Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Tuya Sa, SCH.1.17, Schofield Building, Loughborough University, Loughborough, LE11 3TU. Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Solutions should arrive by 20th September 2023 and will be published in the November 2023 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions must be accompanied by the SPC permission form which is available on the Mathematical Association website

https:www.m-a.org.uk/the-mathematical-gazette

*Note that if permission is not given, a pupil **may still participate and will be eligible for a prize in the same way as others**.*

Problem 2023.3 (**Gregory Dresden**)

Show, without using series, that

$$\lim_{n \to \infty} \left(\cot \frac{x}{n+1} - \cot \frac{x}{n-1} \right) = \frac{2}{x}.$$

Problem 2023.4 (**Paul Stephenson**)

Show that the sum of the divisors of a positive integer (including 1 and the number itself) divides the sum of their cubes.

Solutions to 2023.1 and 2023.2

Both problems were solved by Pediredla Suhaas and Sajin Yeasir Khan.

Problem 2023.1 (**Christopher Starr**)

A shot-putter releases a shot with speed V from a height h at an angle A to the horizontal. Find an expression for the value of the angle that gives the greatest possible horizontal distance.

Solution (Pediredla Suhaas)

The horizontal and vertical components of the initial velocity can be expressed as:

$$V_x = V \cos A, \qquad V_y = V \sin A.$$

The vertical displacement of the shot can be expressed as:

$$y = h + V_y t - \tfrac{1}{2} g t^2$$

where g is the acceleration due to gravity and t is the time of flight.

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The time of flight can be determined by setting y equal to zero and solving for t:

$$0 = h + V_y t - \tfrac{1}{2}gt^2$$
$$t = \frac{V_y \pm \sqrt{V_y^2 + 2gh}}{g}$$

where the ± sign indicates that there are two possible solutions for t, corresponding to the upward and downward portions of the motion.

The horizontal distance travelled by the shot can be expressed as:

$$R = V_x t.$$

Substituting the expressions for V_x and t gives:

$$R = V \cos A \left(\frac{V_y \pm \sqrt{V_y^2 + 2gh}}{g} \right)$$
$$= \frac{V^2 \cos A \sin A + V \cos A \sqrt{V^2 \sin^2 A + 2gh}}{g}$$
$$= \frac{V^2 \cos A}{g} \left(\sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right)$$
$$\Rightarrow \frac{d}{dA} \left[\frac{V^2 \cos A}{g} \left(\sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right) \right] = 0$$
$$\Rightarrow \left\{ \frac{d}{dA} \left(\frac{V^2 \cos A}{g} \right) \right\} \left\{ \sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right\} + \left\{ \frac{V^2 \cos A}{g} \right\} \left\{ \frac{d}{dA} \left(\sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right) \right\} = 0$$
$$\Rightarrow \frac{V^2}{g} \{ - \sin A \} \left\{ \sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right\} + \frac{V^2 \cos A}{g} \left\{ \frac{d}{dA} \left(\sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right) \right\} = 0$$
$$\Rightarrow -\frac{V^2 \sin A}{g} \left\{ \sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right\} + \frac{V^2 \cos^2 A}{g} \left\{ 1 + \sin A \sqrt{\sin^2 A + \frac{2gh}{V^2}}^{\,-1} \right\} = 0$$
$$\Rightarrow \frac{V^2 \cos^2 A}{g} \left\{ 1 + \sin A \sqrt{\sin^2 A + \frac{2gh}{V^2}}^{\,-1} \right\} = \frac{V^2 \sin A}{g} \left\{ \sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right\}$$
$$\Rightarrow \frac{V^2 \cos^2 A}{g} \left\{ \sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right\} \sqrt{\sin^2 A + \frac{2gh}{V^2}}^{\,-1}$$
$$= \frac{V^2 \sin A}{g} \left\{ \sin A + \sqrt{\sin^2 A + \frac{2gh}{V^2}} \right\}$$
$$\Rightarrow \cos^2 A = \sin A \sqrt{\sin^2 A + \frac{2gh}{V^2}}$$

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$$\Rightarrow 1 - \sin^2 A = \sin A \sqrt{\sin^2 A + \frac{2gh}{V^2}}$$

$$\Rightarrow \left(1 - \sin^2 A\right)^2 = \left(\sin A \sqrt{\sin^2 A + \frac{2gh}{V^2}}\right)^2$$

$$\Rightarrow \sin^4 A - 2\sin^2 A + 1 = \sin^2 A \left\{\sqrt{\sin^2 A + \frac{2gh}{V^2}}\right\}^2$$

$$\Rightarrow \sin^4 A - 2\sin^2 A + 1 = \sin^2 A \left(\sin^2 A + \frac{2gh}{V^2}\right)$$

$$\Rightarrow \sin^4 A - 2\sin^2 A + 1 = \sin^4 A + \sin^2 A \frac{2gh}{V^2}$$

$$\Rightarrow 2\sin^2 A + \sin^2 A \frac{2gh}{V^2} = 1$$

$$\Rightarrow 2\sin^2 A \left(1 + \frac{gh}{V^2}\right) = 1$$

$$\Rightarrow \sin^2 A = \frac{1}{2\left(1 + \frac{gh}{V^2}\right)}$$

$$\Rightarrow \sin A = \frac{1}{\sqrt{2\left(1 + \frac{gh}{V^2}\right)}} \Rightarrow A = \sin^{-1}\left\{\frac{1}{\sqrt{2\left(1 + \frac{gh}{V^2}\right)}}\right\}$$

where A is the optional angle that gives the greatest possible horizontal distance.

Problem 2023.2 **(Geoffrey Strickland)**

Given that a and b are strictly positive integers such that $a^3 - 3b^2 = 1$ and $x = 2a - 3b$, $y = 2b - a$, prove that $x^2 - 3y^2 = 1$ and $0 < x < a$, $0 \leqslant y < b$.

Solution (Pediredla Suhaas)

Let us assume for $a^3 - 3b^2 = 1$, $x = 2a - 3b$ and $y = 2b - a$, $x^2 - 3y^2 = 1$ is true.

Then by substituting $x = 2a - 3b$ and $y = 2b - a$, we obtain:

$$\begin{aligned} x^2 - 3y^2 &= (2a - 3b)^2 - 3(2b - a)^2 \\ &= \left(4a^2 - 12ab + 9b^2\right) - 3\left(4b^2 - 4ab + a^2\right) \\ &= a^2 - 3b^2. \end{aligned}$$

Then $a^2 - 3b^2 \neq a^3 - 3b^2$, so it cannot be proved.

Prize Winners

A prize of £25 is awarded to Pediredla Suhaas.

TUYA SA

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