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A step towards a general density Corrádi–Hajnal Theorem*

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Abstract. **For a nondegenerate r-graph F, large n, and t in the regime $[0, c_F n]$, where $c_F > 0$ is a constant depending only on F, we present a general approach for determining the maximum number of edges in an n-vertex r-graph that does not contain $t+1$ vertex-disjoint copies of F. In fact, our method results in a rainbow version of the above result and includes a characterization of the extremal constructions.**

Our approach applies to many well-studied hypergraphs (including graphs) such as the edge-critical graphs, the Fano plane, the generalized triangles, hypergraph expansions, the expanded triangles, and hypergraph books. Our results extend old results of Erdős [12], Simonovits [76], and Moon [58] on complete graphs, and can be viewed as a step towards a general density version of the classical Corrádi–Hajnal [10] and Hajnal–Szemerédi [32] Theorems.

Our method relies on a novel understanding of the general properties of nondegenerate Turán problems, which we refer to as smoothness and boundedness. These properties are satisfied by a broad class of nondegenerate hypergraphs and appear to be worthy of future exploration.

1 Introduction

1.1 Motivation

Fix an integer $r \geq 2$, an r-graph $\mathcal{H}$ is a collection of r-subsets of some finite set V. We identify a hypergraph $\mathcal{H}$ with its edge set and use $V(\mathcal{H})$ to denote its vertex set. The size of $V(\mathcal{H})$ is denoted by $v(\mathcal{H})$.

Given two r-graphs F and $\mathcal{H}$ we use $\nu(F, \mathcal{H})$ to denote the maximum of $k \in \mathbb{N}$ such that there exist k vertex-disjoint copies of F in $\mathcal{H}$. We call $\nu(F, \mathcal{H})$ the F-**matching number** of $\mathcal{H}$. If $F = K_r^r$ (i.e. an edge), then we use $\nu(\mathcal{H})$ to represent $\nu(F, \mathcal{H})$ for simplicity. The number $\nu(\mathcal{H})$ is also known as the **matching number** of $\mathcal{H}$.

The study of the following problem encompasses several central topics in Extremal Combinatorics. Given an r-graph F and integers $n, t \in \mathbb{N}$:

What constraints on an n-vertex r-graph $\mathcal{H}$ force it to satisfy $\nu(F, \mathcal{H}) \geq t+1$?

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For $r = 2$ and $F = K_2$, the celebrated Erdős–Gallai Theorem [14] states that for all integers $n, \ell \in \mathbb{N}$ with $t + 1 \leq n/2$ and for every n-vertex graph G,

$$|G| > \max\left\{\binom{2t+1}{2}, \binom{n}{2} - \binom{n-t}{2}\right\} \quad \Rightarrow \quad \nu(G) \geq t + 1.$$

Here we use the symbol $\Rightarrow$ to indicate that the constraint on the left side forces the conclusion on the right side.

Extending the Erdős–Gallai Theorem to r-graphs for $r \geq 3$ is a major open problem, and the following conjecture of Erdős is still open in general (see e.g. [21, 22, 23, 38] for some recent progress on this topic).

Conjecture Suppose that $n, t, r \in \mathbb{N}$ satisfy $r \geq 3$ and $t + 1 \leq n/r$. Then for every n-vertex r-graph $\mathcal{H}$,

$$|\mathcal{H}| > \max\left\{\binom{r(t+1)-1}{r}, \binom{n}{r} - \binom{n-t}{r}\right\} \quad \Rightarrow \quad \nu(\mathcal{H}) \geq t + 1.$$

■

For general r-graphs F, determining the minimum number of edges in an n-vertex r-graph $\mathcal{H}$ that guarantees $\nu(F, \mathcal{H}) \geq 1$ is closely related to the Turán problem. For our purpose in this work, let us introduce the following notions.

Fix an r-graph F, we say another r-graph $\mathcal{H}$ is F-**free** if $\nu(F, \mathcal{H}) = 0$. In other words, $\mathcal{H}$ does not contains F as a subgraph. The **Turán number** $\mathrm{ex}(n, F)$ of F is the maximum number of edges in an F-free r-graph on n vertices. The **Turán density** of F is defined as $\pi(F) := \lim_{n\to\infty} \mathrm{ex}(n, F)/\binom{n}{r}$, the existence of the limit follows from a simple averaging argument of Katona, Nemetz, and Simonovits [41] (see Proposition 3.2).

An r-graph F is called **nondegenerate** if $\pi(F) > 0$. We use $\mathrm{EX}(n, F)$ to denote the collection of all n-vertex F-free r-graphs with exactly $\mathrm{ex}(n, F)$ edges, and call members in $\mathrm{EX}(n, F)$ the **extremal constructions** of F. The study of $\mathrm{ex}(n, F)$ and $\mathrm{EX}(n, F)$ is a central topic in Extremal Combinatorics.

Much is known when $r = 2$, and one of the earliest results in this regard is Mantel's theorem [57], which states that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$. For every integer $\ell \geq 2$ let $T(n, \ell)$ denote the balanced complete ℓ-partite graph on n vertices. Here, balanced means that the sizes of any two parts differ by at most one. We call $T(n, \ell)$ the **Turán graph**, and use $t(n, \ell)$ to denote the number of edges in $T(n, \ell)$. The seminal Turán Theorem states that $\mathrm{EX}(n, K_{\ell+1}) = \{T(n, \ell)\}$ for all integers $n \geq \ell \geq 2$. Later, Turán's theorem was extended to general graphs F in the celebrated Erdős–Stone–Simonovits Theorem [15, 18], which says that $\pi(F) = (\chi(F) - 2)/(\chi(F) - 1)$. Here $\chi(F)$ is the **chromatic number** of F.

For $r \geq 3$, determining $\mathrm{ex}(n, F)$ or even $\pi(F)$ for an r-graph F is known to be notoriously hard in general. The problem of determining $\pi(K_\ell^r)$ raised by Turán [78], where K_ℓ^r is the complete r-graph on ℓ vertices, is still wide open for all $\ell > r \geq 3$. Erdős offered \$500 for the determination of any $\pi(K_\ell^r)$ with $\ell > r \geq 3$ and \$1000 for all $\pi(K_\ell^r)$ with $\ell > r \geq 3$. We refer the reader to an excellent survey [42] by Keevash for related results before 2011.

Another related central topic in Extremal Combinatorics is the Factor Problem. We say an r-graph $\mathcal{H}$ has an F-**factor** if it contains a collection of vertex-disjoint copies of F that covers all vertices in $V(\mathcal{H})$. In other words, $\nu(F, \mathcal{H}) = \frac{v(\mathcal{H})}{v(F)}$ (in particular, $v(F) \mid v(\mathcal{H})$).

For an r-graph $\mathcal{H}$ and a vertex $v \in V(\mathcal{H})$ the **degree** $d_{\mathcal{H}}(v)$ of v in $\mathcal{H}$ is the number of edges in $\mathcal{H}$ containing v. We use $\delta(\mathcal{H})$, $\Delta(\mathcal{H})$, and $d(\mathcal{H})$ to denote the **minimum degree**, the **maximum degree**, and the **average degree** of $\mathcal{H}$, respectively. We will omit the subscript $\mathcal{H}$ if it is clear from the context.

A classical theorem of Corrádi and Hajnal [10] implies the following result for K_3.

***Theorem* 1.2 (*Corrádi–Hajnal [10]*)** *Suppose that $n, t \in \mathbb{N}$ are integers with $t \le n/3$. Then for every n-vertex graph G,*

$$\delta(G) \ge t + \left\lfloor \frac{n-t}{2} \right\rfloor \quad \Rightarrow \quad \nu(K_3, G) \ge t.$$

In particular, if $3 \mid n$, then every n-vertex graph G with $\delta(G) \ge 2n/3$ contains a K_3-factor.

Later, Theorem 1.2 was extended to all complete graphs in the classical Hajnal–Szemerédi Theorem [32], which implies that for all integers $n \ge \ell \ge 2$, $t \le \lfloor n/(\ell+1) \rfloor$, and for every n-vertex graph G,

$$\delta(G) \ge t + \left\lfloor \frac{\ell-1}{\ell}(n-t) \right\rfloor \quad \Rightarrow \quad \nu(K_{\ell+1}, G) \ge t.$$

For further related results, we refer the reader to a survey [48] by Kühn and Osthus.

In this work, we are interested in density constraints that force an r-graph to have large F-matching number, where F is a nondegenerate r-graph. Since our results are closely related to the Turán problem of F, we abuse the use of notation by letting $\mathrm{ex}(n, (t+1)F)$ denote the maximum number of edges in an n-vertex r-graph $\mathcal{H}$ with $\nu(F, \mathcal{H}) < t + 1$.

Given two r-graphs $\mathcal{G}$ and $\mathcal{H}$ whose vertex sets are disjoint, we define the **join** $\mathcal{G} \boxtimes \mathcal{H}$ of $\mathcal{G}$ and $\mathcal{H}$ to be the r-graph obtained from $\mathcal{G} \sqcup \mathcal{H}$ (the vertex-disjoint union of $\mathcal{G}$ and $\mathcal{H}$) by adding all r-sets that have nonempty intersection with both $V(\mathcal{G})$ and $V(\mathcal{H})$. For simplicity, we define the join of an r-graph $\mathcal{H}$ and a family $\mathcal{F}$ of r-graphs as $\mathcal{H} \boxtimes \mathcal{F} := \{\mathcal{H} \boxtimes \mathcal{G} \colon \mathcal{G} \in \mathcal{F}\}$.

Erdős [12] considered the density problem for K_3 and proved the following result.

***Theorem* 1.3 (*Erdős [12]*)** *Suppose that $n, t \in \mathbb{N}$ and $t \le \sqrt{n/400}$. Then*

$$\mathrm{EX}(n, (t+1)K_3) = \{K_t \boxtimes T(n-t, 2)\}.$$

Later, Moon [58] extended it to all complete graphs.

***Theorem* 1.4 (*Moon [58]*)** *Suppose that integers $n, t, \ell \in \mathbb{N}$ satisfy $\ell \ge 2$, $t \le \frac{2n-3\ell^2+2\ell}{\ell^3+2\ell^2+\ell+1}$, and $\ell \mid (n-t)$. Then*

$$\mathrm{EX}(n, (t+1)K_{\ell+1}) = \{K_t \boxtimes T(n-t, \ell)\}. \tag{1.1}$$

It is worth mentioning that, in fact, for $\ell = 2$, Moon proved that the constraint $\ell \mid (n-t)$ can be removed, and moreover, (1.1) holds for all $t \leq \frac{2n-8}{9}$. For $\ell \geq 3$, Moon remarked in [58] that there are some difficulties to remove the constraint $\ell \mid (n-t)$. Nevertheless, the divisibility constraint is not required in our results. Meanwhile, Simonovits [76] also considered this problem and proved that if $t \geq 1$ and $\ell \geq 2$ are fixed integers, then (1.1) holds for all sufficiently large n.

It becomes much more complicated when extending Theorem 1.4 to larger t. Indeed, a full density version of the Corrádi–Hajnal Theorem was obtained only very recently by Allen, Böttcher, Hladký, and Piguet [2] for large n. Their results show that, interestingly, there are four different extremal constructions for four different regimes of t, and the construction $K_t \boxtimes T(n-t, 2)$ is extremal only for $t \leq \frac{2n-6}{9}$. For the other three extremal constructions, we refer the reader to their paper for details. For larger complete graphs, it seems that there are even no conjectures for the extremal constructions in general (see remarks in the last section of [2]).

The objective of this work is to provide a general approach to determine $\mathrm{ex}(n, (t+1)F)$ for nondegenerate hypergraphs (including graphs) F when n is sufficiently large and t is within the range of $[0, c_F n]$, where $c_F > 0$ is a small constant depending only on F. It is worth mentioning that general methods of this nature are rare for hypergraph Turán-type problems, with only a few notable recent instances, as exemplified by [65, 9, 51]. Our main results are stated in the next section after the introduction of some necessary definitions. We hope our results could shed some light on a full generalization of the density version of the Corrádi–Hajnal and Hajnal–Szemerédi Theorems.

1.2 Main results

Given an r-graph F and an integer $n \in \mathbb{N}$ define

$$\delta(n, F) := \mathrm{ex}(n, F) - \mathrm{ex}(n-1, F) \quad \text{and} \quad d(n, F) := \frac{r \cdot \mathrm{ex}(n, F)}{n}.$$

Observe that $d(n, F)$ is the average degree of hypergraphs in $\mathrm{EX}(n, F)$, and $\delta(n, F)$ is a lower bound for the minimum degree of hypergraphs in $\mathrm{EX}(n, F)$ (see Fact 4.1).

The following two definitions are crucial for our main results. The first definition concerns the maximum degree of a near-extremal F-free r-graph.

Definition 1.1 (**Boundedness**) Let $f_1, f_2 \colon \mathbb{N} \to \mathbb{R}$ be two nonnegative functions. An r-graph F is (f_1, f_2)-bounded if every F-free r-graph $\mathcal{H}$ on n vertices with average degree at least $d(n, F) - f_1(n)$ satisfies $\Delta(\mathcal{H}) \leq d(n, F) + f_2(n)$, i.e.

$$d(\mathcal{H}) \geq d(n, F) - f_1(n) \quad \Rightarrow \quad \Delta(\mathcal{H}) \leq d(n, F) + f_2(n).$$

Remark. For our purposes, it suffices to take $f_1(n) = \varepsilon n^{r-1}$ and $f_2(n) = \delta n^{r-1}$ for some small constants $\varepsilon, \delta > 0$.

Later we will prove that families with certain stability properties also have good boundedness (see Theorem 1.9).

The next definition concerns the smoothness of the Turán function $\mathrm{ex}(n, F)$.

Definition 1.2 (**Smoothness**) Let $g\colon \mathbb{N} \to \mathbb{R}$ be a nonnegative function. The Turán function $\mathrm{ex}(n, F)$ of an r-graph F is g-smooth if

$$|\delta(n, F) - d(n-1, F)| \le g(n) \quad \text{holds for all } n \in \mathbb{N}.$$

Remark. Similarly, for our results, it suffices to take $g(n) = \gamma n^{r-1}$ for some small constant $\gamma > 0$.

Assumptions on the smoothness of $\mathrm{ex}(n, F)$ were used by several researchers before. See e.g. [3, 39] for degenerate graphs and see e.g. [43, Theorem 1.4] for nondegenerate hypergraphs.

Now we are ready to state our main result.

Theorem 1.5 *Fix integers $m \ge r \ge 2$ and a nondegenerate r-graph F on m vertices. Suppose that there exists a constant $c > 0$ such that for all sufficiently large $n \in \mathbb{N}$:*

(a) F is $\left(c\binom{n}{r-1}, \frac{1-\pi(F)}{4m}\binom{n}{r-1}\right)$-bounded, and

(b) $\mathrm{ex}(n, F)$ is $\frac{1-\pi(F)}{8m}\binom{n}{r-1}$-smooth.

Then there exists N_0 such that for all integers $n \ge N_0$ and $t \le \min\left\{\frac{c}{4erm}n, \frac{1-\pi(F)}{64rm^2}n\right\}$, we have

$$\mathrm{EX}\,(n, (t+1)F) = K_t^r \boxtimes \mathrm{EX}(n-t, F), \tag{1.2}$$

and, in particular,

$$\mathrm{ex}\,(n, (t+1)F) = \binom{n}{r} - \binom{n-t}{r} + \mathrm{ex}(n-t, F). \tag{1.3}$$

Remarks.

- Assumption (a) cannot be removed, as demonstrated by the following example: let $F = 2K_3$ and let $t \ge 2$, then

$$\begin{aligned}\mathrm{ex}(n, (t+1)F) = \mathrm{ex}(n, (2t+2)K_3) &\ge \binom{n}{2} - \binom{n-2t-1}{2} + \left\lfloor \frac{(n-2t-1)^2}{4} \right\rfloor \\ &> \binom{n}{2} - \binom{n-t}{2} + \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1 \\ &= \binom{n}{2} - \binom{n-t}{2} + \mathrm{ex}(n-t, F).\end{aligned}$$

  A less obvious example is the triangle-blowup of cycles, which can be deduced similarly from the results in recent work [54, Theorem 1.9].
- Assumption (b) can probably be omitted, as it was conjectured[1] that every F is $o(n^{r-1})$-smooth and this is true for $r = 2$ by a classic result of Simonovits (see [76, p.317]).

[1] This conjecture arose in a previous project of Dhruv Mubayi, Christian Reiher, and the third author, although it did not appear in the literature.

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Fix an r-graph F on m vertices. We say a collection $\{\mathcal{H}_1, \dots, \mathcal{H}_{t+1}\}$ of r-graphs on the same vertex set V has a **rainbow F-matching** if there exists a collection $\{S_i : i \in [t+1]\}$ of pairwise disjoint m-subsets of V such that $F \subset \mathcal{H}_i[S_i]$ for all $i \in [t+1]$.

Recently, there has been considerable interest in extending some classical results to a rainbow version. See e.g. [1, 31, 38, 47, 55, 56] for some recent progress on the rainbow version of the Erdős Matching Conjecture. Here we include the following rainbow version of Theorem 1.5.

***Theorem* 1.6** *The following holds under the assumption of Theorem 1.5. If a collection $\{\mathcal{H}_1, \dots, \mathcal{H}_{t+1}\}$ of n-vertex r-graphs on the same vertex set satisfies*

$$|\mathcal{H}_i| > \binom{n}{r} - \binom{n-t}{r} + \mathrm{ex}(n-t, F) \quad \text{for all } i \in [t+1],$$

then $\{\mathcal{H}_1, \dots, \mathcal{H}_{t+1}\}$ contains a rainbow F-matching.

Observe that (1.3) follows immediately by letting $\mathcal{H}_1 = \cdots = \mathcal{H}_{t+1}$ in Theorem 1.6. In fact, we will prove Theorem 1.6 first (which yields (1.3)), and then we prove (1.2) by adding some further argument.

1.3 Boundedness and smoothness

In this subsection, we present some simple sufficient conditions for an r-graph to have good boundedness and smoothness. Before stating our results, let us introduce some necessary definitions.

For most nondegenerate Turán problems where the exact value of the Turán number is known, the extremal constructions have simple structures. We use the following notions to encode the structural information of a hypergraph.

Let an r-**multiset** mean an unordered collection of r elements with repetitions allowed. Let E be a collection of r-multisets on $[k]$. Let $V_1, \dots, V_k$ be disjoint sets and let $V := V_1 \cup \cdots \cup V_k$. The **profile** of an r-set $X \subseteq V$ (with respect to $V_1, \dots, V_k$) is the r-multiset on $[k]$ that contains $i \in [k]$ with multiplicity $|X \cap V_i|$. For an r-multiset $Y \subseteq [k]$, let $Y((V_1, \dots, V_k))$ consist of all r-subsets of V whose profile is Y. The r-graph $Y((V_1, \dots, V_k))$ is called the **blowup** of Y (with respect to $V_1, \dots, V_k$) and the r-graph

$$E((V_1, \dots, V_k)) := \bigcup_{Y \in E} Y((V_1, \dots, V_k))$$

is called the **blowup** of E (with respect to $V_1, \dots, V_k$).

An **(r-uniform) pattern** is a pair $P = (k, E)$ where k is a positive integer and E is a collection of r-multisets on $[k]$. It is clear that pattern is a generalization of r-graphs, since an r-graph is a pattern in which E consists of only simple r-sets. If it is clear from the context, we will use E to represent the pattern P for simplicity (like what we did for hypergraphs). Moreover, if E consists of a single element, we will use this element to represent E.

We say an r-graph $\mathcal{G}$ is a P-**construction** on a set V if there exists a partition $V = V_1 \cup \cdots \cup V_k$ such that $\mathcal{G} = E((V_1, \dots, V_k))$. An r-graph $\mathcal{H}$ is a P-**subconstruction** if it is a subgraph of some P-construction. For example, the Turán graph $T(n, \ell)$ is a K_ℓ-construction on $[n]$, and an ℓ-partite graph is a K_ℓ-subconstruction.

Let $\Lambda(P, n)$ denote the maximum number of edges in a P-construction with n vertices and define the **Lagrangian** of P as the limit

$$\lambda(P) := \lim_{n\to\infty} \frac{\Lambda(P, n)}{\binom{n}{r}}.$$

Using a simple averaging argument, one can show that $\Lambda(P, n)/\binom{n}{r}$ is nonincreasing, and hence, the limit exists (see [69, Lemma 10]). We say a pattern $P = (k, E)$ is **minimum** if $\lambda(P - i) < \lambda(P)$ for all $i \in [k]$, where $P - i$ denotes the new pattern obtained from P by removing i from $[k]$ and removing all r-multisets containing i from E. Note that the Lagrangian of a pattern is a generalization of the well-known **hypergraph Lagrangian** (see e.g. [5, 26]) that has been successfully applied to Turán-type problems, with the basic idea going back to Motzkin and Straus [59].

Remark. The notion of pattern was introduced by Pikhurko in [69] to study the general properties of nondegenerate hypergraph Turán problems, and it was also used very recently in [52, 53]. Note that the definition of pattern in [69] is more general by allowing recursive parts. Our results about patterns in this work can be easily extended to this more general setting.

Let F be an r-graph and P be a pattern. We say (F, P) is a **Turán pair** if every P-construction is F-free and every maximum F-free construction is a P-construction. For example, it follows from the Turán Theorem that $(K_{\ell+1}, K_\ell)$ is a Turán pair for all $\ell \geq 2$. It is easy to observe that for a Turán pair (F, P), we have

$$\pi(F) = \lambda(P). \tag{1.4}$$

For hypergraphs in Turán pairs, we have the following result concerning the smoothness of their Turán functions.

Theorem 1.7 *Suppose that F is an r-graph and P is a minimal pattern such that (F, P) is a Turán pair. Then* $\mathrm{ex}(n, F)$ *is* $4\binom{n-1}{r-2}$*-smooth.*

The boundedness of F is closely related to the stability of F. So we introduce some definitions related to stability. Suppose that (F, P) is a Turán pair.

- We say F is **edge-stable** with respect to P if for every $\delta > 0$ there exist constants N_0 and $\zeta > 0$ such that for every F-free r-graph $\mathcal{H}$ on $n \geq N_0$ vertices with at least $(\pi(F) - \zeta)\binom{n}{r}$ edges, there exists a subgraph $\mathcal{H}' \subset \mathcal{H}$ with at least $(\pi(F) - \delta)\binom{n}{r}$ edges such that $\mathcal{H}'$ is a P-subconstruction.
- We say F is **vertex-extendable** with respect to P if there exist constants N_0 and $\zeta > 0$ such that for every F-free r-graph $\mathcal{H}$ on $n \geq N_0$ vertices satisfing $\delta(\mathcal{H}) \geq (\pi(F) - \zeta)\binom{n-1}{r-1}$ the following holds: if $\mathcal{H} - v$ is a P-subconstruction for some vertex $v \in V(\mathcal{H})$, then $\mathcal{H}$ is also a P-subconstruction.
- We say F is **weakly vertex-extendable** with respect to P if for every $\delta > 0$ there exist constants N_0 and $\zeta > 0$ such that for every F-free r-graph $\mathcal{H}$ on $n \geq N_0$ vertices satisfying $\delta(\mathcal{H}) \geq (\pi(F) - \zeta)\binom{n-1}{r-1}$ the following holds: if $\mathcal{H} - v$ is a P-subconstruction for some vertex $v \in V(\mathcal{H})$, then $d_{\mathcal{H}}(v) \leq (\pi(F) + \delta)\binom{n-1}{r-1}$.

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For simplicity, if P is clear from the context, we will simply say that F is edge-stable, vertex-extendable, and weakly vertex-extendable, respectively.

The first stability theorem which states that $K_{\ell+1}$ is edge-stable with respect to K_ℓ was proved independently by Erdős and Simonovits [76], and it was used first by Simonovits [76] to determine the exact Turán number $\mathrm{ex}(n, F)$ of an edge-critical graph F for large n. Later, Simonovits' method (also known as the Stability Method) was used by many researchers to determine the Turán numbers of a large collection of hypergraphs (see Section 2 for more details).

The definition of vertex-extendability was introduced by Mubayi, Reiher, and the third author in [51] for a unified framework for proving the stability of a large class of hypergraphs.

The definition of weak vertex-extendability seems to be new, and it is clear from (1.4) and the following lemma that for a Turán pair (F, P) the vertex-extendability implies the weak vertex-extendability. There are several examples showing that the inverse is not true in general (see e.g Section 2.6). It seems interesting to explore the relations between the weak vertex-extendability and other types of stability (see [51] for more details).

***Lemma* 1.8** *([52, Lemma 21])* *Suppose that P is a minimal pattern. Then for every $\delta > 0$ there exist N_0 and $\varepsilon > 0$ such that every P-subconstruction $\mathcal{H}$ on $n \geq N_0$ vertices with $\delta(\mathcal{H}) \geq (\lambda(P) - \varepsilon)\binom{n-1}{r-1}$ satisfies $\Delta(\mathcal{H}) \leq (\lambda(P) + \delta)\binom{n-1}{r-1}$.*

Let us add another remark about the weak vertex-extendability that might be useful for readers who are familiar with the stability method. In a standard stability argument in determining the exact value of $\mathrm{ex}(n, F)$, one usually defines a set $\mathcal{B}$ of bad edges and a set $\mathcal{M}$ of missing edges, and then tries to prove that $|\mathcal{M}| > |\mathcal{B}|$. One key step in this argument is to prove that the maximum degree of $\mathcal{B}$ is small (more specifically, $\Delta(B) = o(n^{r-1})$), which, informally speaking, usually implies the weak vertex-extendability of F.

For a Turán pair (F, P) with the weak vertex-extendability, we have the following result concerning the boundedness of F.

***Theorem* 1.9** *Suppose that F is an r-graph and P is a minimal pattern such that F is edge-stable and weakly vertex-extendable (or vertex-extendable) with respect to P. Then there exists a constant $c > 0$ such that F is $\left(c\binom{n-1}{r-1}, \frac{1-\pi(F)}{8m}\binom{n-1}{r-1}\right)$-bounded for large n.*

Remark. It seems possible to extend Theorems 1.7 and 1.9 to nonminimal patterns, but we do not aware of any r-graph F whose extremal construction is a P-construction for some nonminimal pattern P. However, there does exist a finite family $\mathcal{F}$ of r-graphs whose extremal construction is a P-construction for some nonminimal pattern P (see [37] for more details).

In many cases, (weak) vertex-extendability of F follows from a stronger type of stability that was studied by many researchers before. Suppose that (F, P) is a Turán pair. We say F is **degree-stable** with respect to P if there exists $\zeta > 0$ such that for large n every n-vertex F-free r-graph $\mathcal{H}$ with $\delta(\mathcal{H}) \geq (\pi(F) - \zeta)\binom{n-1}{r-1}$ is a P-subconstruction. It is easy to observe from the definition that if F is degree-stable with respect to P, then F is edge-stable and vertex-extendable with respect to P. Therefore, we have the following corollary of Theorems 1.7 and 1.9.

***Corollary* 1.10** *Suppose that F is an r-graph and P is a minimal pattern such that F is degree-stable with respect to P. Then there exists a constant $c > 0$ such that*

(a) $\mathrm{ex}(n, F)$ *is* $4\binom{n-1}{r-2}$*-smooth, and*
(b) F *is* $\left(c\binom{n-1}{r-1}, \frac{1-\pi(F)}{8m}\binom{n-1}{r-1}\right)$*-bounded.*

In the next section, we show some applications of Theorems 1.5, 1.7 and 1.9, and Corollary 1.10. We omit the applications of Theorem 1.6 since they are quite straightforward to obtain once we present the corresponding applications of Theorem 1.5. The proofs for Theorems 1.5 and 1.6 are included in Section 3. The proofs for Theorems 1.7 and 1.9 are included in Section 4.

2 Applications

Combining some known stability results with Theorems 1.5, 1.7, and 1.9 (or Corollary 1.10) we can immediately obtain results in this section. To demonstrate a way to apply Theorems 1.5, 1.7, and 1.9 in general, we include the short proof for the weak vertex-extendability of $\mathbb{F}_{3,2}$ as defined in Section 2.7 (even though it can be deduced from results in [28]).

2.1 Edge-critical graphs

Recall that for a graph F its chromatic number is denoted by $\chi(F)$. We say a graph F is **edge-critical** if there exists an edge $e \in F$ such that $\chi(F - e) < \chi(F)$. Using the stability method, Simonovits proved in [76] that if a graph F is edge-critical and $\chi(F) \geq 3$, then $\mathrm{EX}(n, F) = \{T(n, \chi(F) - 1)\}$ for all sufficiently large n.

Extending the classical Andrásfai–Erdős–Sós Theorem [4], Erdős and Simonovits [17] proved that every edge-critical graph with chromatic number at least 3 is degree-stable. Theorefore, combined with Theorem 1.5 and Corollary 1.10, we obtain the following result.

***Theorem* 2.1** *Suppose that F is an edge-critical graph with $\chi(F) \geq 3$. Then there exist constants N_0 and $c_F > 0$ such that for all integers $n \geq N_0$ and $t \in [0, c_F n]$ we have*

$$\mathrm{EX}(n, (t+1)F) = \{K_t \boxtimes T(n-t, \chi(F) - 1)\}.$$

Remarks.

- For Theorem 2.1 and all other theorems in this section, we did not try to optimize the constant c_F, but it seems possible to obtain a reasonable bound[2] for c_F by a more careful analysis of the proof for Theorem 1.9 (and the proof for the (weak) vertex-extendability of F in some cases).

[2] It seems possible to get a polynomial dependency between c_F and $\frac{1}{rm}$.

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- The case when F is an odd cycle was also considered in a recent paper [19, Theorem 1.1].
- It might be true that Theorem 2.1 holds for a broader class of graphs, and it would be interesting to characterize the class of graphs for which Theorem 2.1 holds.

2.2 The Fano plane

The **Fano plane** $\mathbb{F}$ is a 3-graph with vertex set $\{1,2,3,4,5,6,7\}$ and edge set

$$\{123, 345, 561, 174, 275, 376, 246\}.$$

Let $[n] = V_1 \cup V_2$ be a partition with $|V_1| = \lfloor n/2 \rfloor$ and $|V_2| = \lceil n/2 \rceil$. Let $B_3(n)$ denote the 3-graph on $[n]$ whose edge set consists of all triples that have a nonempty intersection with both V_1 and V_2. Note that $|B_3(n)| \sim \frac{3}{4}\binom{n}{3}$.

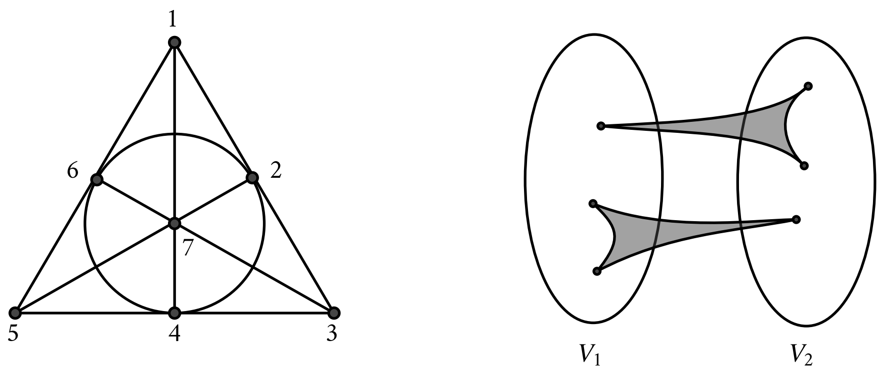

Figure 1: The Fano plane and the complete bipartite 3-graph $B_3(n)$..

It was conjectured by Sós [77] and famously proved by De Caen and Füredi [11] that $\pi(\mathbb{F}) = 3/4$. Later, using a stability argument, Keevash and Sudakov [46], and independently, Füredi and Simonovits [30] proved that $\mathrm{EX}(n, \mathbb{F}) = \{B_3(n)\}$ for all sufficiently large n. Recently, Bellmann and Reiher [6] proved that $\mathrm{ex}(n, \mathbb{F}) = |B_3(n)| = \frac{n-2}{2}\lfloor \frac{n^2}{4} \rfloor$ for all $n \geq 7$, and moreover, they proved that $B_3(n)$ is the unique extremal construction for all $n \geq 8$.

It follows from the result of Keevash and Sudakov [46], and independently, Füredi and Simonovits [30] that $\mathbb{F}$ is degree-stable. Therefore, we obtain the following result.

Theorem 2.2 *There exist constants N_0 and $c_{\mathbb{F}} > 0$ such that for all integers $n \geq N_0$ and $t \in [0, c_{\mathbb{F}} n]$ we have*

$$\mathrm{EX}(n, (t+1)\mathbb{F}) = \left\{ K_t^3 \boxtimes B_3(n-t) \right\}.$$

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2.3 Generalized triangles

The (r-uniform) **generalized triangle** $\mathbb{T}_r$ is the r-graph with vertex set $[2r-1]$ and edge set

$$\{\{1,\ldots,r-1,r\},\{1,\ldots,r-1,r+1\},\{r,r+1,\ldots,2r-1\}\}.$$

Note that $\mathbb{T}_2$ is simply a triangle.

Fix $n \geq r \geq 2$ and $\ell \geq r$. Let $[n] = V_1 \cup \cdots \cup V_\ell$ be a partition such that $|V_i| \in \left\{\lfloor \frac{n}{\ell} \rfloor, \lceil \frac{n}{\ell} \rceil\right\}$ for all $i \in [\ell]$. The **generalized Turán r-graph** $T_r(n,\ell)$ is the r-graph on $[n]$ whose edge set consists of all r-sets that contain at most one vertex from each V_i. Note that $T_2(n,\ell)$ is the Turán graph $T(n,\ell)$. Let $t_r(n,\ell)$ denote the number of edges in $T_r(n,\ell)$.

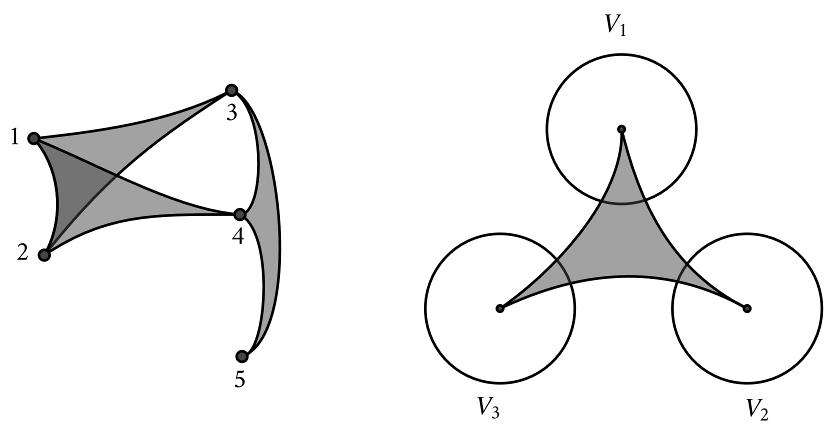

Figure 2: The generealized triangle $\mathbb{T}_3$ and the Turán 3-graph $T_3(n,3)$..

Katona conjectured and Bollobás [8] proved that $\mathrm{EX}(n,\{\mathbb{T}_3, K_4^{3-}\}) = \{T_3(n,3)\}$ for all $n \in \mathbb{N}$, where K_4^{3-} is the unique 3-graph with 4 vertices and 3 edges. Later, Frankl and Füredi [24] sharpened the result of Bollobás by showing that $\mathrm{EX}(n,\mathbb{T}_3) = \{T_3(n,3)\}$ for all $n \geq 3000$. In [44], Keevash and Mubayi proved the edge-stability of $\mathbb{T}_3$ and improved the lower bound of n from 3000 to 33. A short proof for the edge-stability with a linear dependency between the error parameters can be found in [49].

The vertex-extendability of $\mathbb{T}_3$ can be easily obtained from the proof of Lemma 4.4 in [51] (also see the Concluding Remarks in [51]). Therefore, we obtain the following result.

***Theorem* 2.3** *There exist constants N_0 and $c_{\mathbb{T}_3}$ such that for all integers $n \geq N_0$ and $t \in [0, c_{\mathbb{T}_3} n]$ we have*

$$\mathrm{EX}(n,(t+1)\mathbb{T}_3) = \left\{K_t^3 \mathbin{\bar{\times}} T_3(n-t,3)\right\}.$$

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For $r = 4$, improving a result of Sidorenko in [74], Pikhurko proved in [67] that $\mathrm{EX}(n, \mathbb{T}_4) = \{T_4(n, 4)\}$ for all sufficiently large n.

Similarly, the vertex-extendability of $\mathbb{T}_4$ can be obtained from the proof of Lemma 4.4 in [51] (also see the Concluding Remarks in [51]). Therefore, we obtain the following result.

***Theorem* 2.4** *There exist constants N_0 and $c_{\mathbb{T}_4}$ such that for all integers $n \geq N_0$ and $t \in [0, c_{\mathbb{T}_4}n]$ we have*

$$\mathrm{EX}(n, (t+1)\mathbb{T}_4) = \left\{K_t^4 \barwedge T_4(n-t, 4)\right\}.$$

The situation becomes complicated when $r \geq 5$. Let $\mathbb{W}_5$ denote the unique 5-graph with 11 vertices such that every 4-set of vertices is contained in exactly one edge. Let $\mathbb{W}_6$ denote the unique 6-graph with 12 vertices such that every 5-set of vertices is contained in exactly one edge. Let $\mathbb{W}_5(n)$ and $\mathbb{W}_6(n)$ denote the maximum $\mathbb{W}_5$-construction and $\mathbb{W}_6$-construction on n vertices, respectively. Some calculations show that $\mathbb{W}_5(n) \sim \frac{6}{11^4}n^5$ and $\mathbb{W}_6(n) \sim \frac{11}{12^5}n^6$.

In [25], Frankl and Füredi proved that $\mathrm{ex}(n, \mathbb{T}_r) \leq |\mathbb{W}_r(n)| + o(n^r)$ for $r = 5, 6$. Much later, using a sophisticated stability argument, Norin and Yepremyan [65] proved that $\mathbb{T}_5$ and $\mathbb{T}_6$ are edge-stable with respect to $\mathbb{W}_5$ and $\mathbb{W}_6$ respectively, and moreover, $\mathrm{EX}(n, \mathbb{T}_r) = \{\mathbb{W}_r(n)\}$ for $r = 5, 6$ and large n.

It was observed by Pikhurko [67] that both $\mathbb{T}_5$ and $\mathbb{T}_6$ fail to be degree-stable (or vertex-extendable). However, from Lemmas 7.2 and 7.4 in [65] one can easily observe that $\mathbb{T}_5$ and $\mathbb{T}_6$ are weakly vertex-extendable. Therefore, we obtain the following theorem.

***Theorem* 2.5** *For $r \in \{5, 6\}$ there exist constants N_0 and $c_{\mathbb{T}_r} > 0$ such that for all integers $n \geq N_0$ and $t \in [0, c_{\mathbb{T}_r}n]$ we have*

$$\mathrm{EX}(n, (t+1)\mathbb{T}_r) = \left\{K_t^r \barwedge \mathbb{W}_r(n-t)\right\}.$$

It seems that there are even no conjectures for the extremal constructions of $\mathbb{T}_r$ when $r \geq 7$. We refer the reader to [25] for some lower and upper bounds for $\pi(\mathbb{T}_r)$ in general.

2.4 The expansion of complete graphs

Fix integers $\ell \geq r \geq 2$. The **expansion** $H_{\ell+1}^r$ of the complete graph $K_{\ell+1}$ is the r-graph obtained from $K_{\ell+1}$ by adding a set of $r-2$ new vertices into each edge of $K_{\ell+1}$, and moreover, these new $(r-2)$-sets are pairwise disjoint. It is clear from the definition that $H_{\ell+1}^r$ has $\ell + 1 + (r-2)\binom{\ell+1}{2}$ vertices and $\binom{\ell+1}{2}$ edges.

The r-graph $H_{\ell+1}^r$ was introduced by Mubayi [60] as a way to generalize Turán's theorem to hypergraphs. These hypergraphs provide the first explicitly defined examples which yield an infinite family of numbers realizable as Turán densities for hypergraphs. In [60], Mubayi determined the Turán density of $H_{\ell+1}^r$ for all integers $\ell \geq r \geq 3$, and proved that $H_{\ell+1}^r$ is edge-stable. In [68], Pikhurko refined Mubayi's result and proved that $\mathrm{EX}(n, H_{\ell+1}^r) = \{T_r(n, \ell)\}$ for all integers $\ell \geq r \geq 3$ when n is sufficiently large.

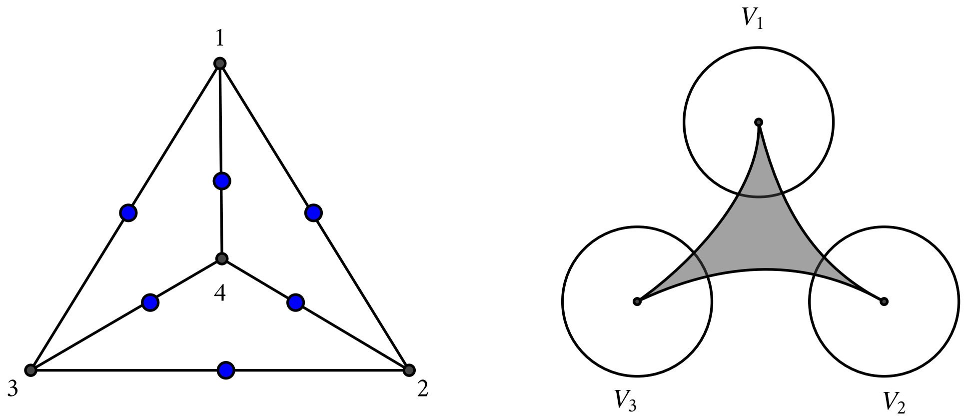

Figure 3: The expansion H_4^3 of K_4 and the Turán 3-graph $T_3(n,3)$..

The vertex-extendability of $H_{\ell+1}^r$ can be easily obtained by a small modification of the proof of Lemma 4.8 in [51] (also see the Concluding Remarks in [51]). Therefore, we obtain the following result.

***Theorem* 2.6** *Fix integers $\ell \geq r \geq 2$. There exist constants N_0 and $c = c(\ell, r) > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$ we have*

$$\mathrm{EX}(n, (t+1)H_{\ell+1}^r) = \left\{K_t^r \boxtimes T_r(n-t, \ell)\right\}.$$

Remarks. The definition of expansion can be extended to all graphs as follows. Fix a graph F, let the r-graph H_F^r be obtained from F by adding a set of $r-2$ new vertices into each edge of F, and moreover, these new $(r-2)$-sets are pairwise disjoint. Similar to Theorem 2.1, one could obtain a corresponding result for the expansion of all edge-critical graphs. We omit its statement and proof here.

2.5 The expansion of hypergraphs

Given an r-graph F with $\ell+1$ vertices, the **expansion** $H_{\ell+1}^F$ of F is the r-graph obtained from F by adding, for every pair $\{u, v\} \subset V(F)$ that is not contained in any edge of F, an $(r-2)$-set of new vertices, and moreover, these $(r-2)$-sets are pairwise disjoint. It is easy to see that the expansion of the empty r-graph on $\ell+1$ vertices (here empty means that the edge set is empty) is the same as the expansion of the complete graph $K_{\ell+1}$ defined in the previous subsection. However, in general, these two definitions are different.

Our first result in this subsection is about the expansion of the expanded trees. Given a tree T on k vertices, define the $(r-2)$-**expansion** $\mathrm{Exp}(T)$ of T as

$$\mathrm{Exp}(T) := \{e \cup A : e \in T\},$$

where A is a set of $r-2$ new vertices that is disjoint from $V(T)$.

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Given a tree T on k vertices, we say T is an **Erdős–Sós tree** if it satisfies the famous Erdős–Sós conjecture on trees. In other words, T is contained in every graph with average degree more than $k-2$. In [75], Sidorenko proved that for large k, if T is an Erdős–Sós tree on k vertices, then $\mathrm{ex}(n, H_{k+r-2}^{\mathrm{Exp}(T)}) \le t_r(n, k+r-3) + o(n^r)$. Much later, Norin and Yepremyan [66], and independently, Brandt, Irwin, and Jiang [9], improved Sidorenko's result by showing that, under the same setting, $H_{k+r-2}^{\mathrm{Exp}(T)}$ is edge-stable with respect to K_{k+r-3}^r and $\mathrm{EX}(n, H_{k+r-2}^{\mathrm{Exp}(T)}) = \{T_r(n, k+r-3)\}$ for large n. In fact, it follows easily from Lemmas 3.5 and 4.1 in [66] that $H_{k+r-2}^{\mathrm{Exp}(T)}$ is weakly vertex-extendable with respect to K_{k+r-3}^r. Hence, we obtain the following result.

***Theorem* 2.7** *For every integer $r \ge 3$ there exists M_r such that if T is an Erdős–Sós tree on $k \ge M_r$ vertices, then there exist N_0 and $c_T > 0$ such that for all integers $n \ge N_0$ and $t \le c_T n$, we have*

$$\mathrm{EX}\left(n, (t+1)H_{k+r-2}^{\mathrm{Exp}(T)}\right) = K_t^r \boxtimes T_r(n-t, k+r-3).$$

Next, we consider the expansion of a different class of hypergraphs. Let $B(r, \ell+1)$ be the r-graph with vertex set $[\ell+1]$ and edge set

$$\{[r]\} \cup \{e \subset [2, \ell+1] \colon |e| = r \text{ and } |e \cap [2, r]| \le 1\}.$$

Recall that the Lagrangian of an r-graph $\mathcal{H}$ (by viewing $\mathcal{H}$ as a pattern) is denoted by $\lambda(\mathcal{H})$. For integers $\ell \ge r \ge 2$ let the family $\mathcal{F}_{\ell+1}^r$ be the collection of r-graphs F with the following properties:

(a) $\sup\left\{\lambda(\mathcal{H}) \colon \mathcal{H} \text{ is } F\text{-free and not a } K_\ell^r\text{-subconstruction}\right\} < \frac{\ell\cdots(\ell-r+1)}{\ell^r}$, and
(b) either F has an isolated vertex or $F \subset B(r, \ell+1)$.

For every $F \in \mathcal{F}_{\ell+1}^r$ the vertex-extendability[3] of the expansion $H_{\ell+1}^F$ can be easily obtained by a small modification of the proof of Lemma 4.8 in [51] (also see the Concluding Remarks in [51]). Hence, we obtain the following result.

***Theorem* 2.8** *Suppose that $\ell \ge r \ge 2$ are integers and $F \in \mathcal{F}_{\ell+1}^r$. Then there exist constants N_0 and $c_F > 0$ such that for all integers $n \ge N_0$ and $t \in [0, c_F n]$, we have*

$$\mathrm{EX}\left(n, (t+1)H_{\ell+1}^F\right) = \left\{K_t^r \boxtimes T_r(n-t, \ell)\right\}.$$

Remarks.

- In [63], Mubayi and Pikhurko considered the Turán problem for the r-graph Fan^r (the generalized Fan), which is the expansion of the r-graph on $r+1$ vertices with only one edge. It is easy to see that Fan^r is a member in $\mathcal{F}_{r+1}^r$.
- The Turán problem for the expansion of certain class of r-graphs (which is a proper subfamily of $\mathcal{F}_{\ell+1}^r$) were studied previously in [9] and [66].

[3] The weak vertex-extendability of $F \in \mathcal{F}_{\ell+1}^r$ with an isolated vertex also follows from Lemma 3.4 in [66].

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- Let M_k^r denote the r-graph consisting of k vertex-disjoint edges (i.e. a matching of size k) and let L_k^r denote the r-graph consisting of k edges having one vertex, say v, in common, and every pair of edges interest only at v (i.e. a k-edge sunflower with the center v). By results in [33, 40], if F is isomorphic to M_k^3 (see [33] for $k = 2$ and [40] for $k \geq 3$), L_k^3 (see [40]), or L_k^4 (see [40]), where $k \geq 2$ is an integer, then F is contained in $\mathcal{F}_{\ell+1}^r$.

Now we focus on the expansion of r-uniform matching of size two with $r \geq 4$. We say an r-graph is **semibipartite** if its vertex set can be partitioned into two parts V_1 and V_2 such that every edge contains exactly one vertex in V_1. Let $S_r(n)$ denote the semibipartite r-graph on n vertices with the maximum number of edges. Simple calculations show that $|S_r(n)| \sim \left(\frac{r-1}{r}\right)^{r-1}\binom{n}{r}$.

Confirming a conjecture of Hefetz and Keevash [33], Bene Watts, Norin, and Yepremyan [7] showed that for $r \geq 4$, $\mathrm{EX}\left(n, H_{2r}^{M_2^r}\right) = \{S_r(n)\}$ for all sufficiently large n.

The vertex-extendability[4] of $H_{2r}^{M_2^r}$ can be easily obtained by a small modification of the proof of Lemma 4.12 in [51] (also see the Concluding Remarks in [51]). Hence we have the following result.

***Theorem* 2.9** *For every integer $r \geq 4$, there exist constants N_0 and $c = c(r) > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$, we have*

$$\mathrm{EX}\left(n, (t+1)H_{2r}^{M_2^r}\right) = \left\{K_t^r \boxtimes S_r(n-t)\right\}.$$

Remark. It is quite possible that Theorem 1.5 applies to the expansion of other hypergraphs, for example, the 3-graph defined in [79] which provides the first example of a single hypergraph whose Turán density is an irrational number.

2.6 Expanded triangles

Let C_3^{2r} denote the $2r$-graph with vertex set $[3r]$ and edge set

$$\{\{1,\ldots,r,r+1,\ldots,2r\},\{r+1,\ldots,2r,2r+1,\ldots,3r\},\{1,\ldots,r,2r+1,\ldots,3r\}\}.$$

Let $[n] = V_1 \cup V_2$ be a partition such that $|V_1| = \lfloor n/2 \rfloor + m$. Let $B_{2r}^{\mathrm{odd}}(n,m)$ denote the $2r$-graph on $[n]$ whose edge set consists of all $2r$-sets that interest V_1 in odd number of vertices. Some calculations show that $\max_m |B_{2r}^{\mathrm{odd}}(n,m)| \sim \frac{1}{2}\binom{n}{2r}$. Let $B_{2r}^{\mathrm{odd}} = (2, E)$ denote the pattern such that E consists of all $2r$-multisets that contain exactly odd number of 1s. Note that $B_{2r}^{\mathrm{odd}}(n,m)$ is a B_{2r}^{odd}-construction.

The Turán problem for C_3^{2r} was first considered by Frankl [20], who proved that $\pi(C_3^{2r}) = 1/2$. Later, Keevash and Sudakov [45] proved that C_3^{2r} is edge-stable with respect to B_{2r}^{odd}, and moreover, $\mathrm{EX}(n, C_3^{2r}) \subset \left\{B_{2r}^{\mathrm{odd}}(n,m) : m \in [0, n/2]\right\}$. Simple constructions[5] show that C_3^{2r} is not degree-stable (or vertex-extendable) with respect

[4] The weak vertex-extendability of $H_{2r}^{M_2^r}$ also follows from Theorem 3.2 in [7]

[5] For example, choose a set S of $2r$ vertices from V_1 in $B_{2r}^{\mathrm{odd}}(n,0)$, then remove all edges in $B_{2r}^{\mathrm{odd}}(n,0)$ that contain at least two vertices in S and add S to the edge set.

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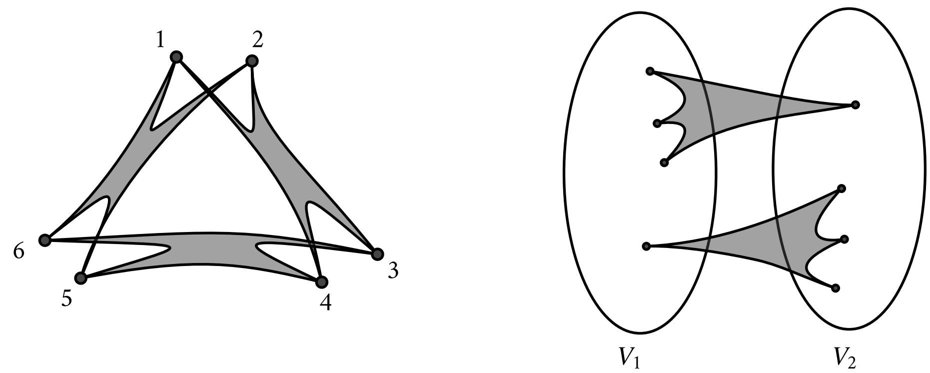

Figure 4: The 4-graph C_3^4 (expanded triangle) and the 4-graph $B_4^{\mathrm{odd}}(n)$..

to B_{2r}^{odd}. However, using Claim 3.5 in [45], one can easily show that C_3^{2r} is weakly vertex-extendable with respect to B_{2r}^{odd}. Hence, we have the following theorem.

***Theorem* 2.10** *For every integer $r \geq 2$ there exist constants N_0 and $c > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$, we have*

$$\mathrm{EX}\left(n, (t+1)C_3^{2r}\right) \subset K_t^{2r} \mathbin{\boxtimes} \left\{B_{2r}^{\mathrm{odd}}(n-t, m) \colon m \in \left[0, \sqrt{2r(n-t)}\right]\right\}.$$

Remarks.

- Calculations in [45] show that if $B_{2r}^{\mathrm{odd}}(n, m)$ is an optimal B_{2r}^{odd}-construction, then $m < \sqrt{2rn}$. So it suffices to consider m in the range $\left[0, \sqrt{2r(n-t)}\right]$ for Theorem 2.10.
- In general, one could consider the expanded $K_{\ell+1}$ for $\ell \geq 3$. It seems that the above theorem can be extended to these hypergraphs in some cases. We refer the reader to [73] and [45] for more details.

2.7 Hypergraph books

Let F_7 (4-book with 3-pages) denote the 3-graph with vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ and edge set

$$\{1234, 1235, 1236, 1237, 4567\}.$$

Let $B_4^{\mathrm{even}}(n)$ denote the maximum $B_4^{\mathrm{even}} := (2, \{1, 1, 2, 2\})$-construction on n vertices. Simply calculations show that $|B_4(n)| \sim \frac{3}{8}\binom{n}{4}$.

Füredi, Pikhurko, and Simonovits [29] proved that $\mathrm{EX}(n, F_7) = \{B_4(n)\}$ for all sufficiently large n. Moreover, they proved that F_7 is degree-stable. Hence, we obtain the following result.

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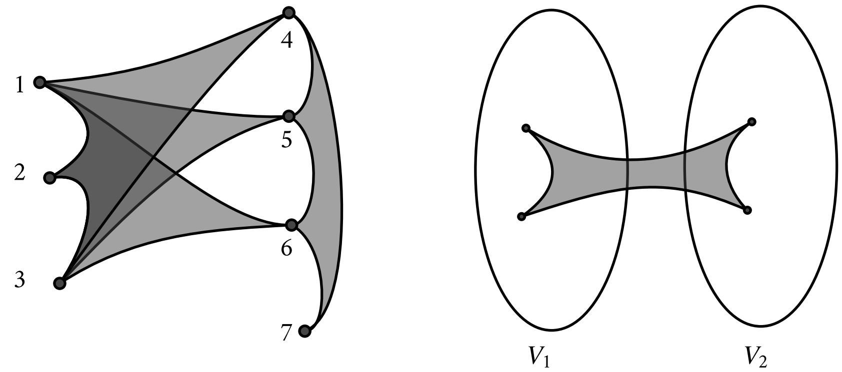

Figure 5: The 4-graph F_7 (4-book with 3 pages) and the 4-graph $B_4^{\mathrm{even}}(n)$..

Theorem 2.11 *There exist constants N_0 and $c > 0$ such that for all integers $n \geq N_0$ and $t \in [0, cn]$, we have*

$$\mathrm{EX}\,(n, (t+1)F_7) = \left\{K_t^4 \boxtimes B_4^{\mathrm{even}}(n-t)\right\}.$$

Let $\mathbb{F}_{4,3}$ denote the 4-graph with vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ and edge set

$$\{1234, 1235, 1236, 1237, 4567\}.$$

Let $B_4^{\mathrm{odd}}(n, m)$ denote the $B_4^{\mathrm{odd}} := (2, \{\{1, 2, 2, 2\}, \{1, 1, 1, 2\}\})$-construction on n vertices with one part of size $\lfloor n/2 \rfloor + m$. Recall from the previous subsection that $\max_m |B_4^{\mathrm{odd}}(n, m)| \sim \frac{1}{2}\binom{n}{4}$.

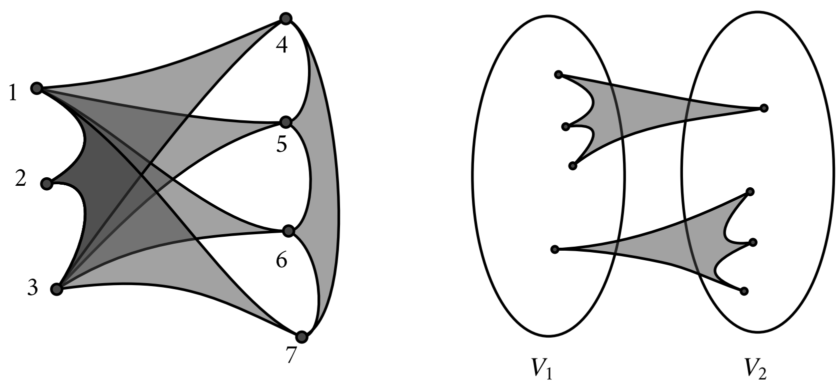

Figure 6: The 4-graph $\mathbb{F}_{4,3}$ and the 4-graph $B_4^{\mathrm{odd}}(n)$..

Füredi, Mubayi, and Pikhurko [27] proved that $\mathrm{EX}(n, \mathbb{F}_{4,3}) \subset \{B_4^{\mathrm{odd}}(n, m) \colon m \in [0, n/2]\}$ for large n, and moreover, $\mathbb{F}_{4,3}$ is edge-stable with respect to B_4^{odd}. They also

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showed that edge-stable cannot be replaced by degree-stable (or vertex-extendable). However, from Lemma 3.1 in [27] one can easily obtain that $\mathbb{F}_{4,3}$ is weakly edge-stable with respect to B_4^{odd}. Hence, we obtain the following theorem.

***Theorem* 2.12** *There exist constants* N_0 *and* $c > 0$ *such that for all integers* $n \geq N_0$ *and* $t \in [0, cn]$, *we have*

$$\mathrm{EX}\left(n, (t+1)\mathbb{F}_{4,3}\right) \subset K_t^4 \boxtimes \left\{B_4^{\mathrm{odd}}(n-t, m) \colon m \in [0, \sqrt{4(n-t)}]\right\}.$$

Let $\mathbb{F}_{3,2}$ denote the 3-graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set

$$\{123, 124, 125, 345\}.$$

Recall that $S_3(n)$ is the semibipartite 3-graph on n vertices with the maximum number of edges, i.e. the maximum $S_3 := (2, \{1, 2, 2\})$-construction on n vertices.

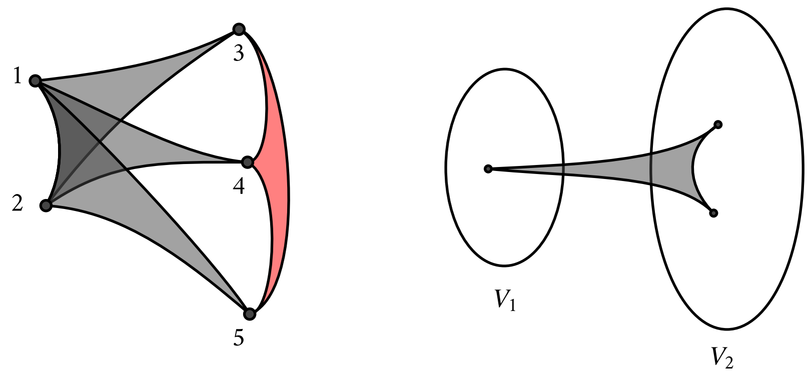

Figure 7: The 3-graph $\mathbb{F}_{3,2}$ and the semibipartite 3-graph $S_3(n)$..

Füredi, Pikhurko, and Simonovits [28] proved that $\mathrm{EX}(n, \mathbb{F}_{3,2}) = \{S_3(n)\}$ for all sufficiently large n. A construction in their paper ([28, Construction 1.2]) shows that $\mathbb{F}_{3,2}$ is not vertex-extendable with respect S_3. But we will present a short proof in Section 5 which shows that $\mathbb{F}_{3,2}$ is weakly vertex-extendable with respect to S_3. Hence, we obtain the following result.

***Theorem* 2.13** *There exist constants* N_0 *and* $c > 0$ *such that for all integers* $n \geq N_0$ *and* $t \in [0, cn]$, *we have*

$$\mathrm{EX}\left(n, (t+1)\mathbb{F}_{3,2}\right) = \left\{K_t^r \boxtimes S_3(n-t)\right\}.$$

3 Proofs of Theorems 1.5 and 1.6

In this section, we prove Theorems 1.5 and 1.6. In fact, we will prove the following more general (but also more technical) version.

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***Theorem* 3.1** *Let $m \geq r \geq 2$ be integers and F be a nondegenerate r-graph on m vertices. Let $f : \mathbb{N} \to \mathbb{R}$ be a nondecreasing function. Suppose that for all sufficiently large $n \in \mathbb{N}$:*

(a) $\mathrm{ex}(n, F)$ *is* $\frac{1-\pi(F)}{8m}\binom{n}{r-1}$*-smooth, and*
(b) F *is* $\left(f(n), \frac{1-\pi(F)}{4m}\binom{n}{r-1}\right)$*-bounded.*

Then there exists N_0 such that the following statements hold for all integers $n, t \in \mathbb{N}$ with

$$n \geq N_0, \quad t \leq \frac{1-\pi(F)}{64rm^2}n, \quad \text{and} \quad 2emt\binom{n-2mt}{r-2} \leq f(n-2mt).$$

(i) If a collection $\{\mathcal{H}_1, \ldots, \mathcal{H}_{t+1}\}$ of n-vertex r-graphs on the same vertex set satisfies

$$|\mathcal{H}_i| > \binom{n}{r} - \binom{n-t}{r} + \mathrm{ex}(n-t, F) \quad \text{for all } i \in [t+1],$$

then $\{\mathcal{H}_1, \ldots, \mathcal{H}_{t+1}\}$ contains a rainbow F-matching.
(ii) We have $\mathrm{EX}(n, (t+1)F) = K_t^r \mathbin{\bar\times} \mathrm{EX}(n-t, F)$.

3.1 Preparations

First, recall the following result due to Katona, Nemetz, and Simonovits [41]

Proposition Fix an r-graph F. The ratio $\frac{\mathrm{ex}(n,F)}{\binom{n}{r}}$ is nonincreasing in n. In particular, $\mathrm{ex}(n, F) \geq \pi(F)\binom{n}{r}$ for all $n \in \mathbb{N}$, and

$$\pi(F) \leq \frac{\mathrm{ex}(v(F), F)}{\binom{v(F)}{r}} \leq \frac{\binom{v(F)}{r} - 1}{\binom{v(F)}{r}} < 1.$$

■

Next, we prove two simple inequalities concerning binomials.

***Lemma* 3.3** *Suppose that $m \leq n/r - 1$. Then*

$$\binom{n}{r} - \binom{n-m}{r} = \sum_{i=1}^{r}\binom{m}{i}\binom{n-m}{r-i} \leq 2m\binom{n-m}{r-1}. \tag{3.1}$$

Proof For every $i \in [2, r]$ we have

$$\frac{\binom{m}{i}\binom{n-m}{r-i}}{\binom{m}{i-1}\binom{n-m}{r-i+1}} = \frac{m-i+1}{i}\frac{r-i+1}{n-m-r+i} \leq \frac{(r-1)m}{2(n-m-r)} \leq \frac{1}{2},$$

where the last inequality follows from the assumption that $m \leq n/r - 1$. Therefore,

$$\sum_{i=1}^{r}\binom{m}{i}\binom{n-m}{r-i} \leq \sum_{i=1}^{r}\left(\frac{1}{2}\right)^{i-1} m\binom{n-m}{r-1} \leq 2m\binom{n-m}{r-1},$$

proving Lemma 3.3. ■

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***Lemma* 3.4** *Suppose that integers $n, b, r \geq 1$ satisfy $b \leq \frac{n-r}{r+1}$. Then*

$$\binom{n}{r} \leq e\binom{n-b}{r}.$$

Proof For every $i \in [b]$ it follows from $b \leq \frac{n-r}{r+1}$ that $\frac{n-i}{n-i-r} = 1 + \frac{r}{n-i-r} \leq 1 + \frac{r}{n-b-r} \leq 1 + \frac{1}{b}$. Therefore,

$$\binom{n}{r} = \prod_{i=0}^{b-1} \frac{n-i}{n-i-r}\binom{n-b}{r} \leq \left(1 + \frac{1}{b}\right)^b \binom{n-b}{r} \leq e\binom{n-b}{r},$$

proving Lemma 3.4. ■

The following lemma says that $d(n, F)$ is well-behaved for every F.

***Lemma* 3.5** *Let F be an r-graph. For every n and $m \leq n/r - 1$ we have*

$$|d(n, F) - d(n-m, F)| \leq 4m\binom{n-m}{r-2}.$$

Proof It follows from Proposition 3.2 that $\mathrm{ex}(n, F)/\binom{n}{r} \leq \mathrm{ex}(n-m, F)/\binom{n-m}{r}$. Therefore,

$$\begin{aligned}
\mathrm{ex}(n, F) - \mathrm{ex}(n-m, F) &\leq \frac{\binom{n}{r}}{\binom{n-m}{r}}\mathrm{ex}(n-m, F) - \mathrm{ex}(n-m, F) \\
&= \frac{\binom{n}{r} - \binom{n-m}{r}}{\binom{n-m}{r}}\mathrm{ex}(n-m, F) \\
&\overset{\text{Lemma 3.3}}{\leq} \frac{2m\binom{n-m}{r-1}}{\binom{n-m}{r}}\mathrm{ex}(n-m, F) = \frac{2mr}{n-m-r+1}\mathrm{ex}(n-m, F).
\end{aligned}$$

Consequently,

$$\begin{aligned}
|d(n, F) - d(n-m, F)| &= \left|\frac{r \cdot \mathrm{ex}(n, F)}{n} - \frac{r \cdot \mathrm{ex}(n-m, F)}{n-m}\right| \\
&= \left|\frac{r}{n}\left(\mathrm{ex}(n, F) - \mathrm{ex}(n-m, F)\right) - \frac{rm}{n(n-m)}\mathrm{ex}(n-m, F)\right| \\
&\leq \max\left\{\frac{2mr^2}{n(n-m-r+1)}, \frac{rm}{n(n-m)}\right\} \cdot \mathrm{ex}(n-m, F) \\
&\leq \frac{2mr^2}{n(n-m-r+1)}\binom{n-m}{r} \leq 4m\binom{n-m}{r-2}.
\end{aligned}$$

This completes the proof of Lemma 3.5. ■

The following lemma deals with a simple case of Theorem 3.1 in which the maximum degree of every r-graph $\mathcal{H}_i$ is bounded away from $\binom{n-1}{r-1}$.

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Lemma 3.6 *Let F be a nondegenerate r-graph with m vertices. Suppose that $\mathrm{ex}(n, F)$ is g-smooth with $g(n) \le \frac{1-\pi(F)}{8m}\binom{n}{r-1}$ for all sufficiently large n. Then there exists N_1 such that the following holds for all integers $n, t \in \mathbb{N}$ with $n \ge N_1$ and $t \le \frac{1-\pi(F)}{64rm^2}n$.*

Suppose that $\{\mathcal{H}_1, \ldots, \mathcal{H}_{t+1}\}$ is a collection of n-vertex r-graphs on the same vertex set V such that

$$|\mathcal{H}_i| \ge \mathrm{ex}(n-t, F) + t\binom{n-t}{r-1} \quad \text{and} \quad \Delta(\mathcal{H}_i) \le d(n-t, F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1}$$

hold for all $i \in [t+1]$. Then $\{\mathcal{H}_1, \ldots, \mathcal{H}_{t+1}\}$ contains a rainbow F-matching.

Proof Given an integer $k \le t+1$, we say a collection $C = \{S_1, \ldots, S_k\}$ of pairwise disjoint m-subsets of V is F-**rainbow** if there exists an injection $f\colon [k] \to [t+1]$ such that $F \subset \mathcal{H}_{f(i)}[S_i]$ for all $i \in [k]$.

Fix a maximal collection $C = \{S_1, \ldots, S_k\}$ of pairwise disjoint m-subsets of V that is F-rainbow. If $k = t+1$, then we are done. So we may assume that $k \le t$. Without loss of generality, we may assume that $F \subset \mathcal{H}_i[S_i]$ for all $i \in [k]$ (i.e. f is the identity map). Let $B = \bigcup_{i=1}^{k} S_i$ and let $b = |B| = mk$.

Let us count the number of edges in $\mathcal{H}_{k+1}$. Observe that every copy of F in $\mathcal{H}_{k+1}$ must contain a vertex from B, since otherwise, it would contradict the maximality of C. Therefore, the induced subgraph of $\mathcal{H}_{k+1}$ on $V_0 := V \setminus B$ is F-free. Hence, by the maximum degree assumption, we obtain

$$\begin{aligned}
|\mathcal{H}_{k+1}| &\le |\mathcal{H}_{k+1}[V_0]| + b\left(d(n-t, F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1}\right) \\
&\le \mathrm{ex}(n-b, F) + b\left(d(n-t, F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1}\right) \\
&= \mathrm{ex}(n-t, F) + t\binom{n-t}{r-1} - (\Delta_1 + \Delta_2),
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1 &:= \mathrm{ex}(n-t, F) - \mathrm{ex}(n-b, F) - (b-t)d(n-t, F), \\
\Delta_2 &:= t\left(\binom{n-t}{r-1} - d(n-t, F)\right) - b\frac{1-\pi(F)}{2m}\binom{n-t}{r-1}.
\end{aligned}$$

Next, we will prove that $\Delta_1 + \Delta_2 > 0$, which implies that $|\mathcal{H}_{k+1}| < \mathrm{ex}(n-t, F) + t\binom{n-t}{r-1}$ contradicting our assumption.

Since $n - t \ge N_1/2$ is sufficiently large and $\lim_{n\to\infty} \mathrm{ex}(n-t, F)/\binom{n-t}{r} = \pi(F)$, we have $\mathrm{ex}(n-t, F) \le \left(\pi(F) + \frac{1-\pi(F)}{5}\right)\binom{n-t}{r}$, and hence,

$$d(n-t, F) = \frac{r \cdot \mathrm{ex}(n-t, F)}{n-t} \le \left(\pi(F) + \frac{1-\pi(F)}{5}\right)\binom{n-t}{r-1}.$$

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Therefore,

$$\begin{aligned}\Delta_2 &\geq t\left(1-\left(\pi(F)+\frac{1-\pi(F)}{5}\right)\right)\binom{n-t}{r-1} - mt\frac{1-\pi(F)}{2m}\binom{n-t}{r-1}\\ &\geq \frac{1-\pi(F)}{4}\binom{n-t}{r-1}t.\end{aligned}$$

On the other hand, by Lemma 3.5, we have

$$d(n-t,F) \leq d(n-b,F) + 4(b-t)\binom{n-b}{r-2} \leq d(n-b,F) + 4mt\binom{n-t}{r-2}.$$

Therefore, it follows from the Smoothness assumption and g is nondecreasing that

$$\begin{aligned}\Delta_1 &= \sum_{i=1}^{b-t}(\mathrm{ex}(n-b+i,F)-\mathrm{ex}(n-b+i-1,F)) - (b-t)d(n-t,F)\\ &\overset{\text{Smoothness}}{\geq} \sum_{i=0}^{b-t-1}(d(n-b+i,F)-g(n-b+i+1)) - (b-t)d(n-t,F)\\ &\overset{\text{Nondecreasing}}{\geq} \sum_{i=0}^{b-t-1}(d(n-b+i,F)-d(n-t,F)) - (b-t)g(n-t)\\ &\overset{\text{Lemma 3.5}}{\geq} -\sum_{i=0}^{b-t-1}4(b-t-i)\binom{n-b+i}{r-2} - (b-t)g(n-t)\\ &\geq -4m^2t^2\binom{n-t-1}{r-2} - mt\cdot g(n-t) = -\frac{4(r-1)m^2t^2}{n-t}\binom{n-t}{r-1} - mt\cdot g(n-t).\end{aligned}$$

Since $t \leq \frac{1-\pi(F)}{64rm^2}n$, we obtain $\frac{4(r-1)m^2t^2}{n-t} < \frac{1-\pi(F)}{8}t$. Together with $g(n-t) \leq \frac{1-\pi(F)}{8m}\binom{n-t}{r-1}$, we obatin

$$\Delta_1 > -\left(\frac{1-\pi(F)}{8}t + mt\frac{1-\pi(F)}{8m}\right)\binom{n-t}{t-1} = -\frac{1-\pi(F)}{4}t\binom{n-t}{r-1}.$$

Therefore, $\Delta_1 + \Delta_2 > 0$. This finishes the proof of Lemma 3.6. ■

3.2 Proof of Theorem 3.1

We prove Theorem 3.1 in this section. Let us prove Part (i) first.

Proof Fix a sufficiently large constant N_0 and suppose that $n \geq N_0$. Let $k \leq t+1$. We say a collection $L := \{v_1, \ldots, v_k\}$ of vertices in V is **heavy-rainbow** if there exists an injection $f\colon [k] \to [t+1]$ such that

$$d_{\mathcal{H}_{f(i)}}(v_i) \geq d(n-t,F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1} \quad \text{for all } i \in [k].$$

Fix a maximal collection $L := \{v_1, \ldots, v_k\}$ of vertices that is heavy-rainbow. Without loss of generality, we may assume that f (defined above) is the identity map.

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Let $V_0 = V \setminus L$ and $\mathcal{H}'_j = \mathcal{H}_j[V_0]$ for all $j \in [k+1, t+1]$. For every $j \in [k+1, t+1]$ observe that there are at most $\binom{n}{r} - \binom{n-k}{r}$ edges in $\mathcal{H}_j$ that have nonempty intersection with L. Hence,

$$\begin{aligned}|\mathcal{H}'_j| &\geq |\mathcal{H}_j| - \left(\binom{n}{r} - \binom{n-k}{r}\right) \\ &\geq \mathrm{ex}(n-t, F) + \binom{n}{r} - \binom{n-t}{r} - \left(\binom{n}{r} - \binom{n-k}{r}\right) \\ &= \mathrm{ex}((n-k)-(t-k), F) + \binom{n-k}{r} - \binom{(n-k)-(t-k)}{r}.\end{aligned}$$

On the other hand, it follows from the maximality of L that

$$\begin{aligned}\Delta(\mathcal{H}'_j) \leq \Delta(\mathcal{H}_j) &\leq d(n-t, F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1} \\ &= d((n-k)-(t-k), F) + \frac{1-\pi(F)}{2m}\binom{(n-k)-(t-k)}{r-1}\end{aligned}$$

holds for all $j \in [k+1, t+1]$. By assumption, $\frac{t-k}{n-k} \leq \frac{t}{n} \leq \frac{1-\pi(F)}{64rm^2}$ and $n-k \geq n/2$ is sufficiently large, so it follows from Lemma 3.6 that there exists a collection $\mathcal{C} = \{S_{k+1}, \ldots, S_{t+1}\}$ of pairwise disjoint m-subsets of V_0 such that $F \subset \mathcal{H}'_j[S_j]$ for all $j \in [k+1, t+1]$.

Next we will find a collection of rainbow copies of F from $\{\mathcal{H}_1, \ldots, \mathcal{H}_k\}$.

Claim For every $i \in [k]$ and for every set $B_i \subset V \setminus \{v_i\}$ of size at most $2mt$ there exists a copy of F in $\mathcal{H}_i[V \setminus B_i]$. ■

Proof Fix $i \in [k]$ and fix a set $B_i \subset V \setminus \{v_i\}$ of size at most $2mt$. We may assume that $|B_i| = 2mt$. Let $V_i = V \setminus B_i$ and $n_i = |V_i| = n - 2mt$. Let $\mathcal{H}'_i = \mathcal{H}_i[V_i]$. Since the number of edges in $\mathcal{H}_i$ containing v_i that have nonempty intersection with B_i is at most $2mt\binom{n-1}{r-2}$, we have

$$\begin{aligned}d_{\mathcal{H}'_i}(v_i) &\geq d(n-t, F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1} - 2mt\binom{n-1}{r-2} \\ &\overset{\text{Lemma 3.5}}{\geq} d(n-2mt, F) - 2mt\binom{n-2mt}{r-2} + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1} - 2mt\binom{n-1}{r-2} \\ &> d(n-2mt, F) + \frac{1-\pi(F)}{4m}\binom{n-2mt}{r-1}, \qquad (3.2)\end{aligned}$$

where the last inequality holds because $t \leq \frac{1-\pi(F)}{64rm^2}n$ and n is sufficiently large.

Similarly, we have

$$\begin{aligned}|\mathcal{H}'_i| \geq |\mathcal{H}_i| - 2mt\binom{n-1}{r-1} &> \mathrm{ex}(n-t, F) + \binom{n}{r} - \binom{n-t}{r} - 2mt\binom{n-1}{r-1} \\ &\overset{\text{Lemma 3.4}}{\geq} \mathrm{ex}(n-2mt, F) - 2emt\binom{n-2mt}{r-1}.\end{aligned}$$

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which, by the assumption $f(n-2mt) \geq 2emt\binom{n-2mt}{r-2}$, implies that

$$d(\mathcal{H}_{i'}) = \frac{r \cdot |\mathcal{H}_i'|}{n-2mt} \geq d(n-2mt, F) - 2emt\binom{n-2mt-1}{r-2}$$
$$> d(n-2mt, F) - f(n-2mt). \tag{3.3}$$

It follows from (3.2), (3.3), and the Boundedness assumption that $F \subset \mathcal{H}_i'$. ■

Let $B = L \cup S_{k+1} \cup \cdots \cup S_{t+1}$. Now we can repeatedly apply Claim 3.7 to find a collection of rainbow copies of F as follows. First, we let $B_1 = B \setminus \{v_1\}$. Since $|B_1| = k-1+m(t+1-k) \leq 2mt$, Claim 3.7 applied to v_1, B_1, and $\mathcal{H}_1$ yields an m-set $S_1 \subset V \setminus B_1$ such that $F \subset \mathcal{H}_1[S_1]$. Suppose that we have define $S_1, \ldots, S_i$ for some $i \in [k-1]$ such that $F \subset \mathcal{H}_j[S_j]$ holds for all $j \leq i$. Then let $B_{i+1} = (B \cup S_1 \cup \cdots \cup S_i) \setminus \{v_{i+1}\}$. Since $|B_{i+1}| = k-1+m(t+1-k)+im \leq 2mt$, Claim 3.7 applied to v_{i+1}, B_{i+1}, and $\mathcal{H}_{i+1}$ yields an m-set $S_{i+1} \subset V \setminus B_{i+1}$ such that $F \subset \mathcal{H}_{i+1}[S_{i+1}]$. At the end of this process, we obtain a collection $\{S_1, \ldots, S_k\}$ of pairwise disjoint sets such that $F \subset \mathcal{H}_i[S_i]$ holds for all $i \in [k]$. Since $S_i \cap S_j = \emptyset$ for all $i \in [k]$ and $j \in [k+1, t+1]$, the set $\{S_1, \ldots, S_{t+1}\}$ yields a rainbow F-matching.

Before proving Part (ii) of Theorem 3.1, we need the simple corollary of Lemma 3.6.

***Lemma* 3.8** *Let F be a nondegenerate r-graph with m vertices. Suppose that $\mathrm{ex}(n, F)$ is g-smooth with $g(n) \leq \frac{1-\pi(F)}{8m}\binom{n}{r-1}$ for all sufficiently large n. Then there exists N_1 such that the following holds for all integers $n, t \in \mathbb{N}$ with $n \geq N_1$ and $t \leq \frac{1-\pi(F)}{64rm^2}n$.*

Suppose that $\mathcal{H}$ is an n-vertex r-graph with

$$\Delta(\mathcal{H}) \leq d(n-t, F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1} \quad \text{and} \quad \nu(F, \mathcal{H}) < t+1.$$

Then

$$|\mathcal{H}| < \mathrm{ex}(n-t, F) + t\binom{n-t}{r-1}.$$

Now we are ready to prove Part (ii).

Proof Let $\mathcal{H}$ be an n-vertex r-graph with $\mathrm{ex}(n, (t+1)F)$ edges and $\nu(F, \mathcal{H}) < t+1$. Note that Theorem 3.1 (i) already implies that $\mathrm{ex}(n, (t+1)F) \leq \binom{n}{r} - \binom{n-t}{r} + \mathrm{ex}(n-t, F)$. So, it suffices to show that $\mathcal{H}$ is isomorphic to $K_t^r \boxtimes \mathcal{G}$ for some $\mathcal{G} \in \mathrm{EX}(n-t, F)$.

Let $V = V(\mathcal{H})$ and define

$$L := \left\{v \in V \colon d_{\mathcal{H}}(v) \geq d(n-t, F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1}\right\}.$$

A similar argument as in the proof of Claim 3.7 yields the following claim.

Claim For every $v \in L$ and for every set $B \subset V \setminus \{v\}$ of size at most $2mt$ there exists a copy of F in $\mathcal{H}[V \setminus B]$. ■

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Let $\ell = |L|$. We have the following claim for ℓ.

Claim We have $\ell \le t$. ■

Proof Suppose to the contrary that $\ell \ge t + 1$. By taking a subset of L if necessary, we may assume that $\ell = t + 1$. Let us assume that $L = \{v_1, \ldots, v_{t+1}\}$. We will repeatedly apply Claim 3.9 to find a collection $\{S_1, \ldots, S_{t+1}\}$ of pairwise disjoint m-sets such that $F \subset \mathcal{H}[S_i]$ for all $i \in [t + 1]$ as follows.

Let $B_1 = L \setminus \{v_1\}$. Since $|B_1| \le 2mt$, it follows from Claim 3.9 that there exists a set $S_1 \subset V \setminus B$ such that $F \subset \mathcal{H}[S_1]$. Now suppose that we have found pairwise disjoint m-sets $S_1, \ldots, S_i$ for some $i \le t$. Let $B_{i+1} = (L \cup S_1 \cup \cdots \cup S_i) \setminus \{v_i\}$. It is clear that $|B_{i+1}| \le 2mt$. So it follows from Claim 3.9 that there exists a set $S_{i+1} \subset V \setminus B$ such that $F \subset \mathcal{H}[S_{i+1}]$. Repeat this process for $t + 1$ times, we find the collection $\{S_1, \ldots, S_{t+1}\}$ that satisfies the assertion. However, this contradicts the assumption that $\nu(F, \mathcal{H}) < t + 1$. ■

Let $V_0 = V \setminus L$ and $\mathcal{H}_0 = \mathcal{H}[V_0]$. The following claim follows from a similar argument as in the last paragraph of the proof of Theorem 3.1.

Claim We have $\nu(F, \mathcal{H}_0) < t - \ell + 1$. ■

If $\ell = t$, then Claim 3.11 implies that $\mathcal{H}_0$ is F-free. Therefore, it follows from

$$|\mathcal{H}_0| \ge |\mathcal{H}| - \left(\binom{n}{r} - \binom{n-t}{r}\right) = \mathrm{ex}(n - t, F)$$

that $\mathcal{H}_0 \in \mathrm{EX}(n-t, F)$ and $d(v) = \binom{n-1}{r-1}$ for all $v \in L$, which implies that $\mathcal{H} = K_t^r \boxtimes \mathcal{G}$ for some $\mathcal{G} \in \mathrm{EX}(n - t, F)$.

If $\ell \le t - 1$, then it follows from $\Delta(\mathcal{H}_0) \le d(n - t, F) + \frac{1-\pi(F)}{2m}\binom{n-t}{r-1}$, $\nu(F, \mathcal{H}_0) < t - \ell + 1$, and Lemma 3.8 that

$$|\mathcal{H}_0| < \mathrm{ex}(n - t, F) + (t - \ell)\binom{n-t}{r-1}.$$

Consequently,

$$\begin{aligned}|\mathcal{H}| \le |\mathcal{H}_0| + \binom{n}{r} - \binom{n-\ell}{r} &< \mathrm{ex}(n - t, F) + (t - \ell)\binom{n-t}{r-1} + \binom{n}{r} - \binom{n-\ell}{r} \\ &\le \mathrm{ex}(n - t, F) + \binom{n}{r} - \binom{n-t}{r},\end{aligned}$$

a contradiction.

4 Proofs of Theorems 1.7 and 1.9

In this section, we prove Theorems 1.7 and 1.9. Before that, let us introduce some definitions and prove some preliminary results.

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4.1 Preliminaries

The following fact concerning $\delta(n, F)$ for all hypergraphs F.

Fact Let F be an r-graph and $n \geq 1$ be an integer. Then every maximum n-vertex F-free r-graph $\mathcal{H}$ satisfies $\delta(\mathcal{H}) \geq \delta(n, F)$. In particular, $d(n, F) \geq \delta(n, F)$. ∎

Proof Let $v \in V(\mathcal{H})$ be a vertex with minimum degree and let $\mathcal{H}'$ be the induced subgraph of $\mathcal{H}$ on $V(\mathcal{H}) \setminus \{v\}$. Since $\mathcal{H}'$ is an $(n-1)$-vertex F-free r-graph, we have $|\mathcal{H}'| \leq \mathrm{ex}(n-1, F)$. On the other hand, since $\mathcal{H}$ is a maximum n-vertex F-free r-graph, we have $\mathrm{ex}(n, F) = |\mathcal{H}|$. Therefore,

$$\delta(n, F) = \mathrm{ex}(n, F) - \mathrm{ex}(n-1, F) \leq |\mathcal{H}| - |\mathcal{H}'| = d_{\mathcal{H}}(v) = \delta(\mathcal{H}),$$

which proves Fact 4.1. ∎

For Turán pairs (F, P) we have the following fact which provides a lower bound for $\delta(n, F)$.

Fact Suppose that (F, P) is a Turán pair and $\mathcal{H}$ is a maximum F-free r-graph on $n-1$ vertices. Then $\delta(n, F) \geq \Delta(\mathcal{H})$. In particular, $\delta(n, F) \geq d(n-1, F)$. ∎

Proof First, notice that $|\mathcal{H}| = \mathrm{ex}(n-1, F)$. On the other hand, it follows from the definition of Turán pair that $\mathcal{H}$ is an $(n-1)$-vertex P-construction. Let $\tilde{\mathcal{H}}$ be an n-vertex P-construction obtained from $\mathcal{H}$ by duplicating a vertex $v \in V(\mathcal{H})$ with maximum degree. In other words, $\tilde{\mathcal{H}}$ is obtained from $\mathcal{H}$ by adding a new vertex u and adding all edges in $\{\{u\} \cup S \colon S \in L_{\mathcal{H}}(v)\}$. It is clear that $\tilde{\mathcal{H}}$ is an n-vertex P-construction, and hence, $\tilde{\mathcal{H}}$ is F-free. So $|\tilde{\mathcal{H}}| \leq \mathrm{ex}(n, F)$. It follows that

$$\delta(n, F) = \mathrm{ex}(n, F) - \mathrm{ex}(n-1, F) \geq |\tilde{\mathcal{H}}| - |\mathcal{H}| = d_{\mathcal{H}}(v) = \Delta(\mathcal{H}) \geq d(\mathcal{H}) \geq d(n-1, F),$$

which proves Fact 4.2. ∎

The following result can be derived with a minor modification to the proof of [50, Lemma 4.2] (see Section A for details).

Fact Let F be an r-graph and let $\mathcal{H}$ be an n-vertex F-free r-graph. If n is large, $\varepsilon > 0$ is small, and $|\mathcal{H}| \geq (\pi(F) - \varepsilon)\binom{n}{r}$, then

(a) the set

$$Z_{\varepsilon}(\mathcal{H}) := \left\{ v \in V(\mathcal{H}) \colon d_{\mathcal{H}}(v) \leq \left(\pi(F) - r\varepsilon^{1/2}\right)\binom{n-1}{r-1} \right\}$$

has size at most $\varepsilon^{1/2} n$, and

(b) the induced subgraph $\mathcal{H}'$ of $\mathcal{H}$ on $V(\mathcal{H}) \setminus Z_{\varepsilon}(\mathcal{H})$ satisfies $\delta(\mathcal{H}') \geq \left(\pi(F) - 2r\varepsilon^{1/2}\right)\binom{n-1}{r-1}$.

∎

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4.2 Proofs of Theorems 1.7 and 1.9

We prove Theorem 1.7 first.

Proof Fix an integer $n \geq 1$. Then

$$|\delta(n,F) - d(n-1,F)| \overset{\text{Fact 4.2}}{=} \delta(n,F) - d(n-1,F) \overset{\text{Fact 4.1}}{\leq} d(n,F) - d(n-1,F) \overset{\text{Lemma 3.5}}{\leq} 4\binom{n-1}{r-2},$$

which proves Theorem 1.7. ■

Next we prove Theorem 1.9.

Proof Fix constants $0 < \varepsilon \ll \varepsilon_1 \ll 1$ and let $n \in \mathbb{N}$ be sufficiently large. Suppose to the contrary that there exists an n-vertex F-free r-graph $\mathcal{H}$ with $d(\mathcal{H}) \geq d(n,F) - \varepsilon\binom{n-1}{r-1}$ and $\Delta(\mathcal{H}) \geq d(n,F) + \frac{1-\pi(F)}{8m}\binom{n-1}{r-1}$. Let $V = V(\mathcal{H})$. Fix a vertex $v \in V$ with $d_{\mathcal{H}}(v) = \Delta(\mathcal{H})$. Let $V_0 = V \setminus \{v\}$ and $\mathcal{H}_0 = \mathcal{H}[V_0]$. Since

$$|\mathcal{H}_0| \geq |\mathcal{H}| - \binom{n-1}{r-1} \geq \mathrm{ex}(n,F) - 2\varepsilon\binom{n}{r},$$

it follows from the edge-stability of F that $\mathcal{H}_0$ contains a subgraph $\mathcal{H}_1$ with at least $\mathrm{ex}(n,F) - \varepsilon_1\binom{n}{r} \geq (\pi(F) - \varepsilon_1)\binom{n}{r}$ edges, and moreover, $\mathcal{H}_1$ is a P-subconstruction.

It follows from Fact 4.3 that the set

$$Z := \left\{v \in V : d_{\mathcal{H}_1}(v) \leq \left(\pi(F) - r\varepsilon_1^{1/2}\right)\binom{n-1}{r-1}\right\}$$

has size at most $\varepsilon_1^{1/2}n$, and moreover, the r-graph $\mathcal{H}_2 := \mathcal{H}_1[V_0 \setminus Z]$ satisfies $\delta(\mathcal{H}_2) \geq \left(\pi(F) - 2r\varepsilon_1^{1/2}\right)\binom{n-1}{r-1}$. Note that $\mathcal{H}_2 \subset \mathcal{H}_1$ is also a P-subconstruction.

Define $\mathcal{H}_3 := \mathcal{H}_2 \cup \{e \in \mathcal{H}[V \setminus Z] : v \in e\}$. Since $|Z| \leq \varepsilon_1^{1/2}n \leq \frac{1-\pi(F)}{72m}\frac{n}{r}$, we have

$$\begin{aligned} d_{\mathcal{H}_3}(v) \geq d_{\mathcal{H}}(v) - |Z|\binom{n-2}{r-2} &\geq d(n,F) + \frac{1-\pi(F)}{8m}\binom{n-1}{r-1} - \frac{1-\pi(F)}{72m}\frac{n}{r}\binom{n-2}{r-2} \\ &\geq d(n,F) + \frac{1-\pi(F)}{8m}\binom{n-1}{r-1} - \frac{1-\pi(F)}{72m}\binom{n-1}{r-1} \\ &\geq \left(\pi(F) + \frac{1-\pi(F)}{9m}\right)\binom{n-1}{r-1}. \end{aligned}$$

Let $n' = |V \setminus Z|$. Note that $\mathcal{H}_3$ is an F-free r-graph on n' vertices with $\delta(\mathcal{H}_3) \geq \delta(\mathcal{H}_2) \geq \left(\pi(F) - 2r\varepsilon_1^{1/2}\right)\binom{n-1}{r-1}$, and $v \in V(\mathcal{H}_3)$ is a vertex such that $\mathcal{H}_3 - v = \mathcal{H}_2$ is a P-subconstruction. However, this contradicts the weak vertex-extendability of F since ε_1 is sufficiently small and $d_{\mathcal{H}_3}(v) \geq \left(\pi(F) + \frac{1-\pi(F)}{9m}\right)\binom{n-1}{r-1}$. ■

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5 Proof of Theorem 2.13

The edge-stability of $\mathbb{F}_{3,2}$ was already proved in [28, Theorem 2.2], so by Theorems 1.5, 1.7, and 1.9, to prove Theorem 2.13 it suffices to prove the following result.

***Theorem* 5.1** *The 3-graph $\mathbb{F}_{3,2}$ is weakly vertex-extendable with respect to the pattern $S_3 :=$ $(2, \{1, 2, 2\})$.*

Proof Fix $\delta > 0$. Let n be sufficiently large and $\zeta > 0$ be sufficiently small. Let $\mathcal{H}$ be an n-vertex $\mathbb{F}_{3,2}$-free 3-graph with $\delta(\mathcal{H}) \geq \left(\frac{4}{9} - \zeta\right)\binom{n-1}{2}$. Suppose that $v \in V$ is a vertex such that $\mathcal{H}_0 := \mathcal{H} - v$ is an S_3-subconstruction (i.e. semibipartite). It suffices to show that $d_{\mathcal{H}}(v) \leq \left(\frac{4}{9} + \delta\right)\binom{n-1}{2}$.

Suppose to the contrary that $d_{\mathcal{H}}(v) > \left(\frac{4}{9} + \delta\right)\binom{n-1}{2}$. Let $V_1 \cup V_2$ be a bipartition of $V_0 := V \setminus \{v\}$ such that every edge in $\mathcal{H}_0$ contains exactly one vertex from V_1. Since $|\mathcal{H}_0| \geq \frac{3}{n}\delta(\mathcal{H}) \geq \left(\frac{4}{9} - \zeta\right)\binom{n}{3}$, it follows from some simple calculations (see e.g. [28, Theorem 2.2 (ii)]) that

$$\max\left\{\left||V_1| - \frac{n}{3}\right|, \left||V_2| - \frac{2n}{3}\right|\right\} \leq \zeta^{1/2} n. \tag{5.1}$$

Recall that the link of a vertex $u \in V(\mathcal{H})$ is defined as

$$L_{\mathcal{H}}(u) := \left\{A \in \binom{V(\mathcal{H})}{r-1} : A \cup \{u\} \in \mathcal{H}\right\}.$$

Let $L = L_{\mathcal{H}}(v)$ for simplicity and let

$$L_1 := L \cap \binom{V_1}{2}, \quad L_2 := L \cap \binom{V_2}{2}, \quad \text{and} \quad L_{1,2} := L \cap (V_1 \times V_2).$$

Here we abuse the use of notation by letting $V_1 \times V_2$ denote the edge set of the complete bipartite graph with parts V_1 and V_2.

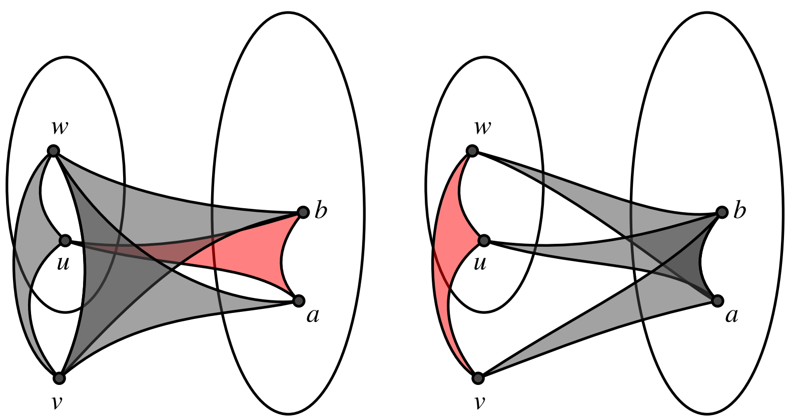

Figure 8: Finding $\mathbb{F}_{3,2}$ in Claim 5.2 (left) and Claim 5.3 (right)..

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Claim We have $|L_2| \geq \frac{\delta}{8}n^2$. ■

Proof Suppose to the contrary that $|L_2| \leq \delta n^2/8$. Then it follows from the inequality

$$\sum_{v' \in V_1} d_L(v') = 2|L_1| + |L_{1,2}| \geq |L| - |L_2| \geq \left(\frac{4}{9}+\delta\right)\binom{n-1}{2} - \frac{\delta}{8}n^2 \geq \left(\frac{2}{9}+\frac{\delta}{4}\right)n^2$$

that there exists a vertex $w \in V_1$ with

$$d_L(w) \geq \frac{\left(\frac{2}{9}+\frac{\delta}{4}\right)n^2}{\left(\frac{1}{3}+\zeta^{1/2}\right)n} \geq \left(\frac{2}{3}+\frac{\delta}{8}\right)n.$$

Therefore, by (5.1), we have

$$\min\left\{|N_L(w) \cap V_1|, |N_L(w) \cap V_2|\right\} \geq \frac{\delta}{16}n.$$

Fix a vertex $u \in N_L(w) \cap V_1$ and let $V_2' = N_L(w) \cap V_2$. Since

$$\binom{|V_2|}{2} - d_{\mathcal{H}_0}(u) \leq \binom{\left(\frac{2}{3}+\zeta^{1/2}\right)n}{2} - \left(\frac{4}{9}-2\zeta\right)\binom{n-1}{2} < \binom{\delta n/16}{2}, \tag{5.2}$$

there exists an edge $ab \in L_{\mathcal{H}}(u) \cap \binom{V_2'}{2}$. However, this implies that $\mathbb{F}_{3,2} \subset \mathcal{H}[\{v,u,w,a,b\}]$ (see Figure 8), a contradiction. ■

Claim We have $L_1 = \emptyset$. ■

Proof Suppose to the contrary that there exists an edge $uw \in L_1$. Note that $|L_2| \geq \delta n^2/8$ from Claim 5.2. Choosing uniformly at random a pair $\{a,b\}$ from $\binom{V_2}{2}$, we obtain

$$\min\left\{\mathbb{P}\left[ab \in L_{\mathcal{H}}(u)\right], \mathbb{P}\left[ab \in L_{\mathcal{H}}(w)\right]\right\} \geq \frac{\delta(\mathcal{H}_0)}{\binom{|V_2|}{2}} > \frac{\left(\frac{4}{9}-2\zeta\right)\binom{n-1}{2}}{\binom{\left(\frac{2}{3}+\zeta^{1/2}\right)n}{2}} > 1 - 10\zeta^{1/2},$$

and

$$\mathbb{P}\left[ab \in L_2\right] = \frac{|L_2|}{\binom{|V_2|}{2}} > \frac{\delta n^2/8}{\binom{\left(\frac{2}{3}+\zeta^{1/2}\right)n}{2}} > \frac{\delta}{8}.$$

So it follows from the Union Bound that

$$\mathbb{P}\left[ab \in L_2 \cap L_{\mathcal{H}}(u) \cap L_{\mathcal{H}}(w)\right] > 1 - \left(10\zeta^{1/2} + 10\zeta^{1/2} + 1 - \frac{\delta}{8}\right) > 0.$$

Hence, there exists an edge $ab \in L_2 \cap L_{\mathcal{H}}(u) \cap L_{\mathcal{H}}(w)$. However, this implies that $\mathbb{F}_{3,2} \subset \mathcal{H}[\{v,u,w,a,b\}]$ (see Figure 8), a contradiction. ■

Let us define

$$U_1 := \left\{v' \in V_2 \colon |N_L(v') \cap V_1| \geq \frac{\delta}{16}n\right\} \quad \text{and} \quad U_2 := \left\{v' \in V_2 \colon |N_L(v') \cap V_2| \geq \frac{\delta}{16}n\right\}.$$

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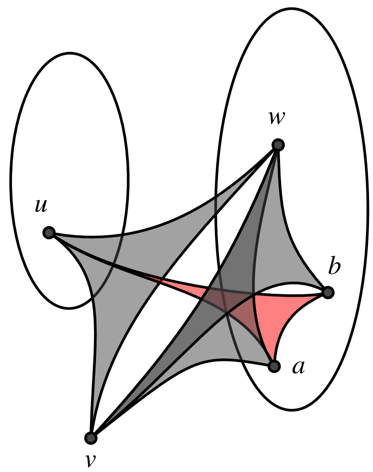

Figure 9: Finding $\mathbb{F}_{3,2}$ when $L_1 = \emptyset$..

It follows from

$$\left(\frac{1}{3}+\zeta^{1/2}\right)n|U_1| \geq \sum_{v' \in U_1} |N_L(v') \cap V_1| \geq |L_{1,2}| - \frac{\delta}{16}n|V_2 \setminus U_1| \geq |L_{1,2}| - \frac{\delta}{16}n^2$$

and

$$\left(\frac{2}{3}+\zeta^{1/2}\right)n|U_2| \geq \sum_{v' \in U_2} |N_L(v') \cap V_2| \geq 2|L_2| - \frac{\delta}{16}n|V_2 \setminus U_2| \geq 2|L_2| - \frac{\delta}{16}n^2$$

that

$$\begin{aligned} |U_1| + |U_2| &\geq \frac{|L_{1,2}| - \frac{\delta}{16}n^2}{\left(\frac{1}{3}+\zeta^{1/2}\right)n} + \frac{2|L_2| - \frac{\delta}{16}n^2}{\left(\frac{2}{3}+\zeta^{1/2}\right)n} \\ &\geq \frac{|L_{1,2}| - \frac{\delta}{16}n^2 + |L_2| - \frac{\delta}{16}n^2}{\left(\frac{1}{3}+\zeta^{1/2}\right)n} \\ &= \frac{|L| - \frac{\delta}{8}n^2}{\left(\frac{1}{3}+\zeta^{1/2}\right)n} \geq \frac{\left(\frac{2}{9}+\frac{\delta}{4}\right)n^2 - \frac{\delta}{8}n^2}{\left(\frac{1}{3}+\zeta^{1/2}\right)n} \geq \left(\frac{2}{3}+\frac{\delta}{8}\right)n. \end{aligned}$$

So it follows from (5.1) that $|U_1 \cap U_2| \geq |U_1| + |U_2| - |V_2| \geq \frac{\delta}{16}n$.

Fix a vertex $w \in U_1 \cap U_2$ and a vertex $u \in N_L(w) \cap V_1$. Let $V_2' = N_L(w) \cap V_2$. Since $|V_2'| \geq \frac{\delta}{16}n$, similar to (5.2), there exists an edge $ab \in L_{\mathcal{H}}(u) \cap \binom{V_2'}{2}$. However, this implies that $\mathbb{F}_{3,2} \subset \mathcal{H}[\{v, u, w, a, b\}]$ (see Figure 9), a contradiction. This completes the proof of Theorem 5.1.

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6 Concluding remarks

By a small modification of the proof, one can easily extend Theorems 1.5 and 1.6 to vertex-disjoint union of different hypergraphs as follows (here we omit the statement for the rainbow version).

***Theorem* 6.1** *Let $m \ge r \ge 2, k \ge 1$ be integers and let $F_1, \dots, F_k$ be nondegenerate r-graphs on at most m vertices. Suppose that there exists a constant $c > 0$ such that for all $i \in [k]$ and large n:*

(a) *F_i is $\left(c\binom{n}{r-1}, \frac{1-\pi(F)}{4m}\binom{n}{r-1}\right)$-bounded, and*

(b) *$\mathrm{ex}(n, F_i)$ is $\frac{1-\pi(F)}{8m}\binom{n}{r-1}$-smooth.*

Then there exist constant N_0 such that for all integers $n \ge N_0$ and $t_1, \dots, t_k \in \mathbb{N}$ with $t+1 := \sum_{i=1}^{k} t_i \in [0, \varepsilon n]$, where $\varepsilon = \min\left\{\frac{c}{4erm}, \frac{1-\pi(F_1)}{64rm^2}, \dots, \frac{1-\pi(F_k)}{64rm^2}\right\}$, we have

$$\mathrm{ex}\left(n, \bigsqcup_{i=1}^{k} t_i F_i\right) \le \binom{n}{r} - \binom{n-t}{r} + \max_{i\in[k]}\left\{\mathrm{ex}(n-t, F_i)\right\}.$$

Moreover, if $\max_{i\in[k]} \mathrm{ex}(n-t, F_i) = \mathrm{ex}(n, \{F_1, \dots, F_k\})$, then the inequality above can be replace by equality.

Recall from Theorem 1.7 that for every r-uniform Turán pair (F, P), the function $\mathrm{ex}(n, F)$ is smooth. This result can be extended in the following ways with a slight modification to the proof (the proof of Theorem 6.2 below is included in the Appendix).

Let $(n_i)_{i=1}^{\infty}$ be an ascending sequence of integers, F be an r-graph, and P be a pattern. We say (F, P) is a $(n_i)_{i=1}^{\infty}$-**Turán pair** if there exists N_0 such that

- every P-construction is F-free, and
- for every $n_i \ge N_0$, there exists an n_i-vertex F-free extremal construction that is a P-construction.

Let $k \ge 1$ be an integer. We say an ascending sequence of integers $(n_i)_{i=1}^{\infty}$ is $\frac{1}{k}$-**dense** if for every integer $m \ge N_0$, we have

$$\{m+1, \dots, m+k\} \cap \{n_i : i \ge 1\} \ne \emptyset.$$

***Theorem* 6.2** *Let $k \ge 1$ be an integer, F be an r-graph, and P be a pattern. Suppose that (F, P) is an $(n_i)_{i=1}^{\infty}$-Turán pair for some $\frac{1}{k}$-dense ascending sequence of integers $(n_i)_{i=1}^{\infty}$. Then $\mathrm{ex}(n, F)$ is $16k^2\binom{n-1}{r-2}$-smooth.*

Given an r-graph F, we say F is 2-**covered** if every pair of vertices in $V(F)$ is contained in some edge of F. In particular, complete r-graphs are 2-covered.

Suppose that F is a 2-covered r-graph. Then it is easy to see that duplicating a vertex in an F-free r-graph does not change its F-freeness. Thus, a proof analogous to that of Fact 4.2 can show that $\delta(n, F) \ge \Delta(\mathcal{H})$ for every maximum F-free r-graph $\mathcal{H}$ on $n-1$

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vertices. Consequently, we have $\delta(n,F) \geq d(n-1,F)$. By combining this result with Fact 4.1 and Lemma 3.5, we can extend Theorem 6.3 as follows:

***Theorem* 6.3** *Suppose that F is a 2-covered r-graph. Then $\mathrm{ex}(n,F)$ is $4\binom{n-1}{r-2}$-smooth.*

Recall that Allen, Böttcher, Hladký, and Piguet [2] determined, for large n, the value of $\mathrm{ex}(n,(t+1)K_3)$ for all $t \leq n/3$. Considering that the situation is already very complicated for K_3, the following question seems very hard in general.

Problem Let $r \geq 2$ be an integer and F be a nondegenerate r-graph with m vertices. For large n determine $\mathrm{ex}(n,(t+1)F)$ for all $t \leq n/m$. ■

A first step towards a full understanding of Problem 6.4 would be determining the regime of t in which members in $K_t^r \boxtimes \mathrm{EX}(n-t,F)$ are extremal. Here we propose the following question, which seems feasible for many hypergraphs (including graphs).

Problem Let $r \geq 2$ be an integer and F be an r-graph with m vertices. For large n determine the maximum value of $s(n,F)$ such that

$$\mathrm{ex}(n,(t+1)F) = \binom{n}{r} - \binom{n-t}{r} + \mathrm{ex}(n-t,F)$$

holds for all $t \in [0, s(n,F)]$. ■

Understanding the asymptotic behavior of $s(n,F)$ would be also very interesting.

Problem Let $r \geq 2$ be an integer and F be an r-graph with m vertices. Let $s(n,F)$ be the same as in Problem 6.5. Determine the value of $\liminf_{n\to\infty} \frac{s(n,F)}{n}$. ■

Note that the result of Allen, Böttcher, Hladký, and Piguet [2] implies that $s(n,K_3) = \frac{2n-6}{9}$ for large n. In particular, $\lim_{n\to\infty} \frac{s(n,K_3)}{n} = \frac{2}{9}$.

It would be also interesting to consider extensions of the density Corrádi–Hajnal Theorem to degenerate hypergraphs such as complete r-partite r-graphs and even cycles[6]. The behavior for degenerate hypergraphs seems very different from nondegenerate hypergraphs, and we refer the reader to e.g. [19, Theorem 1.3] for related results on even cycles.

As pointed out to us by Mubayi, $\mathrm{ex}(n,(t+1)F)$ is also related to the well-known Erdős–Rademacher Problem [16]. More specifically, a lower bound for the number of copies of F in an n-vertex r-graph $\mathcal{H}$ with e edges can provide a lower bound for the number of vertex-disjoint copies of F in $\mathcal{H}$, as revealed by results in Hypergraph Matching Theory. However, this approach is unlikely to give a tight bound for $\mathrm{ex}(n,(t+1)F)$ since these two questions generally have different extremal constructions. We refer the reader to, for example, [70, 71, 64, 61, 62, 72] and references therein for more results related to the Erdős–Rademacher Problem.

[6] This question was explored in recent work [35, 34, 36].

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A Proof of Fact 4.3

Proof Fix a sufficiently small $\varepsilon > 0$ and let n be a sufficiently large integer such that

$$\mathrm{ex}(n - \varepsilon^{1/2}n, F) \leq \pi(F)\binom{\left(1-\varepsilon^{1/2}\right)n}{r} + \varepsilon\binom{n}{r}.$$

Suppose to the contrary that $|Z_\varepsilon(\mathcal{H})| \geq \varepsilon^{1/2}n$. Then fix a set $Z \subset Z_\varepsilon(\mathcal{H})$ of size $\varepsilon^{1/2}n$ and let $U := V(\mathcal{H}) \setminus Z$. It follows from the definition of $Z_\varepsilon(\mathcal{H})$ that

$$\begin{aligned}
|\mathcal{H}[U]| &\geq |\mathcal{H}| - |Z| \cdot \left(\pi(F) - r\varepsilon^{1/2}\right)\binom{n-1}{r-1} \\
&\geq (\pi(F) - \varepsilon)\binom{n}{r} - \frac{r}{n} \cdot |Z| \cdot \left(\pi(F) - r\varepsilon^{1/2}\right)\binom{n}{r} \\
&= (\pi(F) - \varepsilon)\binom{n}{r} - r\varepsilon^{1/2} \cdot \left(\pi(F) - r\varepsilon^{1/2}\right)\binom{n}{r} \\
&= \left((1 - r\varepsilon^{1/2})\pi(F) + (r^2 - 1)\varepsilon\right)\binom{n}{r}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\mathcal{H}[U]| \leq \mathrm{ex}(n - \varepsilon^{1/2}n, F) &\leq \pi(F)\binom{\left(1-\varepsilon^{1/2}\right)n}{r} + \varepsilon\binom{n}{r} \\
&\leq \left(1 - \varepsilon^{1/2}\right)^r \pi(F)\binom{n}{r} + \varepsilon\binom{n}{r} \\
&\leq \left(1 - r\varepsilon^{1/2} + \binom{r}{2}\varepsilon\right)\pi(F)\binom{n}{r} + \varepsilon\binom{n}{r} \\
&\leq \left(\left(1 - r\varepsilon^{1/2}\right)\pi(F) + \left(\binom{r}{2} + 1\right)\varepsilon\right)\binom{n}{r}.
\end{aligned}$$

Here, we used the inequality that $(1-x)^r \leq 1 - rx + \binom{r}{2}x^2$ for $x \in [0,1]$ and $r \geq 2$.

Since $r^2 - 1 > \binom{r}{2} + 1$ for $r \geq 2$, we arrived at a contradiction. Therefore, we have $|Z_\varepsilon(\mathcal{H})| \leq \varepsilon^{1/2}n$. It follows that the induced subgraph $\mathcal{H}'$ of $\mathcal{H}$ on $V(\mathcal{H}) - Z_\varepsilon(\mathcal{H})$ satisfies

$$\begin{aligned}
\delta(\mathcal{H}') &\geq \left(\pi(F) - r\varepsilon^{1/2}\right)\binom{n-1}{r-1} - |Z_\varepsilon(\mathcal{H})|\binom{n-2}{r-2} \\
&\geq \left(\pi(F) - r\varepsilon^{1/2}\right)\binom{n-1}{r-1} - \varepsilon^{1/2}n \cdot \frac{r-1}{n-1}\binom{n-1}{r-1} \\
&\geq \left(\pi(F) - 2r\varepsilon^{1/2}\right)\binom{n-1}{r-1},
\end{aligned}$$

completing the proof of Fact 4.3. ∎

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B Proof of Theorem 6.2

The following fact can be derived from a slight modification of the proof of Fact 4.2: instead of duplicating the vertex v just once, we duplicate it $n - n_{i_*}$ times.

Fact Let $k \geq 1$ be an integer and $(n_i)_{i=1}^{\infty}$ be a $\frac{1}{k}$-dense ascending sequence of integers. Suppose that (F, P) is an r-uniform $(n_i)_{i=1}^{\infty}$-Turán pair and $\mathcal{H}$ is a maximum P-construction on n_{i_*} vertices, where i_* is sufficiently large. Then for every $n \geq n_{i_*}$, we have

$$\mathrm{ex}(n, F) - \mathrm{ex}(n_{i_*}, F) \geq (n - n_{i_*}) \cdot \Delta(\mathcal{H}) \geq (n - n_i) \cdot d(n_i, F).$$

■

Proof Let n be a sufficiently large integer and let i_* be such that $n_{i_*} \leq n \leq n_{i_*} + k$. The existence of such an i_* is guaranteed by the assumption that $(n_i)_{i=1}^{\infty}$ is a $\frac{1}{k}$-dense ascending sequence.

Let $\Phi := \left|\mathrm{ex}(n, F) - \mathrm{ex}(n_{i_*}, F) - (n - n_{i_*}) \cdot d(n_{i_*}, F)\right|$. Notice that

$$\begin{aligned}
\Phi &\overset{\text{Fact } B.1}{=} \mathrm{ex}(n, F) - \mathrm{ex}(n_{i_*}, F) - (n - n_{i_*}) \cdot d(n_{i_*}, F) \\
&= \sum_{j=1}^{n-n_{i_*}} \left(\mathrm{ex}(n_{i_*} + j, F) - \mathrm{ex}(n_{i_*} + j - 1, F) - d(n_{i_*}, F)\right) \\
&\overset{\text{Fact } 4.1}{\leq} \sum_{j=1}^{n-n_{i_*}} \left(d(n_{i_*} + j, F) - d(n_{i_*}, F)\right) \\
&\overset{\text{Lemma } 3.5}{\leq} \sum_{j=1}^{n-n_{i_*}} 4j \binom{n_{i_*}}{r-2} \leq 4\left(n - n_{i_*}\right)^2 \binom{n_{i_*}}{r-2}.
\end{aligned}$$

Additionally,

$$\begin{aligned}
& d(n, F) - \delta(n, F) \\
&\overset{\text{Fact } 4.2}{\leq} (d(n, F) - \delta(n, F)) + \sum_{j=1}^{n-n_{i_*}-1} (d(n-j, F) - \delta(n-j, F)) \\
&= \sum_{j=0}^{n-n_{i_*}-1} d(n-j, F) - \sum_{j=0}^{n-n_{i_*}-1} \delta(n-j, F) \\
&= \sum_{j=0}^{n-n_{i_*}-1} \left(d(n-j, F) - d(n_{i_*}, F)\right) + (n - n_{i_*}) \cdot d(n_{i_*}, F) - \left(\mathrm{ex}(n, F) - \mathrm{ex}(n_{i_*}, F)\right) \\
&\leq \sum_{j=0}^{n-n_{i_*}-1} \left|d(n-j, F) - d(n_{i_*}, F)\right| + \Phi \\
&\overset{\text{Lemma } 3.5}{\leq} 4(n - n_{i_*})^2 \binom{n_{i_*}}{r-2} + 4(n - n_{i_*})^2 \binom{n_{i_*}}{r-2} = 8(n - n_{i_*})^2 \binom{n_{i_*}}{r-2}.
\end{aligned}$$

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Therefore, we obtain that

$$\begin{aligned}|\delta(n,F)-d(n-1,F)| &= |\delta(n,F)-d(n,F)+d(n,F)-d(n-1,F)|\\ &\overset{\text{Lemma 3.5}}{\leq} |\delta(n,F)-d(n,F)|+4\binom{n-1}{r-2}\\ &\leq 8(n-n_{i_*})^2\binom{n_{i_*}}{r-2}+4\binom{n-1}{r-2} \leq 12k^2\binom{n-1}{r-2},\end{aligned}$$

proving Theorem 6.2. ∎

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