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DERIVED MENDELSOHN TRIPLE SYSTEMS

ZORAN STOJAKOVIĆ

Mendelsohn triple system of order v which can be extended to a tetrahedral quadruple system of order $v+1$ we call a derived Mendelsohn triple system. We consider some properties of derived Mendelsohn triple systems and give some results on their existence.

1. INTRODUCTION

Mendelsohn triple systems (MTSs), which represent a generalisation of Steiner triple systems, were introduced in [**6**]. A MTS of order v is a pair (S,T), where S is a finite set of v elements and T is a collection of cyclic triples $\langle abc\rangle = \{(a,b),\ (b,c),\ (c,a)\}$, a,b,c distinct elements from S, such that every ordered pair of distinct elements from S belongs to exactly one cyclic triple from S. By MTS(v) we denote a MTS of order v.

In [14] a class of quadruple systems called tetrahedral quadruple systems (TQSs) was defined. TQSs represent a generalisation of Mendelsohn triple systems different from generalisations in [**11, 13**]. A TQS of order v is a pair (S,T), where S is a finite set of v elements and T is a family of directed quadruples $\langle abcd\rangle$, a,b,c,d distinct elements of S, such that every ordered triple of distinct elements of S belongs to exactly one directed quadruple from T. A directed quadruple $\langle abcd\rangle$ is the following set of 12 ordered triples

$$\langle abcd\rangle = \{(abc),(bca),(cab),(adb),(dba),(bad),\\ (acd),(cda),(dac),(bdc),(dcb),(cbd)\}.$$

By TQS(v) we denote a TQS of order v.

It was proved in [14] that TQSs are equivalent to generalised idempotent alternating symmetric (GIAS) 3-quasigroups, their properties were investigated and some parts of the spectrum of TQSs determined. In [**3**] further investigation of TQSs was carried on and it was proved that the spectrum of TQSs consists of all v such that $v \equiv 1,2,4,5,8,10 \pmod{12}$.

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The sequence $x_m, x_{m+1}, \dots, x_n$ is denoted by $\{x_i\}_{i=m}^n$ or by x_m^n. If $m > n$, then x_m^n will be considered empty.

An n-ary groupoid (n-groupoid) (Q,f) is called an n-quasigroup if the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in \{1, \dots, n\}$.

By S_n we denote the symmetric group of degree n and by A_n its alternating subgroup.

2. Alternating symmetric 3-quasigroups and TQSs

In [**12, 14**] a class of n-groupoids called alternating symmetric n-groupoids was defined and considered.

Definition 1.: A 3-groupoid (S,f) is alternating symmetric (AS) if for every permutation $\sigma \in A_4$ and all $x_1^4 \in S$

$$f(\{x_{\sigma(i)}\}_{i=1}^3) = x_{\sigma(4)} \iff f(x_1^3) = x_4.$$

This definition of AS-3-groupoids can be given also in another equivalent form.

Theorem 1. [14] *A 3-groupoid (S,f) is AS if and only if the following identities are satisfied:*

$$\begin{cases} f(x,y,z) = f(y,z,x), \\ f(y, f(x,y,z), z) = x. \end{cases}$$

Every AS-3-groupoid is necessarily a 3-quasigroup.

A 3-groupoid (S,f) is called generalised idempotent (GI) if and only if for all $x, y \in S$

$$f(x,y,y) = f(y,x,y) = f(y,y,x) = x.$$

An AS-3-groupoid which is GI is called a GIAS-3-groupoid.

So, a 3-groupoid (Q,f) is a GIAS-3-groupoid if and only if it satisfies the following identities:

$$\begin{cases} f(x,y,z) = f(y,z,x), \\ f(y, f(x,y,z), z) = x, \\ f(x,y,y) = x. \end{cases}$$

Hence the class of all GIAS-3-groupoids is a variety.

In [**14**] it is proved that finite GIAS-3-groupoids are equivalent to TQSs.

Theorem 2. [14] *Every TQS of order v defines and is defined by a GIAS-3-groupoid of order v.*

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If (S,T) is a TQS of order v, and f is defined for distinct elements $x,y,z,u \in S$ by

$$f(x,y,z) = u \Longleftrightarrow \langle xyzu \rangle \in T \tag{1}$$

and

$$f(x,y,y) = f(y,x,y) = f(y,y,x) = x, \tag{2}$$

then (S,f) is GIAS-3-groupoid of order v. Conversely, if (S,f) is a GIAS-3-groupoid of order v, then by (1) a TQS (S,T) of order v is defined.

Since Steiner quadruple systems are equivalent to generalised idempotent totally symmetric (GITS) 3-quasigroups (the definition of a totally symmetric 3-quasigroup is obtained if in Definition 1 we replace A_4 by S_4), and every GITS-3-quasigroup is a GIAS-3-quasigroup, we see that TQS represent a generalisation of Steiner quadruple systems. Some questions concerning the algebraic theory of Steiner quadruple systems were considered in [**2**, **5**, **8**, **9**, **10**].

3. Derived MTSs

If (S,T) is a TQS of order v and x is any element in S, we shall denote $S \setminus \{x\}$ by S_x and the set of all cyclic triples $\langle abc \rangle$ such that $\langle xabc \rangle \in T$ by $T(x)$. Then $(S_x, T(x))$ must be a MTS of order $v-1$, which we call a derived MTS (briefly DMTS) of the TQS (S,T).

The DMTSs are equivalent to retracts of GIAS-3-quasigroups. If (S,f) is a GIAS-3-quasigroup, $a \in S$ a fixed element, then by

$$xy = f(a,x,y)$$

a retract $(S,\cdot)$ of (S,f) is defined. Since (S,f) is a 3-quasigroup, $(S,\cdot)$ is a binary quasigroup. But (S,f) is also AS, hence

$$xy = f(a,x,y) = f(x,y,a) = f(y,a,x)$$

and

$$xy = z \Leftrightarrow f(a,x,y) = z \Leftrightarrow f(a,z,x) = y \Leftrightarrow zx = y.$$

A quasigroup satisfying the equivalence $xy = z \Leftrightarrow zx = y$ (or the equivalent identity $(xy)x = y$) is called semisymmetric, hence $(S,\cdot)$ is a semisymmetric quasigroup.

Further, from $f(a,a,x) = f(a,x,a) = x$ and $f(a,x,x) = a$, we get that for all $x \in S$, $ax = xa = x$, $x^2 = a$, that is, $(S,\cdot)$ is a semisymmetric unipotent loop with the unit a. Semisymmetric unipotent loops we shall call M-loops. M-loops which can be obtained as retracts of GIAS-3-quasigroups will be called derived M-loops.

M-loops were considered in [7] where it was proved that every M-loop of order v defines a MTS($v-1$) (and also that every such loop can be defined by one identity $x(((yy)z)x) = z$). As we have seen, derived M-loops of order v are equivalent to DMTSs of order $v-1$. If $(S,\cdot)$ is a derived M-loop with the unit a, and (S,f) is a GIAS-3-quasigroup such that $xy = f(b,x,y)$ for some $b \in S$, then it follows that $a = b$.

An interesting problem about Steiner triple systems (which is far from solved) is whether or not every Steiner triple system is derived of some Steiner quadruple system. The similar question for MTSs - is every MTS derived of some TQS - has a negative answer. Since the spectrum of MTSs consists of all $v \equiv 0,1 \pmod 3$, $v \neq 6$ and the spectrum of TQSs consists of all $v \equiv 1,2,4,5,8,10 \pmod{12}$, we get that no MTS of order $v \equiv 6,10 \pmod{12}$ is derived.

If we assume that two MTSs derived from a TQS for distinct elements $a,b \in Q$ have a triple $\langle xyz \rangle$ in common, then $f(a,x,y) = f(b,x,y)$, hence $a = b$, which is a contradiction. So, for different elements $a,b \in Q$ the two DMTSs must be disjoint.

Theorem 3. *Let (S,T) be a TQS(v). If for every $a \in S$ a DMTS $(S_a, T(a))$ is defined, then a family of disjoint MTSs is obtained such that $C(S) = \bigcup_{a\in S} T(a)$, where $C(S)$ is the set of all possible cyclic triples of elements from S.*

Proof: We have already proved that for $a \neq b$, $T_a \cap T_b = \emptyset$, and by a simple calculation we get that every cyclic triple of elements from S belongs to $\bigcup_{a\in S} T(a)$. □

The following problem, which arises quite naturaly, was considered by several authors (see [1, 4]). If S is a set of v elements, where v is such that there exists a MTS(v), is it possible to partition $C(S)$ into $v-2$ subsets $T_1, \dots, T_{v-2}$ such that each $(S,T_1), \dots, (S,T_{v-2})$ is a MTS? Such a collection of MTSs of order v is called a large set of pairwiswe disjoint MTSs of order v (LSMTS(v)) and so far only partial results on the existence of LSMTS(v) are known.

The family of MTSs obtained in Theorem 3 is not a LSMTS(v), but it is in some sense "large". In fact, it is a partition of $C(S)$ into v MTSs of order $v-1$ and such a family we shall call quasi LSMTS(v) (QLSMTS(v)). So, the next theorem is valid.

Theorem 4. *For every $v \equiv 1,2,4,5,8,10 \pmod{12}$ there exists a QLSMTS(v).*

4. Constructions of DMTs

Theorem 5. *If (S_1,T_1) and (S_2,T_2) are DMTSs of orders v_1 and v_2 respectively, then the MTS of order $v_1v_2 + v_1 + v_2$ equivalent to the direct product of M-loops defined by the given DMTs, is also derived.*

Proof: Let (S_1,T_1) and (S_2,T_2) be DMTSs of orders v_1 and v_2 respectively. If $(\overline{S}_1,\cdot)$ and $(\overline{S}_2,*)$ are M-loops which are equivalent to the given DMTSs, where

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$\overline{S}_1 = S_1 \cup \{a\}$, $a \notin S_1$, $\overline{S}_2 = S_2 \cup \{b\}$, $b \notin S_2$, then, since the class of M-loops is a variety, their direct product $(\overline{S}_1 \times \overline{S}_2, \circ)$ is also an M-loop which defines a MTS of order $v_1v_2 + v_1 + v_2$.

M-loops $(\overline{S}_1, \cdot)$ and $(\overline{S}_2, *)$ are derived, hence there exist GIAS-3-quasigroups $(\overline{S}_1, f_1)$, $(\overline{S}_2, f_2)$ such that $f_1(a,x,y) = xy$, $f_2(b,x,y) = x*y$. Since the class of GIAS-3-quasigroups is a variety, the direct product of $(\overline{S}_1, f_1)$ and $(\overline{S}_2, f_2)$, $(\overline{S}_1 \times \overline{S}_2, g)$, is also a GIAS-3-quasigroup. If we define a retract of g by $g((a,b),x,y)$, then this retract is in fact the direct product of M-loops $(\overline{S}_1, \cdot)$ and $(\overline{S}_2, *)$. □

In the preceding theorem the assumption that (S,T_1) and (S,T_2) are DMTSs of orders v_1 and v_2, implies that $v_1, v_2 \geq 3$. But since there exists an M-loop of order 2 (although there is no MTS which is equivalent to that M-loop, we can consider this M-loop to be equivalent to a DMTS(1) with empty set of triples), Theorem 5 can be extended to the case where $v_1 = 1$. Hence, as a consequence of Theorem 5 we get the next theorem which starting from a DMTS(v) gives a DMTS(2v+1).

Theorem 6. *If (S,T) is a DMTS(v), then there exists a DMTS($2v+1$) (R,K), such that (S,T) is a subsystem of (R,K).*

A subsystem of a MTS (R,K) is a MTS (S,T) such that $S \subseteq R$ and $T \subseteq K$.

Theorem 7. *Let (R,K) be a MTS($2v+1$) having a DMTS(v) (S,T) as a subsystem. If the M-loop $(\overline{R}, \cdot)$ which is equivalent to (R,K) is such that for every $a,b,c,d \in R \setminus S$, $ab = cd \Longrightarrow ba = dc$, then (R,K) is itself derived.*

Proof: Let $S = \{x_1, \dots, x_v\}$ and let $A = R \setminus S = \{a_1, \dots, a_{v+1}\}$. We form the following partition $\overline{A}^2 = A_1 \cup \dots \cup A_v$ of the set of all ordered pairs of distinct elements of A: $(a,b) \in A_i$ if and only if $\langle x_i, a, b\rangle \in K$. In every set A_i, $i = 1, \dots, v$, each element of A appears exactly once as the first and exactly once as the second component of an ordered pair. Also, if we replace every pair (a,b) in A_i by (b,a) we get some class A_j.

Take an element $p \notin R$, let $S^* = S \cup \{p\}$ and (S^*, T^*) be a TQS such that $(S_p^*, T^*(p)) = (S,T)$. Let (A,B) be any TQS on the set A. If y_i is the solution of the equation $a_1 y_i = x_i$, $i = 1, \dots, v$, then $y_i \in A$, and we define a bijection $g : A \to S^*$ by $g(a_1) = p$, $g(y_i) = x_i$, $i = 1, \dots, v$. If in A_i we replace every pair (a,b) by $(g(a), g(b))$, we get a set which will be denoted by S_i. Then $S_1 \cup \dots \cup S_v$ is a partition of the set of all ordered pairs of distinct elements of S^*.

We define a set K^* of directed quadruples on $R^* = S^* \cup A$ as follows. $T^* \cup B \subset K^*$ and if $a,b \in A$, $c,d \in S^*$, then $\langle abcd\rangle \in K^*$ if and only if there exists $i \in \{1, \dots, v\}$ such that $(a,b) \in A_i$, $(c,d) \in S_i$.

We prove that (R^*, K^*) is a TQS($2v+2$). For every i, $|A_i| = |S_i| = v+1$, hence we have formed a list of $v(v+1)^2$ directed quadruples of the form $\langle abcd\rangle$, where

$(a,b) \in A_i$, $(c,d) \in S_i$, but since $\langle abcd \rangle = \langle badc \rangle$ in that list every directed quadruple appears exactly twice, so $K^* \setminus \{T^* \cup B\}$ has $v(v+1)^2/2$ elements. $|T^*| = |B| = (v+1)v(v-1)/12$, hence

$$|K^*| = \frac{(v+1)v(v-1)}{6} + \frac{v(v+1)^2}{2} = \frac{(2v+2)(2v+1)2v}{12}. \tag{3}$$

It is easy to verify that every ordered triple of distinct elements of R^* belongs to at least one directed quadruple of K^*, which is by (3) sufficient for (R^*, K^*) to be a TQS$(2v+2)$.

It is straightforward to check that (R, K) is a DMTS$(2v+1)$ of the TQS (R^*, K^*). $\square$

The choice of TQS (A, B) in the above proof was arbitrary, which means that there exists a large number of TQS$(2v+2)$ having (R, K) as a subsystem.

Remark. Each MTS (R, K) such that its equivalent M-loop is commutative has the property given in the preceding theorem, but there are numerous examples of MTSs satisfying the conditions of Theorem 7, which have noncommutative equivalent M-loops.

If $(\overline{R}, \cdot)$ is an M-loop, then it is not difficult to prove that the implication $ab = cd \Longrightarrow ba = dc$ is equivalent to $(ab)c = c(ba)$.

An open problem is whether Theorem 7 can be proved without any conditions for the MTS (R, K).

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Institute of Mathematics
University of Novi Sad
Trg D. Obradovića 4
21000 Novi Sad
Yugoslavia