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Towards a Generalization of a Mixing-length Model for Nonradially Pulsating Stars: Convection in a Shear

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Abstract. We discuss a generalization of a mixing-length formalism for convection in the presence of a mean flow, and present the convective fluxes for convective cells with the geometry of rolls and of hexagons.

1. Introduction

In the astrophysical community basically two physical pictures of the local mixing-length formulation have emerged. In the first picture a turbulent element is considered as a convective cell, which evolves out of some chaotic state and loses its kinetic energy, working against turbulent drag. In this picture the acceleration terms are neglected and the nonlinear advection terms are approximated appropriately (e.g. Unno 1967). In the second picture a fluid parcel or an eddy accelerates from rest followed by an instantaneous breakup after the element has turned over approximately once. Within this picture the evolution of the fluid properties carried by the turbulent parcels can be approximated by linear growth rates, and the nonlinearities are assumed to be taken into account by the instantaneous breakup of the eddy (e.g., Gough 1978). In nonpulsating stars the two pictures are complementary and lead to the same results; but in a time-dependent treatment, additional information is required to specify how the initial state of a convective element depends on conditions at the time of its creation. Hence, these two models yield different formulae for the turbulent fluxes when applied to pulsating stars. In this contribution we adopt the second picture, and generalize Gough's (1978) formulation for convection in the presence of a shear $\boldsymbol{\mathcal{S}}$. The shearing motion of the mean flow stretches the convective elements and generates off-diagonal terms in the Reynolds stress tensor and lateral components in the convective heat flux. The linearized fluctuation equations are perturbed to first order in $\boldsymbol{\mathcal{S}}$ to obtain the eigenfunctions of the turbulent velocity field in the presence of a mean flow. The turbulent fluxes are then calculated in the manner of Gough (1978) and are presented for convective cells with geometries of rolls and of hexagons.

2. The equations of motion

The equations describing the dynamics in a statistically stationary, one-dimensional flow of an inviscid Boussinesq fluid can be written in Cartesian coordinates $[\boldsymbol{x} = (x, y, z) = (x_1, x_2, x_3)]$ as (e.g., Spiegel & Veronis 1960):

$$\partial_j \hat{u}_j = 0\,, \tag{1}$$

$$\partial_t \hat{u}_i + \hat{u}_j \partial_j \hat{u}_i = -\rho^{-1} \partial_i p' + g\widehat{\alpha} T' \delta_{i3}\,, \tag{2}$$

$$\partial_t T' + \hat{u}_j \partial_j T' - \beta w = -(\rho c_p)^{-1} \partial_i F_i'\,, \tag{3}$$

where $\hat{\boldsymbol{u}}$ denotes the total velocity field, which can be decomposed into the mean flow $\overline{\boldsymbol{U}} = (\overline{U}_1, \overline{U}_2, 0)$ and into the turbulent velocity fluctuations $\boldsymbol{u} = (u, v, w) = (u_1, u_2, u_3)$ ($\overline{\boldsymbol{u}} = \boldsymbol{0}$; overbars denote horizontal averages); T' and p' are the Eulerian temperature and pressure fluctuations respectively. The superadiabatic temperature gradient $\beta \equiv -\mathrm{d}\overline{T}/\mathrm{d}z - g/c_p$, where c_p is the specific heat at constant pressure, and g is the acceleration due to gravity (assumed constant); $\widehat{\alpha}$ is the coefficient of thermal expansion, ρ is the density, F_i denotes the radiative heat flux and δ_{ij} is the Kronecker delta. We assume a simple form for the mean flow with constant shear $\boldsymbol{\mathcal{S}} = \mathrm{d}\overline{\boldsymbol{U}}/\mathrm{d}z$. We follow the standard treatment of these linearized equations, by taking the curl and double curl of Eq. (2) and seeking normal modes proportional to $f(x,y)\exp[q(t-t_0)]$ with f describing the horizontal structure of the flow and where q is the linear growth rate with which the convective eddy, created at the time t_0, grows with time t. There results an eigenvalue problem for the growth rate q consisting of equations for the components of vertical velocity $W(z)$, temperature perturbation $\Theta(z)$ and vertical vorticity $\Omega(z)$:

$$[q + iz(a\mathcal{S}_1 + b\mathcal{S}_2)](\mathrm{D}^2 - a^2)\,W + a^2\widehat{\alpha}\Theta = 0 \tag{4}$$

$$[q + iz(a\mathcal{S}_1 + b\mathcal{S}_2) - \kappa(\mathrm{D}^2 - a^2)]\Theta - \beta\,W = 0 \tag{5}$$

$$[q + iz(a\mathcal{S}_1 + b\mathcal{S}_2)]\Omega + i(a\mathcal{S}_2 - b\mathcal{S}_1)W = 0 \tag{6}$$

with $f(x,y) = \exp(iax + iby)$, $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$, $\mathcal{S}_1 = |\boldsymbol{\mathcal{S}}|\cos\phi$, $\mathcal{S}_2 = |\boldsymbol{\mathcal{S}}|\sin\phi$, in which ϕ is the angle between the direction of the shear $\boldsymbol{\mathcal{S}}$ and the x-coordinate (i.e. direction of the wavenumber a). Radiative transfer is here treated in the diffusion approximation with κ denoting the thermal diffusivity.

3. A model for rolls and hexagons

We first consider the convective pattern of rolls, where the rolls are centred vertically at $z = 0$ and their axes are aligned with the y-direction (i.e., $\partial_y = 0$, or $b = 0$). Thus all quantities depend only on x and z. The planform function for rolls is then $f(x) = \cos ax + i\,\sin ax$, and the fluctuation equations (4) and (5), written as a single fourth-order equation, thus become

$$[q + iaz\mathcal{S}_1 - \kappa(\mathrm{D}^2 - a^2)]\,(q + iaz\mathcal{S}_1)(\mathrm{D}^2 - a^2)W + a^2\alpha W = 0\,. \tag{7}$$

For small amplitudes of the shear $\boldsymbol{\mathcal{S}}$, Eq. (7) can be solved with linear perturbation theory, expanding the eigenvalue q and eigenfunction W in terms of $\boldsymbol{\mathcal{S}}$: $q = q_0 + \mathcal{S}_1\,q_1$ and $W = W_0 + \mathcal{S}_1 W_1$, and subject to the assumed, inviscid boundary conditions: $W = 0$ and $\mathrm{D}^2 W = 0$ at $z = \pm\frac{1}{2}\ell$, with ℓ, the mixing-length, describing the vertical extend of the convective eddy. To first order in $\mathcal{S}_1$ the calculations are straightforward, and the results for the turbulent fluxes, after averaging over all angles ϕ and all possible t_0 in the manner of Gough (1978), become

$$\boldsymbol{F}_\mathrm{c} = (-\frac{1}{4}\Xi\mathcal{S}, 0, 1)\,F_\mathrm{c}\,, \qquad F_\mathrm{c} = \frac{4}{\tau_c^2[k^2\Phi + \frac{1}{2}\mathcal{S}^2(k^2 + I)]}\frac{1}{4}\rho c_p \frac{\Phi}{g\widehat{\alpha}} q_0^3\,, \tag{8}$$

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and

$$\overline{u_i u_j} = \begin{pmatrix} (\Phi - 1) & 0 & -\frac{1}{4}\Xi\mathcal{S} \\ 0 & 0 & 0 \\ -\frac{1}{4}\Xi\mathcal{S} & 0 & 1 \end{pmatrix} \overline{w^2}, \quad \overline{w^2} = \frac{4}{\tau_c^2[k^2\Phi + \frac{1}{2}\mathcal{S}^2(k^2+I)]}\frac{1}{4}\rho q_0^2, \quad (9)$$

with τ_c being a constant of order unity, $k^2 = a^2 + (\pi/\ell)^2$, $\Phi = k^2/a^2$ and $\Xi = 1 + B + (\pi/\ell)\,C$. The coefficients B, C and I are obtained from solving the first-order perturbation equation, and they depend on $\kappa, g, \widehat{\alpha}$ and ℓ; we note that $q_1 = 0$ and the zero-order solution, q_0, is quoted by Gough (1978). Both the heat flux $\boldsymbol{F}_\mathrm{c}$ and the momentum flux $\overline{u_i u_j}$ $(j = 1, 2)$ have horizontal components which are parallel to the shear $\boldsymbol{\mathcal{S}}$. The fluxes in Eq. (8) and (9) are displayed in coordinates (x', y', z') with respect to the shear $\boldsymbol{\mathcal{S}} = (\mathcal{S}, 0, 0)$.

Perhaps a more realistic geometry for convection cells is the hexagon. The planform $\tilde{f}$ for hexagons has been discussed first by Christopherson (1940) for $\boldsymbol{\mathcal{S}} = 0$:

$$\tilde{f}(x,y) = \frac{2}{\sqrt{6}}\left[\cos\frac{1}{2}a(\sqrt{3}\,y + x) + \cos\frac{1}{2}a(\sqrt{3}\,y - x) + \cos ax\right]. \quad (10)$$

The structure of the planform for a hexagon is a superposition of three roll sets with wavevectors having the same modulus a and directed at angles of $2\pi/3$ to one another. Thus, in addition to the periodicity in both the $x-$ and $y-$directions, hexagonal patterns are invariant with respect to rotation by this angle. These properties allow us to calculate the fluxes in the presence of a shear for hexagons in a similar way to that for rolls, by superimposing the solutions of the eigenfunctions w of the three roll sets. For hexagons, the turbulent fluxes in coordinates with respect to the shear $\boldsymbol{\mathcal{S}} = (\mathcal{S}, 0, 0)$ become

$$\boldsymbol{F}_\mathrm{c} = (-\frac{1}{4}\Xi\mathcal{S}, 0, 1)\, F_\mathrm{c}\,, \quad (11)$$

$$\overline{u_i u_j} = \begin{pmatrix} \frac{1}{2}(\Phi - 1) & 0 & -\frac{1}{4}\Xi\mathcal{S} \\ 0 & \frac{1}{2}(\Phi - 1) & 0 \\ -\frac{1}{4}\Xi\mathcal{S} & 0 & 1 \end{pmatrix} \overline{w^2}. \quad (12)$$

Only the normal components of the Reynolds stresses are found to be different between the results for rolls, Eq. (9), and hexagons, Eq. (12). The off-diagonal terms of the stress tensors and the lateral components of the convective heat fluxes, Eqs. (8) and (11), are presented to leading order in $\mathcal{S}$. The magnitude of $\overline{w^2}$ and F_c are modified only when terms of higher order in $\mathcal{S}$ are retained. We therefore plan to expand Eq. (7) to second order in $\mathcal{S}_1$ and discuss the resulting turbulent fluxes in a forthcoming paper (Gough & Houdek 2000).

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