (\mathbf{H} - \mathbf{n})$ and $\mathbf{C} \cap (\mathbf{H} + \mathbf{n})$ are congruent. Since this holds for all $\mathbf{n}$, the set $\mathbf{C}$ is symmetric by reflection in $\mathbf{H}$.

THEOREM 4. *Let* $\mathbf{C}$ *be a convex body which is symmetric by reflection in each of the coordinate hyperplanes. Then if the sections of* $\mathbf{C}$ *parallel to the coordinate hyperplane* $\mathbf{H}$ *are mutually similar, we can compute* $\nu(\mathbf{C}, \mathbf{L})$ *for subspaces* $\mathbf{L} \subset \mathbf{H}$ *from* $\nu(\mathbf{C}\cap\mathbf{H}, \mathbf{L})$.

PROOF. Suppose $\mathbf{H}$ is given by $x_n = 0$. Since $\mathbf{C}$ is convex and symmetric by reflection in $\mathbf{H}$ we have for any $\mathbf{z} = (\mathbf{z}', z_n) \in \frac{1}{2} + \mathbb{Z}^n$ that

$$[\alpha\mathbf{C} + (\mathbf{z}', z_n)] \cap \mathbf{H} \subset \left[\alpha\mathbf{C} + \left(\mathbf{z}', \frac{1}{2}\right)\right] \cap \mathbf{H} = \left[\alpha\mathbf{C} + \left(\mathbf{z}', -\frac{1}{2}\right)\right] \cap \mathbf{H}.$$

By considering the product of reflections in the $n-1$ hyperplanes through the two points $(\mathbf{z}', \pm\frac{1}{2})$ and parallel in turn to the first $n-1$ coordinate hyperplanes, we see that this intersection with $\mathbf{H}$ is a centrally symmetric body centred at $\mathbf{z}' \in \frac{1}{2} + \mathbb{Z}^{n-1}$. These intersection bodies are mutually congruent for fixed α and varying $\mathbf{z}'$. The third condition in the hypothesis implies that they are similar to $\mathbf{C} \cap \mathbf{H}$. This means that we have

$$\left(\alpha\mathbf{C} + \frac{1}{2} + \mathbb{Z}^n\right) \cap \mathbf{H} = \beta(\mathbf{C} \cap \mathbf{H}) + \frac{1}{2} + \mathbb{Z}^{n-1}$$

as required.

Finally we give an example of this type of computation. Suppose that $\mathbf{C}$ is the generalized ℓ_p-ball $\sum_{i=1}^{n}(\frac{2|x_i|}{b_i})^p \leq 1$ with $1 \leq p < \infty$ and $\mathbf{L}$ is the 1-dimensional subspace spanned by $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ with $a_{m+1} = a_{m+2} = \cdots = a_n = 0$. Then $\alpha\mathbf{C} + (z_1, z_2, \ldots, z_m, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ meets the m-flat $x_{m+1} = x_{m+2} = \cdots = x_n = 0$ in

$$\sum_{i=1}^{m}\left(\frac{2|x_i - z_i|}{\alpha b_i}\right)^p \leq 1 - \sum_{j=m+1}^{n}\left(\frac{1}{\alpha b_j}\right)^p.$$

This means that

$$\beta = \alpha\left[1 - \sum_{j=m+1}^{n}\left(\frac{1}{\alpha b_j}\right)^p\right]^{1/p}$$

and

$$\nu^p(\mathbf{C}, \mathbf{L}) = \sum_{j=m+1}^{n}\left(\frac{1}{b_i}\right)^p + \nu^p\left(\mathbf{C} \cap \left(\bigcap_{j=m+1}^{n} \mathbf{H}_j\right), \mathbf{L}\right)$$

where $\mathbf{H}_j$ is the coordinate hyperplane $x_j = 0$. This formula makes a useful addition to the short list of things that are known exactly and we take this opportunity to give a brief summary.

If $\mathbf{C}$ is the cube $\max\{2|x_i|\} \leq 1$ then for any $\mathbf{L}$ lying in a coordinate hyperplane $\mathbf{H}$, $\nu(\mathbf{C}, \mathbf{L}) = 1$. In dimensions $n = 2, 3, 4$ it is known that $\nu(\mathbf{C}, 1) = \frac{1}{3}$ [6], $\frac{1}{2}$ ([6]–[8]), and $\frac{3}{5}$ [9], respectively. See also ([1], [3], [4], [16]) where the same problem is studied in a different way.

If $\mathbf{C}$ is the Euclidean ball $\sum_{i=1}^{n}(2x_i)^2 \leq 1$ then for $n = 2, 3$ it is known that $\nu(\mathbf{C}, 1) = \frac{1}{\sqrt{5}}$ [6], and $\sqrt{\frac{3}{7}}$ [10], (see also [4]) respectively. In dimension $n = 2$ we can add

$$\nu^2\big(\mathbf{C}, \langle(1,0)\rangle\big) = 1$$

and in dimension $n = 3$,

$$\nu^2\big(\mathbf{C}, \langle(1,0,0)\rangle\big) = 2,$$

and

$$\nu^2\big(\mathbf{C}, \langle(1,a,0)\rangle\big) = 1 + \nu^2\big(\mathbf{C} \cap \mathbf{H}_3, \langle(1,a)\rangle\big).$$

In dimension $n = 4$, the spectrum of the ball is not yet understood well enough to determine $\nu(\mathbf{C}, 1)$ but for 1-dimensional subspaces lying in coordinate hyperplanes we can use lower dimensional information to say

$$\nu^2\big(\mathbf{C}, \langle(1,0,0,0)\rangle\big) = 3$$
$$\nu^2\big(\mathbf{C}, \langle(1,a,0,0)\rangle\big) = 2 + \nu^2\big(\mathbf{C} \cap \mathbf{H}_3 \cap \mathbf{H}_4, \langle(1,a)\rangle\big), \text{ and}$$
$$\nu^2\big(\mathbf{C}, \langle(1,a,b,0)\rangle\big) = 1 + \nu^2\big(\mathbf{C} \cap \mathbf{H}_4, \langle(1,a,b)\rangle\big).$$

5. **The dual body $\mathbf{C}^*$.** Let $\mathbf{C}$ be a closed, centrally symmetric convex body centred at the origin in $\mathbb{R}^n$. Then

$$f(\mathbf{x}) = d_c(\mathbf{0}, \mathbf{x}) = \inf\{\alpha > 0 : \mathbf{x} \in \alpha\mathbf{C}\}$$

is a norm on $\mathbb{R}^n$ and in terms of f the body $\mathbf{C}$ is the closed unit ball

$$\mathbf{C} = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq 1\}.$$

Starting either from $\mathbf{C}$ or from f and using the standard inner product $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$ on $\mathbb{R}^n$ we can define a new centrally symmetric convex body $\mathbf{C}^*$ and companion norm f^* in such a way that $\mathbf{C}^{**} = \mathbf{C}$ and $f^{**} = f$. See [2] Chapter IV Section 3 or [15] Section 14. The definitions are

$$\mathbf{C}^* = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{x} \in \mathbf{C}\}$$

and

$$f^*(\mathbf{y}) = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x} \cdot \mathbf{y}}{f(\mathbf{x})} = \sup_{f(\mathbf{x})=1} \mathbf{x} \cdot \mathbf{y};$$

$\mathbf{C}^*$ is called the *polar reciprocal body* of $\mathbf{C}$ or the *dual body* of $\mathbf{C}$ and f^* is called the *dual norm* of f Among the basic properties of these objects, the inequality

$$\mathbf{x} \ \mathbf{y} \leq f(\mathbf{x})f^*(\mathbf{y})$$

is most important for us because it leads to a useful expression for the distance from a point to an $(n-1)$-dimensional subspace

A few proofs will enhance intuition for $\mathbf{C}^*$ and f^* and clarify their definitions The body $\mathbf{C}^*$ is convex because it is the intersection of half-spaces $\mathbf{x} \ \mathbf{y} \leq 1$ with $\mathbf{x}$ in $\mathbf{C}$ The body $\mathbf{C}^{**}$ is therefore also convex and $\mathbf{C} \subset \mathbf{C}^{**}$ because every $\mathbf{x}$ in $\mathbf{C}$ obviously satisfies $\mathbf{x} \ \mathbf{y} \leq 1$ for every $\mathbf{y}$ admitted into $\mathbf{C}^*$ Since $\mathbf{C}$ is full dimensional, we can prove $\mathbf{C} = \mathbf{C}^{**}$ by showing that boundary points of $\mathbf{C}$ are also boundary points of $\mathbf{C}^{**}$ If $\mathbf{x}_0 \in \operatorname{bdy} \mathbf{C}$ there is a $\mathbf{y}_0 \in \mathbb{R}^n$ such that the hyperplane $\mathbf{x} \ \mathbf{y}_0 = 1$ touches $\mathbf{C}$ at $\mathbf{x}_0$ and the half-space $\mathbf{x} \ \mathbf{y}_0 \leq 1$ contains $\mathbf{C}$ These two statements imply $\mathbf{x}_0 \ \mathbf{y}_0 = 1$ and $\mathbf{y}_0 \in \mathbf{C}^*$ But if $\mathbf{y}_0 \in \mathbf{C}^*$, $\mathbf{x}_0 \ \mathbf{y}_0 = 1$ implies $\mathbf{x}_0$ lies on the boundary of one of the half spaces whose intersection comprises $\mathbf{C}^{**}$ Normally this would suggest that $\mathbf{x}_0 \not\in \mathbf{C}^{**}$ but since we know that $\mathbf{x}_0 \in \mathbf{C} \subset \mathbf{C}^{**}$ it says instead that $\mathbf{x}_0 \in \operatorname{bdy} \mathbf{C}^{**}$ as required

Since $\mathbf{C}$ is a proper body, it contains a small Euclidean ball and is contained in a large Euclidean ball This means that there is a constant $k \geq 1$ such that the Euclidean norm satisfies

$$\frac{1}{k}\|\mathbf{x}\| \leq f(\mathbf{x}) \leq k\|\mathbf{x}\|$$

Knowing only that f is a norm satisfying these conditions one can work from the definition of f^* to show that f^* is also a norm When $\mathbf{y} \neq \mathbf{0}$ the string of equivalent definitions of $f^*(\mathbf{y})$ can be extended to include

$$f^*(\mathbf{y}) = \sup_{\mathbf{x} \ \mathbf{y} \ 1} \frac{1}{f(\mathbf{x})}$$

and this helps to show that f^* is the companion norm to $\mathbf{C}^*$ For on any ray $t\mathbf{y}$, $t \geq 0$, there is a vector $\mathbf{y}_0$ such that $\mathbf{y}_0$ is the outward pointing normal of a half-space $\mathbf{x} \ \mathbf{y}_0 \leq 1$ that just contains $\mathbf{C}$ If the boundary hyperplane touches $\mathbf{C}$ at $\mathbf{x}_0$ we have in turn $\mathbf{x}_0 \in \operatorname{bdy} \mathbf{C}$, $\mathbf{y}_0 \in \operatorname{bdy} \mathbf{C}^*$, and $f^*(\mathbf{y}_0) = \frac{1}{f(\mathbf{x}_0)} = 1$ as required for f^* to be the companion norm of $\mathbf{C}^*$ This line of reasoning also shows that

$$\mathbf{x} \ \mathbf{y} \leq f(\mathbf{x})f^*(\mathbf{y})$$

and any fixed value of either variable determines at least one ray in the other variable where equality holds For example if $\mathbf{x}_0$ is fixed, the possible rays in $\mathbf{y}$ are precisely those corresponding to the direction of outward normals to hyperplanes that touch $\mathbf{C}$ at $\frac{\mathbf{x}_0}{f(\mathbf{x}_0)}$

THEOREM 5 *Let* $\mathbf{z}$ *be a point of* $\mathbb{R}^n$ *and let* $\mathbf{L}$ *be the* $(n-1)$*-space* $\mathbf{x} \ \mathbf{c} = 0$ *Then the distance* $d_c(\mathbf{z}, \mathbf{L})$ *is given by* $d_c(\mathbf{z}, \mathbf{L}) = \frac{|\mathbf{z} \ \mathbf{c}|}{f^*(\mathbf{c})}$ *where* f^* *is the norm of the body* $\mathbf{C}^*$

PROOF Let us assume that the normal to the $(n-1)$-space is taken so that $\mathbf{z} \ \mathbf{c} = |\mathbf{z} \ \mathbf{c}| > 0$ In this case we have

$$d_c(\mathbf{z}, \mathbf{L}) = \inf_{\mathbf{x} \in \mathbf{L}} d_c(\mathbf{z}, \mathbf{x}) = \inf_{\mathbf{x} \in \mathbf{L}} f(\mathbf{z} - \mathbf{x}) = \inf_{\mathbf{w} \ \mathbf{c} \ \mathbf{z} \ \mathbf{c}} f(\mathbf{w})$$

But $\mathbf{w}\cdot\mathbf{c} \le f(\mathbf{w})f^*(\mathbf{c})$ and hence $f(\mathbf{w}) \ge \frac{\mathbf{w}\cdot\mathbf{c}}{f^*(\mathbf{c})}$. Moreover with $\mathbf{c}$ fixed there are values of $\mathbf{w}$ that attain equality. It follows that $d_c(\mathbf{z},\mathbf{L}) = \frac{|\mathbf{z}\cdot\mathbf{c}|}{f^*(\mathbf{c})}$ as required.

As a corollary to Theorem 5 we obtain an algorithm for solving the view-obstruction problem completely for $(n-1)$-spaces.

COROLLARY. *For* $\mathbf{c}\in\mathbb{Z}^n$ *with* g. c. d. $\{c_1,c_2,\ldots,c_n\}=1$,

$$\nu(\mathbf{C},\mathbf{c}^\perp) = \begin{cases} 0 & \text{if } \sum_{i=1}^n c_i \equiv 0 \pmod 2 \\ \frac{1}{2f^*(\mathbf{c})} & \text{if } \sum_{i=1}^n c_i \equiv 1 \pmod 2. \end{cases}$$

If the integral vectors $\mathbf{c}$ *satisfying* $\mathbf{c}^\perp\cap\mathbf{P}\neq\emptyset$ *and* $\sum_{i=1}^n c_i\equiv 1 \pmod 2$ *are arranged in order of increasing* $f^*(\mathbf{c})$, *we obtain a display of the full spectrum. The first entry is* $\nu(\mathbf{C},n-1)$.

The rest of this section is devoted to a situation where the approach yields quite explicit results.

LEMMA 7. *The dual of the generalized* ℓ_p*-ball*

$$\mathbf{C}_{p,b} : \sum_{i=1}^n\left(\frac{2|x_i|}{b_i}\right)^p \le 1, \quad b_i>0,\ 1\le p\le\infty$$

is the generalized ℓ_q*-ball* $\sum_{i=1}^n(\frac{b_i|y_i|}{2})^q \le 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

PROOF. This is standard except for the $\mathbf{b}$-parameters and homogeneity demands that they may be placed in f^* as we have them.

By combining Lemma 7 and the corollary to Theorem 5 we obtain

THEOREM 6. *In* $\mathbb{R}^2$ *the bodies* $\mathbf{C}_{p,\mathbf{b}}$ *satisfy*

$$\nu(\mathbf{C}_{p,\mathbf{b}},1) = \max\left\{\frac{1}{[b_1^q+(2b_2)^q]^{1/q}}, \frac{1}{[(2b_1)^q+b_2^q]^{1/q}}\right\}.$$

We note that Theorem 6 extends the previously known values [6] for the square and the disk, namely $\nu(\mathbf{C}_{\infty,(1,1)},1)=\frac{1}{3}$ and $\nu(\mathbf{C}_{2,(1,1)},1)=\frac{1}{\sqrt5}$.

6. **A metric on subspaces.** The object of this section is to establish a useful criterion for deciding when the quantity

$$\nu(\mathbf{C},d) = \sup\{\nu(\mathbf{C},\mathbf{L}) : \mathbf{L}\cap\mathbf{P}\neq\emptyset, \dim\mathbf{L}=d\}$$

is an attained maximum. Continuity and compactness arguments will eventually play a role but there are two immediate difficulties: the function $\nu(\mathbf{C},\mathbf{L})$ is not continuous in $\mathbf{L}$ and its domain is not compact. The remark about the domain stems from the obvious fact that $\mathbf{P}$ is open, but the observation about the function runs deeper. Theorem 1 indicates that a subset of the $\mathbf{L}$'s which is so large that it is bound to be dense in any reasonable topology, satisfies $\mathbf{M}(\mathbf{L})=\mathbb{R}^n$. This condition entails $\nu(\mathbf{C},\mathbf{L})=0$ and hence the function cannot be continuous without being identically equal to zero. To escape this apparent impasse, we recall from general topology that we can still establish the existence of a maximum if we relax the hypothesis on the function from continuity to upper semi-continuity.

A famous metric on the non-empty compact subsets of $\mathbb{R}^n$ is the Hausdorff metric. Starting from the Euclidean metric d on $\mathbb{R}^n$ we define for any non-empty subset $\mathbf{A} \subset \mathbb{R}^n$ and for any $\varepsilon > 0$, the ε-neighbourhood of $\mathbf{A}$

$$N_\varepsilon(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{A}) \leq \varepsilon\}$$

and then for any two bounded, non-empty subsets $\mathbf{A}_1, \mathbf{A}_2 \subset \mathbb{R}^2$, the distance

$$h(\mathbf{A}_1, \mathbf{A}_2) = \inf\{\varepsilon > 0 : N_\varepsilon(\mathbf{A}_1) \supset \mathbf{A}_2 \text{ and } N_\varepsilon(\mathbf{A}_2) \supset \mathbf{A}_1\}.$$

If $\mathbf{A}_1 \subset \mathbf{A}_2 \subset \overline{\mathbf{A}_1}$, $h(\mathbf{A}_1, \mathbf{A}_2) = 0$ but if we restrict our attention to closed subsets, then h becomes a metric. It is an important theorem about the Hausdorff metric h that the family $\mathcal{B}$ of non-empty compact subsets of a fixed compact set $\mathbf{B}$ comprise a compact metric space $(\mathcal{B}, h)$. Moreover the non-empty, closed, convex subsets of $\mathbf{B}$ comprise a compact subspace of $(\mathcal{B}, h)$. This fact is often used in convexity theory where it is known as the Blaschke Selection Principle ([13] p. 64 or [15] p. 11).

If $\mathbf{B}$ is the unit ball $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$ we can use $(\mathcal{B}, h)$ to put a metric on the vector subspaces of $\mathbb{R}^n$: for if $\mathbf{L}$ and $\mathbf{L}'$ are vector subspaces of $\mathbb{R}^n$ then $\mathbf{L} \cap \mathbf{B}$ and $\mathbf{L}' \cap \mathbf{B}$ are non-empty, closed, convex subsets of $\mathbf{B}$ and we can define

$$h(\mathbf{L}, \mathbf{L}') = h(\mathbf{L} \cap \mathbf{B}, \mathbf{L}' \cap \mathbf{B}).$$

This metric on subspaces has a number of pleasant properties.

LEMMA 8. *The Hausdorff-related metric on subspaces of $\mathbb{R}^n$ satisfies:*

(i) If $\mathbf{L}$ and $\mathbf{L}'$ are subspaces of $\mathbb{R}^n$, $h(\mathbf{L}, \mathbf{L}') \leq 1$,

(ii) If $\dim \mathbf{L} \neq \dim \mathbf{L}'$, $h(\mathbf{L}, \mathbf{L}') = 1$,

(iii) If $\mathbf{L}_k$ is a Cauchy sequence of subspaces, then $\mathbf{L}_k$ is eventually m-dimensional for some $0 \leq m \leq n$ and there is an m-dimensional subspace $\mathbf{L}$ for which $\mathbf{L}_k \longrightarrow \mathbf{L}$,

(iv) If $\mathbf{L}$ is a subspace, $\mathbf{x}_0 \in \mathbf{L}$, and $\varrho > 0$ there is a $\delta > 0$ with the property that any subspace $\mathbf{L}'$ with $h(\mathbf{L}, \mathbf{L}') < \delta$ contains a point $\mathbf{x}'$ with $d(\mathbf{x}_0, \mathbf{x}') < \varrho$.

PROOF. (i) Let $\mathbf{L}$ and $\mathbf{L}'$ be subspaces of $\mathbb{R}^n$. Since $\mathbf{0} \in \mathbf{L}$ and $\mathbf{L}' \cap \mathbf{B} \subset \mathbf{B}$ we have

$$N_1(\mathbf{L} \cap \mathbf{B}) \supset N_1(\{\mathbf{0}\}) \supset \mathbf{B} \supset \mathbf{L}' \cap \mathbf{B}$$

and symmetrically, $N_1(\mathbf{L} \cap \mathbf{B}') \supset \mathbf{L} \cap \mathbf{B}$. This proves that $h(\mathbf{L}, \mathbf{L}') \leq 1$.

(ii) Suppose $\dim \mathbf{L} < \dim \mathbf{L}'$. Then $\dim \mathbf{L}^\perp = n - \dim \mathbf{L} > n - \dim \mathbf{L}'$. Since $\dim \mathbf{L}^\perp + \dim \mathbf{L}' > n$ there is a unit vector $\mathbf{x}' \in (\mathbf{L}' \cap \mathbf{B}) \cap \mathbf{L}^\perp$. Under these circumstances $N_\varepsilon(\mathbf{L} \cap \mathbf{B}) \supset \mathbf{L}' \cap \mathbf{B} \supset \{\mathbf{x}'\}$ only if $\varepsilon \geq 1$ and $h(\mathbf{L}, \mathbf{L}') = 1$ as required.

(iii) If $\mathbf{L}_k$ is a Cauchy sequence of subspaces, (ii) implies there is an m with $0 \leq m \leq n$ such that $\dim \mathbf{L}_k = m$ from some point onwards. We can assume that the sequence of convex sets $\mathbf{L}_k \cap \mathbf{B} \longrightarrow \mathbf{K}$ where $\mathbf{K}$ is a non-empty, closed, convex subset of $\mathbf{B}$ The proof that $\mathbf{K} = \mathbf{L} \cap \mathbf{B}$ for some m-dimensional subspace $\mathbf{L}$ is similar to the known argument ([2] p 61) that if $\mathbf{C}_k$ is a Cauchy sequence of convex sets and $\mathbf{C}_k \longrightarrow \mathbf{K}$, then $\mathbf{K}$ is convex

(iv) If $\varrho > \|\mathbf{x}_0\|$, any subspace $\mathbf{L}'$ will serve If $\varrho \leq \|\mathbf{x}_0\|$, the cone generated by $\mathbf{0}$ and the ball with centre $\mathbf{x}_0$ and radius ϱ meets $\mathbf{B}$ in a cone with axis $\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} \in \mathbf{L} \cap \mathbf{B}$ If $\delta = \frac{\varrho}{\|\mathbf{x}_0\|}$ and $h(\mathbf{L}, \mathbf{L}') < \delta$ then $N_\delta(\mathbf{L}' \cap \mathbf{B}) \supset \mathbf{L} \cap \mathbf{B}$ and there is a vector $\mathbf{y} \in \mathbf{L}' \cap \mathbf{B}$, which can write as $\mathbf{y} = \frac{\mathbf{x}}{\|\mathbf{x}_0\|}$, that satisfies $d(\mathbf{y}, \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|}) < \frac{\varrho}{\|\mathbf{x}_0\|}$ It follows that $d(\mathbf{x}', \mathbf{x}_0) < \varrho$ as required

THEOREM 7 *The subspaces of $\mathbb{R}^n$ are compact in the Hausdorff-related metric and $\nu(\mathbf{C}, \mathbf{L})$ is upper semi continuous in $\mathbf{L}$ with respect to this metric*

PROOF The compactness follows from Lemma 8 part (iii)

To prove the upper semi-continuity, let $\varepsilon > 0$ There exists $\varrho = \varrho(\varepsilon) > 0$ such that $d(\mathbf{x}, \mathbf{y}) < \varrho$ implies $d_c(\mathbf{x}, \mathbf{y}) < \frac{\varepsilon}{2}$ Since $\nu(\mathbf{C}, \mathbf{L}) = d_c(\Lambda, \mathbf{L})$ there exist points $\mathbf{x}_0 \in \mathbf{L}$ and $\mathbf{z}_0 \in \Lambda$ such that

$$\nu(\mathbf{C}, \mathbf{L}) = d_c(\Lambda, \mathbf{L}) \leq d_c(\mathbf{z}_0, \mathbf{x}_0) + \frac{\varepsilon}{2}$$

By Lemma 8(iv) there exists a $\delta > 0$ such that for any subspace $\mathbf{L}'$ with $h(\mathbf{L}, \mathbf{L}') < \delta$ there exists a point $\mathbf{x}' \in \mathbf{L}'$ with $d(\mathbf{x}_0, \mathbf{x}') < \varrho$ Our opening remark shows that $d_c(\mathbf{x}_0, \mathbf{x}') < \frac{\varepsilon}{2}$ and therefore

$$\begin{aligned}
\nu(\mathbf{C}, \mathbf{L}') &= d_c(\Lambda, \mathbf{L}') \\
&\leq d_c(\mathbf{z}_0, \mathbf{x}') \\
&\leq d_c(\mathbf{z}_0, \mathbf{x}_0) + d_c(\mathbf{x}_0, \mathbf{x}') \\
&< \left(d_c(\Lambda, \mathbf{L}) + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \\
&= \nu(\mathbf{C}, \mathbf{L}) + \varepsilon
\end{aligned}$$

Theorem 7 shows that $\nu(\mathbf{C}, d)$ is an attained maximum whenever we can restrict the subspaces $\mathbf{L}$ that must be considered to a closed and therefore compact subset of these spaces We shall see in §7 that this is always possible for view-obstruction problems in $\mathbb{R}^2$

7 **View-obstruction in $\mathbb{R}^2$.** Let $\mathbf{C}$ be a closed, centrally symmetric convex body in $\mathbb{R}^2$ Consider the family of bodies $\alpha\mathbf{C} + \frac{1}{2}$ and suppose that as α increases from 0, $\alpha\mathbf{C} + \frac{1}{2}$ meets $\mathbf{L}_1 = \langle(2, 1)\rangle$ for the first time when $\alpha = \alpha_1$ and $\mathbf{L}_2 = \langle(1, 2)\rangle$ for the first time when $\alpha = \alpha_2$ Since $\mathbf{L}_1$ and $\mathbf{L}_2$ are hyperplanes and $\mathbf{z} = \frac{1}{2}$ is one of the points of Λ that is closest to them in the Euclidean sense, the dual body formula assures us that $\nu(\mathbf{C}, \mathbf{L}_i) = \alpha_i$ for $i = 1, 2$ Theorem 6 shows that for a large class of bodies $\nu(\mathbf{C}, 1) = \max\{\alpha_1, \alpha_2\}$ This was obtained from an explicit formula for the dual norm but in general such a formula is hard to obtain We now supply an alternative sufficient condition directly in terms of $\mathbf{C}$ which will ensure this result We say that $\mathbf{C}$ satisfies the *blocking condition* if

$$\left(\alpha_i\mathbf{C} + \frac{1}{2}\right) \cap \mathbf{L}_i \subset [0, 1]^2$$

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for $i = 1, 2$. The blocking condition is satisfied if $\mathbf{C}$ is symmetric by reflection in one and hence both of the coordinate axes or if $\mathbf{C}$ is symmetric by reflection in one and hence both of the lines $x_2 = \pm x_1$.

THEOREM 8. *If $\mathbf{C}$ is a centrally symmetric convex body that satisfies the blocking condition then*

$$\nu(\mathbf{C}, 1) = \max\{\nu(\mathbf{C}, \mathbf{L}_1), \nu(\mathbf{C}, \mathbf{L}_2)\}.$$

Moreover if $\mathbf{L}$ is any subspace satisfying $\mathbf{L} \cap \mathbf{P} \neq \emptyset$ then there is a point $\mathbf{z} = (z_1, z_2) \in \Lambda$ with $z_1 = \frac{1}{2}$ or $z_2 = \frac{1}{2}$ such that $\big(\nu(\mathbf{C}, 1)\mathbf{C} + \mathbf{z}\big) \cap \mathbf{L} \neq \emptyset$.

PROOF. Let us write $\alpha_i = \nu(\mathbf{C}, \mathbf{L}_i)$, $i = 1, 2$, as in the preamble to the theorem. Then certainly $\nu(\mathbf{C}, 1) \geq \max\{\alpha_1, \alpha_2\}$. We proceed to show that $\nu(\mathbf{C}, 1) \leq \max\{\alpha_1, \alpha_2\}$ and also that the other conditions of the theorem are fulfilled. Because of the symmetry between x_1 and x_2 it will suffice to show that if $\mathbf{L}$ lies on the line $x_2 = mx_1$ with slope $m \leq 1$ then there is a point $\mathbf{z} = (z_1, z_2) \in \Lambda$ with $z_2 = \frac{1}{2}$ such that $(\alpha_1\mathbf{C} + \mathbf{z}) \cap \mathbf{L} \neq \emptyset$.

We know that the line $x_2 = \frac{1}{2}x_1$ touches $\alpha_1\mathbf{C} + \frac{1}{2}$ at a point $\mathbf{a}$ of $[0, 1]^2$. By reflecting in the point $\frac{1}{2}$ we see that the parallel line $x_2 = \frac{1}{2}x_1 + \frac{1}{2}$ touches $\alpha_1\mathbf{C} + \frac{1}{2}$ at a point $\mathbf{b}$ of $[0, 1]^2$. The segment $\mathbf{ab}$ is a chord of $\alpha_1\mathbf{C} + \frac{1}{2}$ through $\frac{1}{2}$ and it together with the parallel chords $\mathbf{a}_n\mathbf{b}_n$ of $\alpha_1\mathbf{C} + (n + \frac{1}{2}, \frac{1}{2})$, $n = 1, 2, \ldots$, give a polygonal roof of the form $\frac{1}{2}, \mathbf{a}, \mathbf{b}_1, \mathbf{a}_1, \mathbf{b}_2, \mathbf{a}_2, \ldots$ above the x_1-axis. Every line $x_2 = mx_1$ with slope $0 < m \leq 1$ must pass through this roof in such a way that it meets one of the chords and hence one of the bodies, as claimed.

In the corollary to Theorem 5 we showed that in n dimensions $\nu(\mathbf{C}, n-1)$ is an attained maximum. As an application of Theorem 7 we give an alternative proof that in 2 dimensions $\nu(\mathbf{C}, 1)$ is an attained maximum. This geometrical proof deepens our understanding of the 2-dimensional situation and may lead to an n-dimensional generalization.

THEOREM 9. *If $\mathbf{C}$ is an arbitrary closed, centrally symmetric convex body in $\mathbb{R}^2$ then*

$$\nu(\mathbf{C}, 1) = \sup\{\nu(\mathbf{C}, \mathbf{L}) : \mathbf{L} \cap \mathbf{P} \neq \emptyset, \dim \mathbf{L} = 1\}$$

is an attained maximum.

PROOF. Let $\mathbf{D}$ be the ball $x_1^2 + x_2^2 \leq \frac{1}{4}$. Then there are constants μ and λ such that $\mu\mathbf{D} \subset \mathbf{C} \subset \lambda\mathbf{D}$. It follows that, for any $\alpha_0 > 0$,

$$\alpha_0\mu\mathbf{D} + \Lambda \subset \alpha_0\mathbf{C} + \Lambda \subset \alpha_0\lambda\mathbf{D} + \Lambda.$$

The inclusion on the right shows that if $\alpha_0\lambda < \nu(\mathbf{D}, 1) = \frac{1}{\sqrt{5}}$ there are some subspaces including $\mathbf{L}_1$ and $\mathbf{L}_2$ that are not met by $\alpha_0\mathbf{C} + \Lambda$ and hence $\nu(\mathbf{C}, 1) \geq \alpha_0$. The inclusion on the left shows that every subspace $\mathbf{L}$ with $\nu(\mathbf{D}, \mathbf{L}) \leq \alpha_0\mu$ is met by $\alpha_0\mathbf{C} + \Lambda$. This collection of subspaces includes those which make an angle $\theta < \theta_0$ with one of the coordinate axes where $\tan_0\theta = \frac{\alpha_0\mu}{\sqrt{1-(\alpha_0\mu)^2}} < \frac{1}{2}$. For lines with small slope this is true

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because disks of radius r with centres $(n+\frac{1}{2},\frac{1}{2})$, $n \in \mathbb{N}$, meet every line $x_2 = mx_1$ with slope $0 < m \leq \frac{2r}{\sqrt{1-(2r)^2}}$ It follows that

$$\nu(\mathbf{C},1) = \sup\Big\{\nu(\mathbf{C},\mathbf{L}) \quad \mathbf{L} = \langle(\cos\theta,\sin\theta)\rangle \text{ with } \theta_0 \leq \theta \leq \frac{\pi}{2} - \theta_0\Big\}$$

This reduction to a closed set of subspaces can be combined with Theorem 7 to prove that $\nu(\mathbf{C},1)$ is an attained maximum

Theorem 9 shows that in 2 dimensions every closed, centrally symmetric convex body $\mathbf{C}$ determines a 1-dimensional subspace $\mathbf{L} = \mathbf{L}(\mathbf{C})$ such that $\nu(\mathbf{C},1) = \nu\big(\mathbf{C},\mathbf{L}(\mathbf{C})\big)$ Theorem 1 implies that $\mathbf{L}(\mathbf{C})$ must be rational and Theorem 2 implies that if $\mathbf{L}$ is any rational subspace then there is a point $\mathbf{z} \in \Lambda$ such that $\nu(\mathbf{C},\mathbf{L}) = d_c(\Lambda,\mathbf{L}) = d_c(\mathbf{z},\mathbf{L})$ This point $\mathbf{z}$ is not unique, if $\mathbf{L} = \langle\mathbf{a}\rangle$ with $\mathbf{a} \in \mathbb{Z}^2$ then $\mathbf{z}$ can be replaced by its image under any of the symmetries of Λ and $\mathbf{L}$ generated by $\mathbf{x} \longrightarrow -\mathbf{x}$ and $\mathbf{x} \longrightarrow \mathbf{x}+\mathbf{a}$ Let us denote by $\mathbf{z}(\mathbf{C},\mathbf{L})$ the eligible point in $\Lambda \cap \mathbf{P}$ that lies closest to the boundary of $\mathbf{P}$ If $\mathbf{C}$ satisfies the blocking condition then $\mathbf{z}\big(\mathbf{C},\mathbf{L}(\mathbf{C})\big) = \frac{1}{2}$ The rest of the paper is devoted to showing that this behaviour is not universal We construct bodies $\mathbf{C}$ and subspaces $\mathbf{L}$ such that $\mathbf{z}(\mathbf{C},\mathbf{L})$ and even $\mathbf{z}\big(\mathbf{C},\mathbf{L}(\mathbf{C})\big)$ lies arbitrarily deep in $\mathbf{P}$

Let us suppose that $\mathbf{L} = \langle(a,b)\rangle = \langle(-b,a)\rangle^{\perp}$ with $(a,b) \in \mathbb{N}^2$ normalized by the condition g c d $\{a,b\} = 1$ Theorem 5 and its corollary show that $\nu(\mathbf{C},\mathbf{L}) > 0$ if and only if a and b are of opposite parity and then

$$\nu(\mathbf{C},\mathbf{L}) = \frac{1}{2f^*(-b,a)} = d_c(\mathbf{z},\mathbf{L})$$

if and only if $\mathbf{z} = \frac{1}{2}(x,y) \in \Lambda$ and satisfies $|-bx+ay| = 1$

Our next lemma is motivated by properties of the continued fraction $[1,1,1, \quad]$

LEMMA 9 *Let p_r and q_r, $r \geq -1$, be Fibonacci sequences beginning with $p_{-1} = p_0 = 1$ and with $q_{-1} = 0$, $q_0 = 1$ For any $m \geq 1$, let $a = p_m$ and $b = q_m$ and consider the Diophantine equations $ay - bx = \pm 1$ A particular solution to one of these equations is given by $(x_0,y_0) = (p_{m-1},q_{m-1})$ An arbitrary solution to either of these equations must satisfy $|x| > \frac{1}{3}a$, $|y| > \frac{1}{3}b$*

PROOF It is easy to see that

$$p_m q_{m-1} - q_m p_{m-1} = (-1)^{m+1}$$

This means that the vectors (a,b) and (x_0,y_0) of the lemma satisfy $ay_0 - bx_0 = (-1)^{m+1}$ and the general solution to $ay - bx = \pm 1$ is given by

$$(x,y) = (\pm x_0 + at, \pm y_0 + bt), \qquad t \in \mathbb{Z}$$

The Fibonacci sequence p_r satisfies $p_r = p_{r-1} + p_{r-2} \leq 2p_{r-1}$ This implies that

$$\frac{3}{2}x_0 = \frac{1}{2}x_0 + x_0 \leq p_{m-2} + p_{m-1} = p_m = a$$

and

$$a = p_m = p_{m-1} + p_{m-2} \leq 2p_{m-1} = 2x_0.$$

It follows that

$$\frac{1}{2}a \leq x_0 \leq \frac{2}{3}a.$$

Taking into account the form of general solution (x, y) we have $|x| \geq \frac{1}{3}a$ and similarly $|y| \geq \frac{1}{3}b$.

THEOREM 10. *For every $k > 0$ there is a 1-dimensional subspace $\mathbf{L} = \mathbf{L}(k)$ and a point $\mathbf{z} = \mathbf{z}(k) \in \Lambda$ with $z_1 > k$ and $z_2 > k$ such that for any centrally symmetric convex body $\mathbf{C}$ in $\mathbb{R}^2$ the point $\mathbf{z}(\mathbf{C}, \mathbf{L})$ is equal to $\mathbf{z}(k)$. Moreover it is possible to choose a body $\mathbf{C} = \mathbf{C}(k)$ such that $\mathbf{z}\big(\mathbf{C}, \mathbf{L}(\mathbf{C})\big)$ is equal to $\mathbf{z}(k)$.*

PROOF. The quantities $a = p(m)$ and $b = q(m)$ of Lemma 9 can be made arbitrarily large. Moreover, whenever $m \equiv 1 \pmod 3$ a and b are of opposite parity and x_0 and y_0 are both odd. It follows from the preamble that we take $\mathbf{L} = \langle(a, b)\rangle$ then for any $\mathbf{C}$, $\nu(\mathbf{C}, \mathbf{L}) = d_c(\mathbf{z}, \mathbf{L})$ if and only if $\mathbf{z} = \frac{1}{2}(x, y) \in \Lambda$ satisfies $|-bx + ay| = 1$. Lemma 9 applies again and shows that this equality can hold for $\mathbf{z} \in \Lambda \cap \mathbf{P}$ only if $z_1 = \frac{1}{2}x > \frac{1}{6}a$ and $z_2 = \frac{1}{2}y > \frac{1}{6}b$. It follows that by taking $m \equiv 1 \pmod 3$ sufficiently large the subspace $\mathbf{L}(k) = \langle(a, b)\rangle$ and minimal solution $\mathbf{z}(k)$ satisfy the first part of theorem.

To construct a domain $\mathbf{C}(k)$ which satisfies $\mathbf{L}(k) = \mathbf{L}(\mathbf{C})$ and hence $\mathbf{z}(k) = \mathbf{z}\big(\mathbf{C}, \mathbf{L}(\mathbf{C})\big)$ it is enough to ensure that the primitive vector $(-b, a)$ is the first non-zero lattice point swallowed by the expanding family of sets $\lambda\mathbf{C}^*$, $\lambda > 0$. To this end we can take $\mathbf{C}^*$ to be an ellipse whose major axis runs from $(b, -a)$ to $(-b, a)$ and whose eccentricity is sufficiently close to 1 to exclude all other lattice points. With this explicit choice of $\mathbf{C}^*$ we conclude the theorem by defining $\mathbf{C}(k) = \mathbf{C}^{**}$.

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Centre for Advanced Study ın Mathematıcs
Panjab Unıversıty
Chandıgarh 160014
Indıa

Centre for Advanced Study ın Mathematıcs
Panjab Unıversıty
Chandıgarh 160014
Indıa

Physıcal Scıences Dıvısıon
Scarborough College
Unıversıty of Toronto
West Hıll, Ontarıo
M1C 1A4