Ergod. Th. & Dynam. Sys. (1983), **3**, 613–626
Printed in Great Britain

Embedding some transformation group C^*-algebras into AF-algebras

MIHAI V. PIMSNER
Department of Mathematics, National Institute for Scientific and Technical Creation, Bd. Pacii 220, 79622 *Bucharest, Romania*

(*Received* 27 *November* 1982)

Abstract. For a homeomorphism of a compact metrizable space X, we show that the property that every point of X is pseudo-non-wandering (see definition 2) is equivalent to the possibility of embedding the corresponding transformation group C^*-algebra into an AF-algebra.

Let T be a homeomorphism of the compact metrizable topological space X. The aim of this paper is to give a necessary and sufficient condition on T, in order to embed the corresponding transformation group (crossed-product) C^*-algebra into an AF-algebra. Of course the difficult part is to show that the condition is sufficient. This is done with the same techniques that were used in [**4**], in the particular case of the irrational rotation of the unit circle. The only new difficulty that arises, is how to choose the appropriate AF-algebra. This is achieved by a 'rough coding procedure', based on the study of the periodic pseudo-orbits of the transformation T [**1**]. It turns out that the condition which ensures the embedding is the existence of sufficiently many periodic pseudo-orbits, or, to be more precise, the fact that every point of X is pseudo-non-wandering for T (see definition 2.). The proof of the necessity of this condition relies on the adaptation to C^*-algebras of the notion of quasidiagonality introduced by P. R. Halmos [**2**]. The fact that a C^*-algebra containing a non-unitary isometry is not quasidiagonal was also used by D. Hadwin to characterize those transformation group C^*-algebras having only quasidiagonal quotients. (Of course a transformation group C^*-algebra may be quasidiagonal without having all quotients quasidiagonal.)

Finally let us mention that another generalization of [**4**] has been announced by A. M. Vershik in [**6**]. Using a different approach he states in a slightly more particular (and slightly different) case, a more precise result concerning the embedding.

The author gratefully acknowledges helpful advice from D. Voiculescu.

Throughout this paper X will denote a compact metrizable topological space, $C(X)$ the (separable) C^*-algebra of continuous complex valued functions defined on X, and $T: X \to X$ a homeomorphism of X.

As usual if $\mathscr{V}=(V_i)_{i\in I}$ and $\mathscr{W}=(W_j)_{j\in J}$ are open covers of X, $\mathscr{W}$ will be called finer than $\mathscr{V}$ if there exists a map $f:J\to I$ such that $W_j\subset V_{f(j)}$. This will be denoted

$$\mathscr{V}<_f\mathscr{W}.$$

For any natural m and any open cover $\mathscr{V}=(V_i)_{i\in I}$ of X we shall denote by $\mathscr{V}^{(m)}$ the open cover

$$\mathscr{V}\vee T^{-1}\mathscr{V}\vee\cdots\vee T^{-m+1}\mathscr{V},$$

that is $\mathscr{V}^{(m)}=(V_{i_0,\ldots,i_{m-1}})_{(i_0,\ldots,i_{m-1})}\in I^{(m)}$ where

$$V_{i_0,\ldots,i_{m-1}}=V_{i_0}\cap T^{-1}V_{i_1}\cap\cdots\cap T^{-m+1}V_{i_{m-1}}$$

and $I^{(m)}$ is the subset of the direct product I^m consisting of those elements $(i_0,\ldots,i_{m-1})$ with the property that $V_{i_0,\ldots,i_{m-1}}$ is not empty. Also let $\pi_k:I^{(m)}\to I$, $0\le k\le m-1$, be the maps defined by

$$\pi_k(i_0,\ldots,i_{m-1})=i_k.$$

We shall always consider $\mathscr{V}^{(m)}$ finer than $\mathscr{V}$ by means of the projection π_0 so that we shall simply write $\mathscr{V}<\mathscr{V}^{(m)}$. If $\mathscr{W}=(W_j)_{j\in J}$ is another open cover such that $\mathscr{V}<_f\mathscr{W}$ we shall also denote by $f:J^{(m)}\to I^{(m)}$ the map induced by f so that $\mathscr{V}^{(m)}<_f\mathscr{W}^{(m)}$.

Definition 1. Let $\mathscr{V}=(V_i)_{i\in I}$ be an open cover of X. A sequence $\omega=(\omega(n))_{n\in\mathbb{Z}}$, $\omega(n)\in I$ is called a *$\mathscr{V}$-pseudo-orbit of T* if

$$V_{\omega(n)}\cap T^{-1}(V_{\omega(n+1)})\neq\varnothing\qquad\text{for every } n\in\mathbb{Z}.$$

If the $\mathscr{V}$-pseudo orbit ω is periodic we shall denote by $p(\omega)$ its principal period, that is, the smallest natural number p such that

$$\omega(n+p)=\omega(n)\qquad\text{for every } n\in\mathbb{Z}.$$

If $\mathscr{W}$ is another open cover such that $\mathscr{W}<_f\mathscr{V}$ we shall denote by $f\omega$ the $\mathscr{W}$-pseudo-orbit

$$(f\circ\omega(n))_{n\in\mathbb{Z}}.$$

Suppose now that ω is a $\mathscr{V}^{(m)}$ pseudo-orbit of T^m. We shall denote by ω^m the $\mathscr{V}$ pseudo-orbit of T obtained in the following way: write $n=mq+r$ with $0\le r<m$ and define

$$\omega^m(n)=\pi_r(\omega(q))\in I.$$

It is straightforward from the definitions that

$$f(\omega^m)=[f(\omega)]^m.$$

If k is a natural number that divides m, $m=k\cdot l$, we may also regard ω as a $(\mathscr{V}^{(l)})^{(k)}$ pseudo-orbit of $(T^l)^k$ so that ω^k makes sense and is a $\mathscr{V}^{(l)}$ pseudo-orbit of T^l. Moreover

$$(\omega^{k_1})^{k_2}=\omega^{k_1\cdot k_2}$$

whenever the product $k_1\cdot k_2$ divides m.

Definition 2. A point $x\in X$ is said to be *pseudo-non-wandering* for T if for every open cover $\mathscr{V}=(V_i)_{i\in I}$ and any $i\in I$ such that $x\in V_i$ there exists a periodic $\mathscr{V}$-pseudo-orbit $\omega=(\omega(n))_{n\in\mathbb{Z}}$ such that $\omega(0)=i$.

The set of pseudo-non-wandering points for T will be denoted $X(T)$. (This set coincides with the *chain recurrent set* introduced by C. C. Conley.)

It is an easy consequence of the definition that $X(T)$ is a closed T- and T^{-1}-invariant subset of X. Every non-wandering point is clearly pseudo-non-wandering, the converse being false in general. A simple example is the action of the shift on the one point compactification of $\mathbb{Z}$, i.e. $Tx = x+1$ for $x \in \mathbb{Z}$ and $T(\infty) = \infty$, where the only non-wandering point is ∞, whereas every point is pseudo-non-wandering. (For the definition of non-wandering points see, e.g., [**1**].)

LEMMA 1. $X(T) = X(T^m)$ *for every* $m \in \mathbb{N}$.

Proof. Suppose that $\mathcal{V} = (V_i)_{i \in I}$ is an open cover and that $x \in V_{i_0}$. If $x \in X(T^m)$, then there exists a periodic $\mathcal{V}^{(m)}$ pseudo-orbit for T^m, $\omega(n) \in I^{(m)}$ such that $\pi_0(\omega(0)) = i_0$. The $\mathcal{V}$ pseudo-orbit ω^m of T is then periodic and

$$\omega^m(0) = \pi_0(\omega(0)) = i_0$$

so that $x \in X(T)$. Conversely, let $\mathcal{W} = (W_k)_{k \in K}$, $\mathcal{W} <_f \mathcal{V}$ be an open cover with the property that

$$T^{-i} W_{\omega(n+i)} \subset V_{f(\omega(n))}, \qquad 0 \le i \le m,$$

for every $\mathcal{W}$-pseudo orbit ω of T. To choose $\mathcal{W}$ one may fix a metric on X and choose the W_k's to be balls of sufficiently small radii. We leave the details to the reader. If $x \in X(T)$ and ω is a periodic $\mathcal{W}$ pseudo-orbit for T such that $f(\omega(0)) = i_0$, then

$$\tilde{\omega} = (f \circ \omega(m \cdot n))_{n \in \mathbb{Z}}$$

is a periodic $\mathcal{V}$-pseudo-orbit for T^m so that $x \in X(T^m)$. □

We shall be interested mainly in the case when $X(T) = X$. A typical example when this does not hold is the action of the shift on the two point compactification of $\mathbb{Z}$, i.e. $Tx = x+1$ for $x \in \mathbb{Z}$, $T(+\infty) = +\infty$, $T(-\infty) = -\infty$. The following lemma shows that this example is in some sense generic.

LEMMA 2. *The point x belongs to $X \backslash X(T)$ if and only if there exists an open set U such that $T(\bar{U}) \subset U$ and $x \in U \backslash T(\bar{U})$. (As usual $\bar{U}$ stands for the closure of U.)*

Proof. If the open set U has the above properties, consider the open cover $\mathcal{V} = (V_1, V_2, V_3)$ where $V_1 = U$, $V_2 = U \backslash T(\bar{U})$, $V_3 = X \backslash T(\bar{U})$. Since

$$V_2 \cap T^{-1} V_2 = \varnothing = V_2 \cap T^{-1} V_3$$

and

$$V_1 \cap T^{-1} V_2 = \varnothing = V_1 \cap T^{-1} V_3,$$

any $\mathcal{V}$ pseudo-orbit ω with $\omega(n_0) = 2$ satisfies $\omega(n) = 1$ for every $n > n_0$. So there is no periodic $\mathcal{V}$ pseudo-orbit with $\omega(0) = 2$ which in turn implies that

$$V_2 \subset X \backslash X(T).$$

To prove the converse suppose that there exists an open cover $\mathcal{V} = (V_i)_{i \in I}$ and an index $i_0 \in I$ such that $x \in V_{i_0}$ and that no periodic $\mathcal{V}$ pseudo-orbit ω satisfies $\omega(0) = i_0$. Consider the set J of all indices $j \in I$ with the property that there exists a $\mathcal{V}$

pseudo-orbit $(j(n))_{n\in\mathbb{Z}}$ such that $j(0)=i_0$ and $j(n)=j$ for some $n\geq 0$ and define

$$U=\bigcup_{j\in J} V_j.$$

Any point $y\in T(\bar{U})$ has the property that whenever i is such that $V_i\ni y$ there exists $j\in J$ satisfying

$$V_j\cap T^{-1}V_i\neq\varnothing.$$

This implies on the one hand that any such i belongs to J so that

$$T(\bar{U})\subset U$$

and on the other hand

$$V_{i_0}\cap T(\bar{U})=\varnothing.$$

For otherwise there would be a $\mathcal{V}$ pseudo-orbit ω and an $n\geq 1$ such that $\omega(0)=i_0=\omega(n)$. This would easily imply the existence of a periodic $\mathcal{V}$ pseudo-orbit ω' such that $\omega'(0)=i_0$, in contradiction to the choice of i_0. □

Definition 3. A $\mathcal{V}$ pseudo-orbit ω is said to *split* into the $\mathcal{V}$ pseudo-orbits $\{\eta_k\}_{k\in K}$ if there are increasing maps $\varphi_k:\mathbb{Z}\to\mathbb{Z}$ such that

$$\varphi_k(\mathbb{Z})\cap\varphi_{k'}(\mathbb{Z})=\varnothing \qquad \text{for } k\neq k',$$

$$\bigcup_k \varphi_k(\mathbb{Z})=\mathbb{Z},$$

$$\eta_k(n)=\omega(\varphi_k(n)),$$

$$\omega(\varphi_k(n)-1)=\omega(\varphi_k(n-1)) \qquad \text{for every } n\in\mathbb{Z} \text{ and every } k\in K.$$

Note that no condition is imposed on $\omega(\varphi_k(n)+1)$ and that the possibility of all η_k's being equal is not excluded. We shall also say that $\{\eta_k\}_{k\in K}$ is *a decomposition of* ω. It is clear that if ω splits into the $\mathcal{V}$ pseudo-orbits $\{\eta_k\}_{k\in K}$ and each η_k splits into $\{\mu_l\}_{l\in K(k)}$ then ω also splits into $\{\mu_l\}_{l\in\bigcup_k K(k)}$.

LEMMA 3. *Let $\mathcal{V}=(V_i)_{i\in I}$ be a finite open cover consisting of α open sets and $\omega=(\omega(n))_{n\in\mathbb{Z}}$ be a $\mathcal{V}$ pseudo-orbit of period p. (Thus p is a multiple of $p(\omega)$). Then there exists a decomposition $\{\eta_k\}_{k\in K}$ of ω into periodic $\mathcal{V}$ pseudo-orbits such that $p(\eta_k)\leq\alpha$ for every $k\in K$ and*

$$\bigcup_k \varphi_k([0,p(\eta_k)))=[0,p). \tag{1}$$

(Intervals will always be integer valued.)

Proof. It is enough to show that if $p>\alpha$, then ω splits into η and $\tilde{\eta}$ where $p(\eta)\leq\alpha$, $\tilde{\eta}$ has period $\tilde{p}=p-p(\eta)$ and

$$\varphi_\eta([0,p(\eta)))\subset[0,p),$$
$$\varphi_{\tilde{\eta}}([0,p))\subset[0,p);$$

for then the $\tilde{\eta}$'s can be further decomposed until $\tilde{p}\leq\alpha$. Finally if $\tilde{p}=m\cdot p(\tilde{\eta})$, then $\tilde{\eta}$ splits into m identical copies of $\tilde{\eta}$.

To prove the above assertion choose a and b in $[0,p)$ such that $\omega(a)=\omega(b)$ and $0<b-a\leq\alpha$. This is possible since $p>\alpha$. Moreover one can assume that all indices

$\omega(k)$ with $a<k\le b$ are distinct. Define $\varphi_\eta, \varphi_{\tilde\eta}: \mathbb{Z}\to\mathbb{Z}$ in the following way: If $n=(b-a)q+r$, $0\le r<b-a$ put

$$\varphi_\eta(n)=p\cdot q+a+r+1,$$

and if $n=[p-(b-a)]\tilde q+\tilde r$, $0\le r<p-b+a$ put

$$\varphi_{\tilde\eta}=\begin{cases} p\tilde q+\tilde r & \text{if } \tilde r\le a,\\ p\tilde q+\tilde r+b-a & \text{if } \tilde r>a.\end{cases}$$

It is easy to see that if $\eta(n)=\omega(\varphi_\eta(n))$ and $\tilde\eta(n)=\omega(\varphi_{\tilde\eta}(n))$, then η and $\tilde\eta$ are $\mathcal{V}$ pseudo-orbits satisfying the desired properties. □

Remark. The splitting described in the preceding lemma depends essentially on p. For example the same construction carried out with p replaced by $2p$ yields a different decomposition.

From now on we shall suppose that $X(T)=X$. The AF-algebra ssociated to (X, T) will depend on some rough symbolic which we start to describe. For any finite open cover $\mathcal{V}$ and any positive integer m we shall denote by $\Omega(\mathcal{V}, m)$ the set of all periodic $\mathcal{V}^{(m)}$ pseudo-orbits of T^m whose principal period does not exceed the cardinality of the cover $\mathcal{V}^{(m)}$. Consider a sequence of open covers such that $\mathcal{V}_n <_{f_n} \mathcal{V}_{n+1}$ and a sequence $(m_n)_{n\in\mathbb{Z}}$ of positive integers such that m_n divides m_{n+1} for each n.

For each $\omega\in\Omega(\mathcal{V}_{n+1}, m_{n+1})$ we shall consider $f_n\omega^{m_{n+1}/m_n}$ as a $\mathcal{V}_n^{(m_n)}$ pseudo-orbit of T^{m_n} of period $p(\omega)\cdot(m_{n+1}/m_n)$ and we shall fix a decomposition of $f_n\omega^{m_{n+1}/m_n}$ into pseudo-orbits belonging to $\Omega(\mathcal{V}_n, m_n)$ with the additional properties stated in lemma 3. In order to keep this fixed decomposition in mind, we shall denote by $F_n(\omega)$ the index set which was denoted K in the definition of the splitting. By forcing the notation a little we shall regard the elements of $F_n(\omega)$ as $\mathcal{V}^{m_n}$ pseudo-orbits of T^{m_n}. So equal pseudo-orbits may be distinct as elements of $F_n(\omega)$. Identifying these elements we get a map denoted $\tilde F_n$ from the subsets of $\Omega(\mathcal{V}_{n+1}, m_{n+1})$ to the subsets of $\Omega(\mathcal{V}_n, m_n)$. If $\tilde F_{n,p}$ denotes the composition $\tilde F_n\circ\tilde F_{n+1}\circ\cdots\circ\tilde F_{n+p-1}$ define

$$\Omega_n=\bigcup_{p\in\mathbb{N}}\tilde F_{n,p}(\Omega(\mathcal{V}_{n+p}, m_{n+p})),$$

which is clearly non-void since $\Omega(\mathcal{V}_n, m_n)$ is a finite set. In addition $\tilde F_n(\Omega_{n+1})=\Omega_n$. More precisely the following lemma holds.

LEMMA 4. *For every $x\in X$ and every $j=(i_0,\ldots,i_{m-1})\in I^{(m_n)}$ such that $x\in V_j\in\mathcal{V}_n^{(m_n)}$ there exists a $\mathcal{V}_n^{(m_n)}$ pseudo-orbit ω of T^{m_n} belonging to Ω_n such that $\omega(0)=j$.*

Proof. Choose a sequence $j_p=(i_0,\ldots,i_{m_{n+p}-1})\in I_{n+p}^{(m_{n+p})}$, $p\ge 0$ such that $j_0=j$ and

$$j_{p-1}=(f_{n+p-1}(i_0),\ldots,f_{n+p-1}(i_{m_{n+p-1}-1})).$$

This is easily achieved by looking at the orbit of the point x. Since

$$X=X(T)=X(T^{m_n})$$

by lemma 1, the subset

$$\Omega_{n+p}(x)\subset\Omega(\mathcal{V}_{n+p}, m_{n+p})$$

consisting of those pseudo-orbits satisfying $\eta(0)=j_p$ is non-empty for each n.

Moreover the additional requirement (1) of lemma 3 ensures that

$$\tilde{F}_{n,p}(\Omega_{n+p}(x)) \cap \Omega_{n+p-p'}(x) \neq \varnothing$$

for every p and p', $p' \leq p$ so that finally

$$\Omega_n \cap \Omega_n(x) \neq \varnothing. \qquad \square$$

The AF-algebra associated to $(\mathscr{V}_n)_n(m_n)_n$ and to the decomposition maps $(F_n)_n$ is defined as follows. For each $n \in \mathbb{N}$ let

$$A_n = \bigoplus_{\omega \in \Omega_n} M_\omega$$

where M_ω is the finite-dimensional factor isomorphic to the $p(\omega) \cdot m_n \times p(\omega) \cdot m_n$ matrix algebra over $\mathbb{C}$. The morphism $\phi_n : A_n \to A_{n+1}$ will be constructed by exhibiting for each $\omega \in \Omega_n$ a unital $*$-monomorphism ϕ_ω from $\bigoplus_{\eta \in F_n(\omega)} M_\eta$ to M_ω. Note that since $\tilde{F}_n(\Omega_{n+1}) = \Omega_n$, $\bigoplus_{\omega \in \Omega_{n+1}} \phi_\omega$ will determine a unital embedding of A_n into A_{n+1}.

The explicit construction of the ϕ_ω's will be given below. First, we shall identify each M_ω, $\omega \in \Omega_n \subset \Omega(\mathscr{V}_n, m_n)$ with

$$B(l^2[0, p(\omega)) \otimes l^2[0, m_n)),$$

the algebra of bounded operators acting on the complex Hilbert space

$$l^2[0, p(\omega)) \otimes l^2(0, m_n).$$

(Intervals are integer valued.) Recall that if $\omega \in \Omega_{n+1}$, then $F_n(\omega)$ is a decomposition of $f_n\omega^{m_{n+1}/m_n}$ as described in lemma 3. So that there are $\varphi_\eta : \mathbb{Z} \to \mathbb{Z}$, $\eta \in F_n(\omega)$ such that

$$\varphi_\eta([0, p(\eta))) \subset \left[0, p(\omega)\frac{m_{n+1}}{m_n}\right),$$

$$\varphi_\eta(\mathbb{Z}) \cap \varphi_\mu(\mathbb{Z}) = \varnothing \qquad \textit{for } \eta, \mu \in F_n(\omega), (\eta \neq \mu, \text{ as elements in } F_n(\omega)),$$

$$\bigcup_\eta \varphi_\eta(\mathbb{Z}) = \mathbb{Z},$$

$$\eta(n) = \omega'(\varphi_\eta(n)),$$

$$\omega'(\varphi_\eta(n-1)) = \omega'(\varphi_\eta(n) - 1) \qquad \text{for every } n \in \mathbb{Z} \text{ and every } \eta \in F_n(\omega),$$

where

$$\omega' = f_n\omega^{m_{n+1}/m_n}.$$

For every $s, t \in \mathbb{N}$ we shall denote by $U_{s,t} \in B(l^2[0, s) \otimes l^2[0, t))$ the unitary defined by

$$U_{s,t} e_i \otimes e_j = \begin{cases} e_i \otimes e_{j+1} & \text{if } j+1 \neq t, \\ e_{i+1} \otimes e_0 & \text{if } j+1 = t,\ i+1 \neq s, \\ e_0 \otimes e_0 & \text{if } j+1 = t,\ i+1 = s. \end{cases}$$

If $t = 1$ we shall write U_s instead of $U_{s,1}$.

If $u, v \in \mathbb{N}$ is another pair of natural numbers such that $s \cdot t = u \cdot v$ we shall denote by

$$W_{st}^{uv} : l^2[0, u) \otimes l^2[0, v) \to l^2[0, s) \otimes l^2[0, t)$$

the unitary

$$W^{uv}_{st} e_i \otimes e_j = e_q \otimes e_r,$$

where

$$vi + j = tq + r \qquad 0 \le r < t.$$

Note that

$$W^{uv}_{st} U_{uv} = U_{st} W^{uv}_{st}.$$

Consider next the isometries

$$W_{0,\eta},\ W_{1,\eta} : l^2[0, p(\eta)) \to l^2\left[0, p(\omega)\frac{m_{n+1}}{m_n}\right)$$

$$W_{0,\eta} e_i = e_{\varphi_\eta(i)} \qquad i \in [0, p(\eta))$$

$$W_{1,\eta} = U^*_{p(\omega)m_{n+1}/m_n} W_{0,\eta} U_{p(\eta)}.$$

Note that if $\eta \ne \mu$ (as elements in $F_n(\omega)$),

$$W^*_{0,\eta} W_{0,\mu} = 0 = W^*_{1,\eta} W_{1,\eta},$$

so that

$$W_0 = \bigoplus_\eta W_{0,\eta}$$

and

$$W_1 = \bigoplus_\eta W_{1,\eta}$$

are isometries from $\bigoplus_{\eta \in F_n(\omega)} l^2[0, p(\eta))$ to $l^2[0, p(\omega)m_{n+1}/m_n)$, and taking into account the dimensions of the two spaces they are in fact unitaries. In particular $W_1 W_0^*$ is a unitary element in

$$B\left(l^2\left[0, p(\omega)\frac{m_{n+1}}{m_n}\right)\right)$$

and a simple spectral argument shows the existence of a unitary U in the C^*-algebra generated by $W_1 W_0^*$ such that

$$U^{m_n} = W_1 W_0^*$$

and

$$\|1 - U\| \le 2\pi / m_n.$$

Let

$$W^{(1)}_\omega : \left(\bigoplus_{\eta \in F_n(\omega)} l^2[0, p(\eta))\right) \otimes l^2[0, m_n) \to l^2\left[0, p(\omega)\frac{m_{n+1}}{m_n}\right) \otimes l^2[0, m_n)$$

be the unitary defined by

$$W^{(1)}_\omega (\xi \otimes e_i) = (U^i W_0 \xi) \otimes e_i.$$

https://doi.org/10.1017/S0143385700002182 Published online by Cambridge University Press

The map ϕ_ω is defined to be conjugation with the unitary

$$W_\omega = W_\omega^{(2)} W_\omega^{(1)} \qquad \text{where } W_\omega^{(2)} = W_{p(\omega),m_{n+1}}^{p(\omega)(m_{n+1}/m_n),m_n}.$$

Keeping the identification $M_\omega \simeq B(l^2[0,p(\omega))\otimes l^2[0,m_n))$ for $\omega\in\Omega_n$ in mind we shall denote by $U_\omega\in M_\omega$ the unitary $U_{p(\omega),m_n}$.

LEMMA 5. *Let* $\omega\in\Omega_{n+1}$, *then*

(1) $\left\|U_\omega W_\omega - W_\omega\left(\bigoplus_{\eta\in F_n(\omega)} U_\eta\right)\right\| \le 2\pi/m_n$;

(2) $\left\|\phi_\omega\left(\bigoplus_{\eta\in F_n(\omega)} U_\eta\right) - U_\omega\right\| \le 2\pi/m_n$.

Proof. Obviously, we have to prove only the first assertion. Let

$$\xi\otimes e_j \in \left(\bigoplus_{\eta\in F_n(\omega)} l^2[0,p(\eta))\right)\otimes l^2[0,m_n).$$

Using the definition of W_ω and the intertwining properties of the unitaries W_{st}^{uv}, it follows that

$$\begin{aligned} U_\omega W_\omega(\xi\otimes e_j) &= W_\omega^{(2)} V) U_{p(\omega)(m_{n+1}/m_n),m_n}(U^j W_0\xi\otimes e_j)\\ &= \begin{cases} W_\omega^{(2)}(U^j W_0\xi\otimes e_{j+1}) & \text{if } j+1\neq m_n\\ W_\omega^{(2)}(U_{p(\omega)(m_{n+1}/m_n)}U^{m_n-1}W_0\xi\otimes e_0) & \text{if } j+1=m_n. \end{cases}\end{aligned}$$

On the other hand

$$\begin{aligned} W_\omega\left(\bigoplus_\eta U_\eta\right)(\xi\otimes e_j) &= \begin{cases} W_\omega^{(2)} W_\omega^{(1)}(\xi\otimes e_{j+1}) & \text{if } j+1\neq m_n\\ W_\omega^{(2)} W_\omega^{(1)}((\oplus U_{p(\eta)})\xi)\otimes e_0 & \text{if } j+1=m_n \end{cases}\\ &= \begin{cases} W_\omega^{(2)}(U^{j+1}W_0\xi)\otimes e_{j+1} & \text{if } j+1\neq m_n\\ W_\omega^{(2)}\left(W_0\left(\bigoplus_\eta U_{p(\eta)}\right)\xi\right)\otimes e_0 & \text{if } j+1=m_n. \end{cases}\end{aligned}$$

Since

$$W_0\left(\bigoplus_\eta U_{p(\eta)}\right) = U_{p(\omega)(m_{n+1}/m_n)} W_1$$

it follows finally that

$$\begin{aligned} &W_\omega\left(\bigoplus_\eta U_\eta\right)(\xi\otimes e_j)\\ &\quad= \begin{cases} W_\omega^{(2)}(U^{j+1}W_0\xi)\otimes e_{j+1} & \text{if } j+1\neq m_n\\ W_\omega^{(2)}(U_{p(\omega)(m_{n+1}/m_n)}U^{m_n}W_0\xi)\otimes e_0 & \text{if } j+1=m_n. \end{cases}\end{aligned}$$

Since $\|U^{j+1}-U^j\|\le 2\pi/m_n$ and

$$\left(U_\omega W_\omega - W_\omega\left(\bigoplus_\eta U_\eta\right)\right)(\xi\otimes e_i)$$

is orthogonal to

$$\left(U_\omega W_\omega - W_\omega\left(\bigoplus_\eta U_\eta\right)\right)(\xi\otimes e_j)$$

for $i \neq j$, it follows that

$$\left\| U_\omega W_\omega - W_\omega \left(\bigoplus_\eta U_\eta \right) \right\| \leq 2\pi / m_n. \qquad \square$$

Choose now for each open set $V_i \in \mathscr{V}_n^{(m_n)}$, $i \in I^{(m_n)}$ a point $x_i \in V_i$. To each $\mathscr{V}^{(m_n)}$ pseudo-orbit of T^{m_n}, ω, there corresponds the sequence $(x_\omega(k))_{k \in \mathbf{Z}}$ where $x_\omega(k) = x_{\omega(k)}$.

Note that $x_\omega(k)$ does not depend on ω but only on the index $\omega(k) \in I^{(m_n)}$. Let $\pi_\omega : C(X) \to M_\omega$ be the representation

$$\pi_\omega(f) e_i \otimes e_j = f(T^j x_\omega(i)) \cdot e_i \otimes e_j \qquad \text{for every } i \in [0, p(\omega)), j \in [0, m_n).$$

LEMMA 6. *Let $\omega \in \Omega_{n+1}$ and $f \in C(X)$ then*

(1) $$\left\| \pi_\omega(f) W_\omega - W_\omega \left(\bigoplus_{\eta \in F_n(\omega)} \pi_\eta(f) \right) \right\| \leq \max_{V \in \mathscr{V}_n} \sup_{x, y \in V} |f(x) - f(y)|;$$

(2) $$\left\| \pi_\omega(f) - \phi_\omega \left(\bigoplus_\eta \pi_\eta(f) \right) \right\| \leq \max_{V \in \mathscr{V}_n} \sup_{x, y \in V} |f(x) - f(y)|;$$

(3) $$\|\pi_\omega(f \circ T) - U_\omega^* \pi_\omega(f) U_\omega\| \leq \max_{V \in \mathscr{V}_{n+1}} \sup_{x, y \in V} |f(x) - f(y)| + \max_{V \in \mathscr{V}_{n+1}} \sup_{x, y \in V} |f(Tx) - f(Ty)|.$$

Proof. Note first that if $\pi_{\eta,j}(f) \in l^2[0, p(\eta))$ denotes the restriction of $\pi_\eta(f)$ to the subspace $l^2[0, p(\eta)) \otimes e_j$, i.e.

$$\pi_{\eta,j}(f) e_i = f(T^j x_\eta(i)) e_i,$$

then

$$W_0 \left(\bigoplus_\eta \pi_{\eta,j}(f) \right) W_0^* = W_1 \left(\bigoplus_\eta \pi_{\eta,j}(f) \right) W_1^*.$$

This follows from the fact that

$$f(T^j x_\eta(l)) = f(T^j x_\mu(k))$$

whenever $\varphi_\eta(l) = \varphi_\mu(k+1) - 1$, for then denoting by ω' the pseudo-orbit $f_n \omega^{m_{n+1}/m_n}$ the properties of the splitting imply that

$$\eta(l) = \omega'(\varphi_\eta(l)) = \omega'(\varphi_\mu(k+1) - 1) = \omega'(\varphi_\mu(k)) = \mu(k)$$

so that

$$x_\eta(l) = x_\mu(k).$$

This implies that $W_0(\oplus \pi_{\eta,j}) W_0^*$ commutes with $W_1 W_0^*$ and so with U. Note further that

$$\pi_\omega(f) W_\omega^{(2)} = W_\omega^{(2)} \tilde{\pi}_\omega(f)$$

where

$$\tilde{\pi}_\omega(f) e_i \otimes e_j = f(T^r x_\omega(q)) e_i \otimes e_j$$

where $i m_n + j = q m_{n+1} + r$, $(0 \leq r < m_{n+1})$. Now let $\xi \otimes e_j$ be a vector in

https://doi.org/10.1017/S0143385700002182 Published online by Cambridge University Press

$(\bigoplus_\eta l^2[0, p(\eta)))\otimes l^2[0, m_n)$. Then

$$\pi_\omega(f)W_\omega\xi\otimes e_j = W_\omega^{(2)}\tilde{\pi}_\omega(f)(U^jW_0\xi)\otimes e_j.$$

$$\begin{aligned} W_\omega(\oplus\pi_\eta(f))\xi\otimes e_j &= W_\omega^{(2)}W_\omega^{(1)}\left(\bigoplus_\eta \pi_{\eta,j}(f)\xi\right)\otimes e_j \\ &= W_\omega^{(2)}\left(U^jW_0\left(\bigoplus_\eta \pi_{\eta,j}(f)\right)\xi\right)\otimes e_j \\ &= W_\omega^{(2)}\left(W_0\left(\bigoplus_\eta \pi_{\eta,j}(f)\right)W_0^*U^jW_0\xi\right)\otimes e_j \\ &= W_\omega^{(2)}(W_0\otimes 1)\left(\bigoplus_\eta \pi_\eta(f)\right)(W_0^*\otimes 1)(U^jW_0\xi\otimes e_j). \end{aligned}$$

Thus

$$\left\|\pi_\omega(f)W_\omega - W_\omega\left(\bigoplus_\eta \pi_\eta(f)\right)\right\| = \left\|\tilde{\pi}_\omega(f) - W_0\otimes 1\left(\bigoplus_\eta \pi_\eta(f)\right)W_0^*\otimes 1\right\|$$

and since the difference is a diagonal operator the above norm is equal to

$$\begin{aligned} &\max_{i,j}\left\|\tilde{\pi}_\omega(f)e_i\otimes e_j - (W_0\otimes 1)\left(\bigoplus_\eta \pi_\eta(f)\right)(W_0^*\otimes 1)e_i\otimes e_j\right\| \\ &\quad = \max_{i,j}|f(T^rx_\omega(q)) - f(T^jx_\eta(k))| \end{aligned}$$

where

$$im_n + j = qm_{n+1} + r, \quad 0\le r < m_{n+1}, \quad \text{and} \quad \varphi_\eta(k) = i.$$

Recall that $\{\eta\}$ was a decomposition of $f_n\omega^{m_{n+1}/m_n}$. In particular

$$\eta(k) = f_n\omega^{m_{n+1}/m_n}(\varphi_\eta(k)).$$

From the definition of $f_n\omega^{m_{n+1}/m_n}$ we see that

$$f_n\omega^{m_{n+1}/m_n}(\varphi_\eta(k)) = \pi_{\tilde{r}}(f_n\cdot\omega(\tilde{q}))$$

where

$$\varphi_\eta(k) = \tilde{q}(m_{n+1}/m_n) + \tilde{r} \qquad 0\le\tilde{r} < m_{n+1}/m_n.$$

Since

$$\varphi_\eta(k)m_n + j = qm_{n+1} + r \qquad 0\le r < m_{n+1}$$

and

$$\varphi_\eta(k)m_n + j = \tilde{q}m_{n+1} + \tilde{r}m_n + j \qquad 0\le\tilde{r} < m_{n+1}/m_n,$$

it follows that $\tilde{q} = q$ and $\tilde{r}m_n + j = r$. So that

$$\eta(k) = \pi_{\tilde{r}}(f_n\omega(q)).$$

This implies that $x_\eta(k)$ and $T^{\tilde{r}m_n}x_\omega(q)$ lie in the same open set of $\mathscr{V}_n^{(m_n)}$ so that $T^jx_\eta(k)$ and $T^{j+\tilde{r}m_n}x_\omega(q) = T^rx_\omega(q)$ lie in the same open set of $\mathscr{V}_n$. This proves (1) and (2).

To prove (3) note that operator $\pi_\omega(f\circ T) - U_\omega\pi_\omega(f)U_\omega^*$ is diagonal, so that

$$\begin{aligned} &\|\pi_\omega(f\circ T) - U_\omega^*\pi_\omega(f)U_\omega\| \\ &\quad = \max_{i,j}\|\pi_\omega(f\circ T)e_i\otimes e_j - U_\omega^*\pi_\omega(f)U_\omega e_i\otimes e_j\|. \end{aligned}$$

But

$$\|\pi_\omega(f\circ T)e_i\otimes e_j - U_\omega^*\pi_\omega(f)U_\omega e_i\otimes e_j\|$$
$$=\begin{cases} 0 & j+1\neq m_{n+1} \\ |f(T^{m_{n+1}}x_\omega(i))-f(x_\omega(i+1))| & \text{if } j+1=m_{n+1},\ i+1\neq p(\omega) \\ |f(T^{m_{n+1}}x_\omega(p(\omega)-1))-f(x_\omega(0))| & \text{if } j+1=m_{n+1},\ i+1=p(\omega). \end{cases}$$

Recall that ω is $p(\omega)$ periodic, so that $x_\omega(0)=x_\omega(p(\omega))$, so that it suffices to estimate

$$|f(T^{m_{n+1}}x_\omega(k))-f(x_\omega(k+1))|$$

for every $k\in\mathbb{Z}$. Since ω is a $\mathcal{V}_{n+1}^{(m_{n+1})}$ pseudo-orbit of $T^{m_{n+1}}$ there exists $y\in X$ such that $y\in V_{\omega(k)}$ and $T^{m_{n+1}}y\in V_{\omega(k+1)}$, so that

$$\begin{aligned}|f(T^{m_{n+1}}x_\omega(k))-f(x_\omega(k+1))|\\ &\leq |f(T^{m_{n+1}}x_\omega(k))-f(T^{m_{n+1}}y)|+|f(T^{m_{n+1}}y)-f(x_\omega(k+1))|\\ &\leq \max_{V\in\mathcal{V}_{n+1}}\sup_{x,y\in V}|f(Tx)-f(Ty)|+\max_{V\in\mathcal{V}_{n+1}}\sup_{x,y\in V}|f(x)-f(y)|.\end{aligned}$$

This concludes the proof of (3) □

Recall that the AF-algebra associated to $(\mathcal{V}_n)_n$, $(m_n)_n$ and to the decomposition maps $(F_n)_n$ is the inductive limit

$$\longrightarrow A_n \xrightarrow{\phi_n} A_{n+1} \longrightarrow \cdots,$$

where $A_n=\bigoplus_{\omega\in\Omega_n} M_\omega$ and $\phi_n=\bigoplus_{\omega\in\Omega_{n+1}}\phi_\omega$. Define $U_n=\bigoplus_{\omega\in\Omega_n}U_\omega$ and

$$\pi_n: C(X)\to A_n \quad\text{by}\quad \pi_n(f)=\bigoplus_{\omega\in\Omega_n}\pi_\omega(f).$$

Recall also that α_T is the automorphism of $C(X)$ defined as

$$\alpha_T(f)=f\circ T^{-1}$$

THEOREM 7. *Let T be a homeomorphism of the compact topological metrizable space X with the property that every point $x\in X$ is pseudo-non-wandering. Then there exists a sequence of finite open covers $(\mathcal{V}_n)_n$ and a sequence of positive numbers $(m_n)_n$ such that for any decomposition maps F_n, $C(X)\times_{\alpha_T}\mathbb{Z}$ may be unitally embedded into the* AF-*algebra associated to $(\mathcal{V}_n)_n$, $(m_n)_n$ and $(F_n)_n$.*

Proof. Let S be a countable dense subset of $C(X)$ such that $\alpha_T(S)=S$ and choose the sequence of finite open covers $(\mathcal{V}_n)_n$ to have the property that

$$\sum_n\left(\max_{V\in\mathcal{V}_n}\sup_{x,y\in V}|f(x)-f(y)|\right)<\infty \qquad \text{for every } f\in S.$$

Suppose also that the m_n's satisfy

$$\sum_n\frac{1}{m_n}<\infty.$$

Combining lemmas 5 and 6 one sees that the sequences $\{U_n\}_{n\in\mathbb{N}}$, $\{\pi_n(f)\}_{n\in\mathbb{N}}$, $f\in S$ are norm convergent and that

$$\|\pi_n(\alpha_T(f))-U_n\pi_n(f)U_n^*\|\to 0 \qquad \text{for every } f\in S.$$

This implies that $\pi(f)=\lim_n \pi_n(f)$ exists for every $f\in C(X)$ and that, denoting $U=\lim_n U_n$, the pair (π, U) is a covariant representation of the C^*-dynamical system $(C(X), \alpha_T, \mathbb{Z})$. This pair generates a unital $*$-representation

$$\rho: C(X)\times_{\alpha_T}\mathbb{Z}\to A,$$

so all we have to prove is that ρ is faithful. This will follow once we show that for each finite sum $\sum_{i=0}^N f_i u^i \in C(X)\times_{\alpha_T}\mathbb{Z}$ with $f_i\in S$,

$$\left\|\sum_{i=0}^N \pi_n(f_i)U_n^i\right\| \xrightarrow[n\to\infty]{} \left\|\sum_{i=0}^N f_i u^i\right\|.$$

If we represent $C(X)$ faithfully as multiplication operators on $l^2(X)$, then [**3**, corollary 7.7.8] shows that $\bigoplus_{x\in X}\pi_x$ is a faithful representation of $C(X)\times_{\alpha_T}\mathbb{Z}$ on $\bigoplus_{x\in X} H_x$ where

$$H_x \simeq l^2(\mathbb{Z})$$

with canonical basis $e(n)$,

$$\pi_x(f)e(n)=f(T^n x)e(n), \qquad f\in C(X)$$

and

$$\pi_x(u)e(n)=e(n+1), \qquad n\in\mathbb{Z} \quad\text{and}\quad x\in X.$$

Thus

$$\left\|\sum_{i=0}^N f_i u^i\right\| = \sup_{x\in X}\left\|\pi_x\left(\sum_{i=0}^N f_i u^i\right)\right\|$$

so that for a given $\varepsilon>0$, we may find $x\in X$ and a sequence $(\xi_k)_{k\in\mathbb{Z}}$ such that $\sum|\xi_k|^2=1$ and

$$\left\|\sum_{i=0}^N f_i u^i\right\|^2 \le \sum_{k\in\mathbb{Z}}\left|\sum_{i=0}^N \xi_{k-i} f_i(T^k x)\right|^2+\varepsilon.$$

Moreover we may suppose (by replacing the point x too if necessary) that $\xi_k=0$ if k does not belong to some interval $[0, M]$. Let δ be positive and choose n big enough to get

$$\max_{V\in\mathcal{V}_n}\sup_{x,y\in V}|f_i(x)-f_i(y)|<\delta \qquad i=0,\dots,N,$$

and

$$m_n \ge M+N.$$

By lemma 4 there exists a $\mathcal{V}_n^{(m_n)}$ pseudo-orbit of T^{m_n}, $\omega\in\Omega_n$, such that $x\in V_{\omega(0)}$. In other words, for every $k\in[0, M+N]$, $x_\omega(k)$ and $T^k x$ lie in the same open set

$V \in \mathcal{V}_n$, so that

$$\begin{aligned}\left\|\sum_{i=0}^{N} \pi_n(f_i)U_n^i\right\|^2 &\geq \left\|\sum_{i=0}^{N} \pi_\omega(f_i)U_\omega^i\right\|^2 \\ &\geq \left\|\sum_{i=0}^{N} \pi_\omega(f_i)U_\omega^i\left(\sum_0^M \xi_k e_k\right)\right\|^2 \\ &= \sum_{k=0}^{M+N}\left|\sum_{i=0}^{N} \xi_{k-i} f_i(x_\omega(k))\right|^2 \\ &\geq \sum_{k=0}^{M+N}\left|\sum_{i=0}^{N} \xi_{k-i} f_i(T^k x)\right|^2 - (M+N)N^2\delta^2 \\ &\geq \left\|\sum_{i=0}^{N} f_i u^i\right\|^2 - \varepsilon - (M+N)N^2\delta^2.\end{aligned}$$

Choosing δ and ε small enough we get the desired result. □

We conclude this paper by showing that the condition $X = X(T)$ in the above theorem is essential.

PROPOSITION 8. *If T acts on the compact space X such that $X \neq X(T)$, then there exists a non-unitary isometry in $C(X) \times_{\alpha_T} \mathbb{Z}$.*

Proof. As in the proof of theorem 7 we represent $C(X) \times_{\alpha_T} \mathbb{Z}$ faithfully on $\bigoplus_{x\in X} H_x$ where

$$H_x \simeq l^2(\mathbb{Z}),$$

with canonical bases $e(n)$,

$$\begin{aligned}\pi_x(f)e(n) &= f(T^n x)e(n), && f \in C(X)\\ \pi_x(u)e(n) &= e(n+1) && n \in \mathbb{Z} \text{ and } x \in X.\end{aligned}$$

By lemma 2 there exists an open set U such that $T(\bar{U}) \subset U$ and $U \backslash T(\bar{U}) \neq \emptyset$. Let $f \in C(X)$ be such that $f(x) = \frac{1}{2}$ for $x \in X \backslash U$, $f(x) = 2$ for $x \in T(\bar{U})$ and $\frac{1}{2} \leq f(x) \leq 2$ for every $x \in X$. Thus $\pi_x(f \cdot u)$ is a weighted shift whose weights satisfy the properties that if $\alpha_k < 2$ then $\alpha_n = \frac{1}{2}$ for every $n < k$ and if $\alpha_k > \frac{1}{2}$ then $\alpha_n = 2$ for every $n > k$. Since there exists $x \in X$ such that $\pi_x(f \cdot u)$ has weights both $\frac{1}{2}$ and 2 and any two such shifts are similar by means of an invertible S satisfying

$$\|S\| \leq 2, \qquad \|S^{-1}\| \leq 2$$

it follows that $\bigoplus_{x\in X} \pi_x(1 - fu)$ is an injective semi-Fredholm operator of negative index. Hence the isometry in the polar decomposition of $\bigoplus_{x\in X} \pi_x(1 - fu)$ is in $\bigoplus_{x\in X} \pi_x(C(X) \times_{\alpha_T} \mathbb{Z})$. □

Recall from [**5**] that a unital separable C^*-algebra A is called *quasidiagonal* if there is a unital $*$-monomorphism

$$\rho: A \to B(H)$$

such that

$$\rho(A) \cap K(H) = 0,$$

($K(H)$ denotes the compact operators on H), and a sequence $\{P_n\}_{n\in\mathbb{N}}$ of finite dimensional orthogonal projections in $B(H)$ such that

$$\cdots \leq P_n \leq P_{n+1} \leq \cdots, \qquad \overline{\bigcup_n P_n(H)} = H$$

and

$$\|P_n\rho(a) - \rho(a)P_n\| \xrightarrow[n]{} 0 \qquad \text{for every } a \in A.$$

That the definition does not depend on the representation ρ follows from the non-commmutative Weyl–von Neumann-type theorem of D. Voiculescu [**7**]. In particular, any subalgebra of a quasidiagonal algebra is again quasidiagonal. Since any AF-algebra is quasidiagonal, non quasidiagonality is an obstruction to the embedding into an AF-algebra.

The next theorem shows that this is the only obstruction in the case of $C(X) \times_{\alpha_T} \mathbb{Z}$.

THEOREM 9. *Let T be a homeomorphism of the compact metrizable space X. The following are equivalent*:

(1) $X = X(T)$;

(2) $C(X) \times_{\alpha_T} \mathbb{Z}$ *is quasidiagonal*;

(3) *there exists a unital embedding of* $C(X) \times_{\alpha_T} \mathbb{Z}$ *into an* AF-*algebra.*

Proof. (1)$\Rightarrow$(3) and (3)$\Rightarrow$(2) are already proved, while (2)$\Rightarrow$(1) follows from proposition 8 combined with the result of P. R. Halmos, [**2**], that a non-unitary isometry is not a quasidiagonal operator. □

REFERENCES

[**1**] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lecture Notes in Mathematics No. 470. Springer-Verlag: Heidelberg-New York-London, 1975.

[**2**] P. R. Halmos. Quasitriangular operators. *Acta Sci. Math. (Szeged)* **29** (1968), 283–293.

[**3**] G. K. Pedersen. *C*-Algebras and Their Automorphism Groups.* Academic Press: London–New York–San Francisco, 1979.

[**4**] M. V. Pimsner & D. V. Voiculescu. Imbedding the irrational rotation C*-algebra into an AF-algebra. *J. Operator Theory* **4** (1980), 201–210.

[**5**] N. Salinas. Homotopy invariance of Ext(A). *Duke Math. J.* **44** (1977), 777–794.

[**6**] A. M. Vershik. Uniform algebraic approximation of the multiplication and translation operators. (In Russian.) *Doklady Akad. Nauk.* (1981).

[**7**] D. V. Voiculescu. A non-commutative Weyl–von Neumann theorem. *Rev. Roum. Math. Pures et Appl.* **21** (1976), 97–113.