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Odd viscous flow past a sphere at low but non-zero Reynolds numbers

Ruben Lier[1,2,3,†]

[1]Institute for Theoretical Physics, University of Amsterdam, 1090 GL Amsterdam, The Netherlands

[2]Dutch Institute for Emergent Phenomena (DIEP), University of Amsterdam, 1090 GL Amsterdam, The Netherlands

[3]Institute for Advanced Study, University of Amsterdam, Oude Turfmarkt 147, 1012 GC Amsterdam, The Netherlands

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The measurement of lift on symmetrically shaped obstacles immersed in low Reynolds number flow is the quintessential way to signal odd viscosity. For flow past cylinders, such a lift force does not arise if incompressibility and no-slip boundary conditions are fulfilled, whereas for spheres, a lift force has been found in Stokes flow, which is valid for cases where the Reynolds numbers are negligible and convection can be ignored. When considering the role of convection at low but non-zero Reynolds numbers, two hurdles arise, the Whitehead paradox and the breaking of axial symmetry, which are overcome by the method of matched asymptotic expansions and the Lorentz reciprocal theorem, respectively. We also consider the case where axial symmetry is preserved because the translation of the sphere is aligned with the axis of chirality of odd viscosity. We find that while lift vanishes, the interplay between odd viscosity and convection gives rise to a stream-induced torque.

Key words: Stokesian dynamics

1. Introduction

In the ever-continuing study of fluids, the exploration of flows induced by non-dissipative, parity-breaking viscosities has emerged as a new frontier. Such transport coefficients, which fall under the heading of odd viscosity, result from the spinning of the microscopic or mesoscopic particles that constitute the fluid (Banerjee *et al.* 2017; Markovich & Lubensky 2021; Fruchart, Scheibner & Vitelli 2023). Odd viscosity gives rise to a wide

† Email address for correspondence: r.lier@uva.nl

range of novel physical phenomena, such as azimuthal flow of sedimenting particles (Khain *et al.* 2022), torque induced by the rate of area change (Lapa & Hughes 2014), non-reciprocal self-induced flow fields for swimmers (Hosaka, Golestanian & Daddi-Moussa-Ider 2023*a*), formation of inertial-like waves (Kirkinis & Olvera de la Cruz 2023), Hall-like transport in fluids (Lou *et al.* 2022), oscillating vortical boundary layers (Abanov *et al.* 2020), and many more (Lucas & Surówka 2014). Furthermore, odd viscosity has implications across various scales and disciplines, including electron fluids (Delacrétaz & Gromov 2017; Berdyugin *et al.* 2019), diatomic gases (Korving *et al.* 1966; Hulsman *et al.* 1970), equatorial waves (Souslov *et al.* 2019; Tauber, Delplace & Venaille 2019), colloidal rotors (Soni *et al.* 2019; Hargus *et al.* 2020; Han *et al.* 2021; Hargus, Epstein & Mandadapu 2021; Poggioli & Limmer 2023; Matus, Lier & Surówka 2024) and biological systems (Tan *et al.* 2022). Odd viscosity is also studied at different Reynolds numbers, ranging from zero Reynolds number Stokes flow (Khain *et al.* 2022; Hosaka, Golestanian & Vilfan 2023*b*; Yuan & Olvera de la Cruz 2023) or Oseen flow (Lier *et al.* 2023) to fully developed turbulence (de Wit *et al.* 2024).

A key tool for understanding fluid behaviour at low Reynolds numbers is the Lorentz reciprocal theorem (Lorentz 1896), which in its original form holds only for even viscous fluids, but was recently generalized to fluids with odd viscosity (Hosaka *et al.* 2023*b*). One of the many things that this generalized Lorentz reciprocal theorem allows one to compute is the odd viscosity induced lift force on a translating object. In two dimensions, such a lift force is zero for incompressible fluids with no-slip boundary conditions, but it can take on a non-vanishing value at quadratic order in slip length (Lier 2024) for compressible fluids (Hosaka, Komura & Andelman 2021*b*; Lier *et al.* 2022; Duclut *et al.* 2024) or liquid domains in a membrane (Hosaka, Komura & Andelman 2021*a*).

The force and torque for odd Stokes flow past a sphere were computed by Hosaka *et al.* (2023*b*) up to linear order in odd viscosity, and by Everts & Cichocki (2024) nonlinearly in odd viscosity. Stokes flow, also known as creeping flow, means that the role of inertia, which enters through the convective term in the Navier–Stokes equation, is ignored entirely. The reason why it is worthwhile to not ignore this convective terms is twofold. First, to ignore the convective term requires the Reynolds number to be negligible, which is not the case in many fluid systems. Second, for translating obstacles, convective effects always dominate over viscous ones when one considers the flow sufficiently far from the obstacle (Proudman & Pearson 1957; Van Dyke 1975; Veysey & Goldenfeld 2007). In two dimensions, this is what causes the Stokes paradox, which bars one from finding a solution to the Stokes equation (Lamb 1932). In three dimensions, one can still obtain a Stokes solution but one is faced with the Whitehead paradox (Whitehead 1889), which makes the low Reynolds number expansion a singular perturbation theory. Although resolving the Whitehead paradox leads one to find that the Stokes solution accurately describes the flow at leading order in Reynolds number, this impediment further motivates the need to understand the role of convection in the case of odd viscous flow. Considering the role of convection for three-dimensional odd viscous flow is what is done in this work.

2. Odd viscous Navier–Stokes equation

Let us consider an incompressible fluid system with free-stream velocity U_i, shear viscosity η_s, and constant density ρ_0, and a rigid no-slip sphere with radius a. We use these to non-dimensionalize the coordinate x_i, fluid velocity u_i and stress tensor σ_{ij}. In addition, there is a single three-dimensional odd viscosity η_o which is assumed to be small compared to η_s. We start from the steady, incompressible Navier–Stokes equation

$$\nabla_i u_i = 0, \quad Re\, u_j \nabla_j u_i - \nabla_j \sigma_{ij} = 0, \tag{2.1a,b}$$

where Re is the Reynolds number, given by $Re = \rho_0 \, |U| \, a/\eta_s$. To keep the computations tractable at non-zero Reynolds numbers, we consider only the simplest possible three-dimensional odd viscosity η_o (Yuan & Olvera de la Cruz 2023; Everts & Cichocki 2024), namely the odd viscosity that arises upon coarse-graining a system of particles with an intrinsic rotation whose rotation axis is pointed in direction ℓ_i (Markovich & Lubensky 2021, 2022). We will call this vector the axis of chirality. Without loss of generality, we take the axis of chirality to be pointed in the z-direction, i.e. $\ell_i = \delta_i^z$. For such an odd viscosity and the isotropic shear viscosity, the stress σ_{ij} constitutes

$$\sigma_{ij} = -p\delta_{ij} + \nabla_i u_j + \nabla_j u_i + \gamma_o(\delta_{ik}\varepsilon_{jl} + \delta_{jk}\varepsilon_{il})(\nabla_k u_l + \nabla_l u_k), \tag{2.2}$$

where p is the dimensionless pressure, $\gamma_o = \eta_o/\eta_s$, and ε_{ij} is the two-dimensional anisotropic Levi–Civita tensor, i.e. $\varepsilon_{ij} = \epsilon_{ijk}\ell_k$, with ϵ_{ijk} being the three-dimensional Levi–Civita tensor. Plugging (2.2) into (2.1*a*,*b*) yields (Khain *et al.* 2023; Yuan & Olvera de la Cruz 2023)

$$\nabla_i u_i = 0, \quad Re\, u_j \nabla_j u_i - (\delta_{ij} + \gamma_o \varepsilon_{ij})\, \Delta u_j + \nabla_i p_m = 0, \tag{2.3a,b}$$

where we introduced the modified pressure (Ganeshan & Abanov 2017) given by $p_m = p - \gamma_o \varepsilon_{ij} \nabla_i u_j$, and Δ is the Laplacian operator.

3. Odd Oseenlet

We now consider the effect of a non-vanishing Reynolds number on the flow past a stationary sphere. When considering low Reynolds number flow past a sphere, one runs into the Whitehead paradox (Whitehead 1889), which is the finding that it is impossible to obtain a low Reynolds number correction to Stokes flow when one assumes the convective term proportional to Re in (2.3*a*,*b*) to be globally small. To resolve this, we use the method of matched asymptotic expansions (Kaplun 1957; Proudman & Pearson 1957), which means that one iteratively finds inner and outer solutions to distinct expansions of (2.3*a*,*b*), which are matched to each other at the boundary of their respective domains of validity. This begins with the zeroth outer solution, which is given by the dimensionless free-stream velocity $e_i = U_i/|U|$. The zeroth outer solution is then matched to the zeroth inner solution, which is the stokeslet. This zeroth inner solution is then matched to the first outer solution, which is given by the Oseenlet.

3.1. *Zeroth inner solution*

For the inner solution, we expand our fluid velocity as

$$u_i = u_i^{(0)} + Re\, u_i^{(1)} + O(\log(Re)\, Re^2), \quad p_m = p_m^{(0)} + Re\, p_m^{(1)} + O(\log(Re)\, Re^2), \tag{3.1a,b}$$

where we note that we added possible corrections of $O(\log(Re)\, Re^2)$ as it was found that in the absence of odd viscosity, the second inner solution is of this form (Proudman &

Pearson 1957). Equations (3.1*a*,*b*) mean that at leading order, (2.1*a*,*b*) reduce to

$$\nabla_i u_i^{(0)} = 0, \quad (\delta_{ij} + \gamma_o \varepsilon_{ij})\, \Delta u_j^{(0)} - \nabla_i p_m^{(0)} = 0. \tag{3.2a,b}$$

Working in the co-moving frame with the sphere's centre as the origin, the boundary conditions that we impose are given by

$$\lim_{r\to\infty} u_i = e_i, \quad u_i|_{r=1} = 0, \tag{3.3a,b}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and $e_i = U_i/|U|$. Solving up to $O(\gamma_o^2)$ leads to (Hosaka *et al.* 2023*b*)

$$u_i^{(0)} = e_i - \left(1 + \tfrac{1}{6}\Delta\right)(G_{ij}^{(0,s)} + \gamma_o G_{ij}^{(0,o)})\mathcal{F}_j^{(0)} + O(\gamma_o^2), \tag{3.4}$$

where we introduced the even and odd Oseen tensor (Yuan & Olvera de la Cruz 2023)

$$G_{ij}^{(0,s)} = \frac{1}{8\pi}(\delta_{ij}\Delta - \nabla_i\nabla_j)r, \quad G_{ij}^{(0,o)} = -\frac{1}{8\pi}(\varepsilon_{ij}\delta_{kl} + \delta_{li}\varepsilon_{jk} - \delta_{lj}\varepsilon_{ik})\,\nabla_k\nabla_l r. \tag{3.5a,b}$$

The boundary conditions require the dimensionless Stokes force $\mathcal{F}_j^{(0)}$ to be

$$\mathcal{F}_i^{(0)} = (C_D^{(0)}\delta_{ij} + C_L^{(0)}\varepsilon_{ij})e_j, \tag{3.6}$$

with drag and lift coefficients given by (Hosaka *et al.* 2023*b*)

$$C_D^{(0)} = 6\pi + O(\gamma_o^2), \quad C_L^{(0)} = 3\pi\gamma_o + O(\gamma_o^3). \tag{3.7a,b}$$

Restoring the dimensionality and defining the total force F_i, (3.7*a*,*b*) means that we have a drag force

$$F_D^{(0)} = F_i^{(0)} e_i = 6\pi a \eta_s\,|U| + O(\gamma_o^2), \tag{3.8}$$

and lift force

$$F_L^{(0)} = \varepsilon_{ij} F_i^{(0)} e_j = 3\pi a \eta_o\,|U| + O(\gamma_o^3). \tag{3.9}$$

3.2. *First outer solution*

Having found the zeroth inner solution, we connect this solution to the outer region to obtain the first outer solution. The boundary of the outer region is characterized by $Re\, r = O(1)$, so that in order to consistently match small Reynolds number corrections, we work with the Oseen coordinate $\tilde{r} = Re\, r$. For the outer solution, we expand the fields as

$$\tilde{u}_i = e_i + Re\, \tilde{u}_i^{(1)} + O(Re^2), \quad \tilde{p}_m = Re^2\, \tilde{p}_m^{(1)} + O(Re^3). \tag{3.10a,b}$$

The equation that the first outer solution must obey is the Oseen equation (Lamb 1932; Oseen 1910), which is given by

$$\tilde{\nabla}_i \tilde{u}_i^{(1)} = 0, \quad (\delta_{ij} + \gamma_o \varepsilon_{ij})\, \tilde{\Delta}\tilde{u}_j^{(1)} - \tilde{\nabla}_i p_m^{(1)} = e_j\, \tilde{\nabla}_j \tilde{u}_i^{(1)}. \tag{3.11a,b}$$

The method of matched asymptotic expansions dictates that $\tilde{u}_i^{(1)}$ must be connected to the zeroth inner solution. We find the corresponding inner boundary condition to be given by

$$\tilde{u}_i^{(1)} \to -t(\tilde{G}_{ij}^{(0,s)} + \gamma_o \tilde{G}_{ij}^{(0,o)})\mathcal{F}_j^{(0)} + O(\gamma_o^2) \quad \text{as } r \to 0, \tag{3.12}$$

where

$$\tilde{G}^{(0,s)}_{ij} = \frac{1}{8\pi}\,(\delta_{ij}\,\tilde{\Delta} - \tilde{\nabla}_i\tilde{\nabla}_j)\tilde{r}, \quad \tilde{G}^{(0,o)}_{ij} = -\frac{1}{8\pi}\,(\varepsilon_{ij}\delta_{kl} + \delta_{li}\varepsilon_{jk} - \delta_{lj}\varepsilon_{ik})\,\tilde{\nabla}_k\tilde{\nabla}_l\tilde{r}. \quad (3.13a,b)$$

Condition (3.12), together with the outer boundary condition and (3.10a,b), forces $\tilde{u}^{(1)}_i$ to be given by

$$\tilde{u}^{(1)}_i = -(\tilde{G}^{(1,s)}_{ij} + \gamma_o\tilde{G}^{(1,o)}_{ij})\mathcal{F}^{(0)}_j + O(\gamma_o^2), \tag{3.14}$$

where $\tilde{G}^{(1,s)}_{ij}$ represents the tensor corresponding to the even Oseenlet, which is of the form (Pozrikidis 1996)

$$\tilde{G}^{(1,s)}_{ij} = \frac{1}{4\pi}\,(\delta_{ij}\tilde{\Delta} - \tilde{\nabla}_j\tilde{\nabla}_i)\int_0^{\tilde{\zeta}} \frac{1-\exp(-\xi)}{\xi}\,\mathrm{d}\xi, \tag{3.15}$$

with $\tilde{\zeta} = \frac{1}{2}(\tilde{r} - \tilde{x}_i e_i)$. Here, $\tilde{G}^{(1,o)}_{ij}$ is given by (see Appendix A for a derivation)

$$\tilde{G}^{(1,o)}_{ij} = \frac{1}{4\pi}\,(\varepsilon_{ij}\delta_{kl} + \delta_{li}\varepsilon_{jk} - \delta_{lj}\varepsilon_{ik})\,\tilde{\nabla}_k\tilde{\nabla}_l\exp(-\tilde{\zeta}). \tag{3.16}$$

The tensor $\tilde{G}^{(1,o)}_{ij}$ can be understood as corresponding to the odd part of the solution to the Oseen equation in the presence of a singular force. We therefore name this solution the odd Oseenlet.

4. First inner solution

Having found the first outer solution, we use the method of matched asymptotic expansions to obtain the outer boundary condition for $u^{(1)}_i$. Matching of the inner solution to the outer solution at $O(Re)$ leads to the outer boundary condition

$$su^{(1)}_i \to \frac{1}{8\pi}\,\mathcal{F}^{(0)}_j\left[\frac{1}{2}\,(\delta_{ij}\,\Delta - \nabla_j\nabla_i) - \gamma_o(\varepsilon_{ij}\delta_{kl} + \delta_{li}\varepsilon_{jk} - \delta_{lj}\varepsilon_{ik})\,\nabla_k\nabla_l\right]\zeta^2 \\ + O(\gamma_o^2) \quad \text{as } r\to\infty, \tag{4.1}$$

where $\zeta = \frac{1}{2}(r - x_i e_i)$. Expanding (2.1$a$,$b$) up to $O(Re)$, one finds that the first inner solution must satisfy

$$\nabla_i u^{(1)}_i = 0, \quad (\delta_{ij} + \gamma_o\varepsilon_{ij})\,\Delta u^{(1)}_j - \nabla_i p^{(1)}_m = u^{(0)}_j\,\nabla_j u^{(0)}_i. \quad (4.2a,b)$$

Until now, we have made no assumption about the direction of the free-stream velocity with respect to ℓ_i. In order to compute the first inner solution, it is important that we do so. Let us first consider the longitudinal case where $e_i = \ell_i$. The longitudinal case is simpler than the transverse case as we retain axial symmetry. For the case with transverse free-stream velocity, we refrain from computing the first inner solution. Using the Lorentz reciprocal theorem, we instead directly compute the lift force, which exists only for the transverse case.

4.1. *Longitudinal free-stream velocity*

For the case $e_i = \ell_i$, let us first solve (4.2*a,b*) at zeroth order in γ_o. For this, we use the axial symmetry of the longitudinal free-stream velocity to take the stream function ansatz (Happel & Brenner 1983)

$$u_\varpi^{(1,s)} = -\frac{\partial_z \psi^{(1,s)}}{\varpi}, \quad u_\phi^{(1,s)} = 0, \quad u_z^{(1,s)} = \frac{\partial_\varpi \psi^{(1,s)}}{\varpi}, \tag{4.3a–c}$$

where ϖ, ϕ and z are cylindrical coordinates as visualized in figure 1. The solution that obeys (4.2*a,b*) in the absence of odd viscosity and satisfies the inner and outer boundary is given by (Proudman & Pearson 1957)

$$\psi^{(1,s)} = \frac{3}{32}\varpi^2\left(\frac{1}{r^3} - \frac{3}{r} + 2\right) - \frac{3}{32}\varpi^2 z\left(\frac{1}{r^5} - \frac{1}{r^4} + \frac{1}{r^3} - \frac{3}{r^2} + \frac{2}{r}\right), \tag{4.4}$$

with $r = \sqrt{\varpi^2 + z^2}$. Now we move to $u_i^{(1,o)}$. As a first step, let us define $u_i^{(1,o)} = u_i'^{(1,o)} + u_i''^{(1,o)}$, with

$$u_\varpi'^{(1,o)} = 0, \quad u_\phi'^{(1,o)} = -\frac{\partial_z \psi^{(1,s)}}{\varpi}, \quad u_z'^{(1,o)} = 0, \tag{4.5a–c}$$

so that at $O(\gamma_o)$, (4.2*a,b*) reduce to

$$\Delta u_i''^{(1,o)} - \nabla_i p_m^{(1,o)} = u_j^{(0,s)}\,\nabla_j u_i^{(0,o)} + u_j^{(0,o)}\,\nabla_j u_i^{(0,s)}. \tag{4.6}$$

The right-hand side of (4.6) only has a non-vanishing ϕ-component, thus $u_i^{(1,o)}$ also only has a non-vanishing ϕ-component. The only possible contribution from pressure therefore also comes from the ϕ gradient, which vanishes due to axial symmetry. Equation (4.6) thus simplifies to

$$\Delta u_\phi''^{(1,o)} = u_j^{(0,s)}\,\nabla_j u_\phi^{(0,o)}. \tag{4.7}$$

The solution to (4.7) is given by

$$\begin{aligned} u_\phi''^{(1,o)} = {} & \frac{1}{32}\varpi\left(-\frac{1}{r^6} + \frac{A_1}{r^5} - \frac{12}{r^4} + \frac{A_2}{r^3}\right) \\ & + \frac{1}{32}\varpi^3\left(\frac{3}{4r^8} - \frac{5A_1}{4r^7} + \frac{18}{r^6} - \frac{12}{r^5} - \frac{9}{2r^4} + \frac{6}{r^3}\right), \end{aligned} \tag{4.8}$$

where the coefficients A_1 and A_2 must be found by imposing the boundary conditions. For the outer boundary condition, we must consider (4.1), which together with (4.5*a*–*c*) yields the outer boundary condition

$$u_\phi''^{(1,o)} = \frac{3\varpi^3}{16r^3} \quad \text{as } r \to \infty. \tag{4.9}$$

This boundary condition is satisfied regardless of A_1 and A_2. However, requiring that the inner boundary condition is satisfied leads one to find $A_1 = \frac{33}{5}$ and $A_2 = \frac{32}{5}$. Thus the

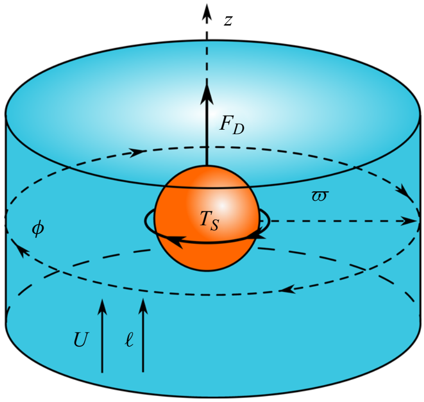

Figure 1. Graphical representation of cylindrical coordinates ϖ, ϕ, z for a sphere immersed in a fluid, with streaming velocity U_i pointed in the z-direction, and an odd viscosity whose axis of chirality ℓ_i is also pointed in the z-direction. The picture also shows the drag force F_D and the stream-induced torque T_S whose value is given in (4.12).

complete solution is given by

$$u_\phi^{\prime\prime(1,o)} = \frac{1}{32}\,\varpi\left(-\frac{1}{r^6}+\frac{33}{5r^5}-\frac{12}{r^4}+\frac{32}{5r^3}\right) + \frac{1}{32}\,\varpi^3\left(\frac{3}{4r^8}-\frac{33}{4r^7}+\frac{18}{r^6}-\frac{12}{r^5}-\frac{9}{2r^4}+\frac{6}{r^3}\right). \tag{4.10}$$

Having obtained a complete solution up to first order in odd viscosity and Reynolds number, we compute the force on the sphere corresponding to this solution. As follows from axial symmetry, there is no lift force. There is only the convective correction to drag force, which leads to a total drag force

$$F_D = F_i e_i = 6\pi a\,|U|\,\eta_s\left(1+\tfrac{3}{8}\,Re\right) + O(\gamma_o^2, \log(Re)\,Re^2). \tag{4.11}$$

However, we do find odd viscous effects through the dimensionless torque $\mathcal{T}_k^{(1)} = \int_B \epsilon_{ijk} x_i \sigma_{jl}^{(1)}\,\mathrm{d}S_l$. Restoring dimensions, we acquire the total stream-induced torque T_S whose value is given by

$$T_S = \ell_i T_i = \frac{2\pi a^2}{5}\,|U|\,\eta_o\,Re + O(\gamma_o^3, \gamma_o \log(Re)\,Re^2). \tag{4.12}$$

This torque is displayed graphically in figure 1. We thus learn that convection can allow for a sphere to feel torque when moving through an odd viscous fluid in an axially symmetry way. Reflection symmetry prevents such a torque from arising for Stokes flow past a sphere (Khain *et al.* 2023).

4.2. *Transverse free-stream velocity*

For the case where the free-stream velocity is transverse to ℓ_i, to explicitly solve (4.2*a,b*) in a way that satisfies the inner and outer boundary conditions is challenging. Fortunately, it turns out that that thanks to the Lorentz reciprocal theorem, we need not know more about the first inner solution than (4.1) and (4.2*a,b*) in order to obtain the first convective correction to drag and lift force on a translating sphere. The Lorentz reciprocal theorem connects two fluid systems, with the first fluid system being the one that we wish to better understand, and an auxiliary fluid system that is used to accomplish this by means of the Lorentz reciprocal theorem. The first fluid system is one corresponding to the first inner solution represented by fluid velocity $u_i^{(1)}$. The auxiliary fluid system labelled by $*$ is identical to the first fluid system except for the following three things.

(i) We consider for the auxiliary fluid system a dimensionless free-stream velocity e_i^* that satisfies $(e_i^*)^2 = 1$ and $e_i^* \ell_i = 0$, but is not necessarily parallel to e_i, as a parallel e_i^* does not allow one to extract lift force (Lier 2024).

(ii) For the auxiliary fluid system, we consider Stokes flow, i.e. flow that corresponds to the zeroth inner solution represented by fluid velocity $u_i^{*(0)}$.

(iii) In order for the Lorentz reciprocal theorem to hold for odd fluids, we require that odd viscosity of the second fluid system is given by $\gamma_o^* = -\gamma_o$ (Hosaka *et al.* 2023*b*).

These differences are displayed graphically in figure 2. Starting from (4.2*a*,*b*), the Lorentz reciprocal theorem gives the relation (Brenner & Cox 1963; Masoud & Stone 2019)

$$\int_{S_\infty} (u_j^{*(0)} \sigma_{ij}^{(1)} - u_j^{(1)} \sigma_{ij}^{*(0)})\, \mathrm{d}S_i = \int_{V_\infty} u_i^{(0)}\, \nabla_i u_j^{(0)} u_j^{*(0)}\, \mathrm{d}V, \tag{4.13}$$

where the normal of the surface integral points outwardly. Here, S_∞ is the surface of a sphere centred at x_i that encloses the volume V_∞ of the fluid system, whose radius will eventually be taken to infinity. Furthermore, V_∞ surrounds B, the boundary of the sphere (see figure 2). To work out (4.13), we introduce the definitions

$$I_1 = \int_{S_\infty} u_j^{*(0)} \sigma_{ij}^{(1)}\, \mathrm{d}S_i, \quad I_2 = -\int_{S_\infty} u_j^{(1)} \sigma_{ij}^{*(0)} \mathrm{d}S_i, \quad I_3 = \int_{V_\infty} u_i^{(0)}\, \nabla_i u_j^{(0)} u_j^{*(0)}\, \mathrm{d}V. \tag{4.14a–c}$$

For the first two terms, it holds that because the sphere with surface S_∞ will be considered infinitely large, any contribution to the integrand that is $O(r^{-3})$ can be discarded. To take advantage of this fact, let us introduce for a general field f the notation

$$f = \sum_{n=0} (f)_{-n}\, r^{-n}, \tag{4.15}$$

where $(f)_{-n}$ has no r-dependence. We first consider I_1, which in the limit of infinite sphere radius can be written as

$$I_1 = \int_{S_\infty} (\sigma_{ij}^{(1)})_{-1} [(u_j^{*(0)})_0\, r^{-1} + (u_j^{*(0)})_{-1}\, r^{-2}]\, \mathrm{d}S_i + \int_{S_\infty} (\sigma_{ij}^{(1)})_{-2}\, (u_j^{*(0)})_0\, r^{-2}\, \mathrm{d}S_i. \tag{4.16}$$

Here, $u_j^{*(0)}$ is fully even under $x_i \to -x_i$, and since (4.16) is a surface integral, $(\sigma_{ij}^{(1)})_{-1}$ or $(\sigma_{ij}^{(1)})_{-2}$ must be odd under $x_i \to -x_i$ in order for this contribution to I_1 to be non-vanishing. This cannot come from the pressure term as it is fully even. For $(\sigma_{ij}^{(1)})_{-1}$ to give a non-vanishing contribution thus requires $(u_j^{(1)})_0$ to be even under $x_i \to -x_i$.

From (4.1), it follows that

$$(u_j^{(1)})_0 = J_{jk}\mathcal{F}_k^{(0)} + \text{terms odd under } x_i \rightarrow -x_i, \tag{4.17}$$

with J_{jk} given by

$$J_{ij} = \frac{1}{32\pi}\,(3\delta_{ij} - e_i e_j - 2\gamma_o \varepsilon_{ij}). \tag{4.18}$$

Because this term is constant, it vanishes when computing the corresponding viscous stress. Therefore, (4.16) reduces to

$$I_1 = e_j^* \int_{S_\infty} \sigma_{ij}^{(1)}\, \mathrm{d}S_i. \tag{4.19}$$

Now I_1 can be related to the desired $\mathcal{F}_i^{(1)}$ whose expression is given by $\mathcal{F}_i^{(1)} = \int_B \sigma_{ij}^{(1)}\, \mathrm{d}S_j$. Specifically, from (4.2*a*,*b*), it follows that

$$\int_{B+S_\infty} (\sigma_{ij}^{(1)} - u_j^{(0)} u_j^{(0)})\, \mathrm{d}S_j = 0. \tag{4.20}$$

Equation (4.20) simplifies by noting that $u_j^{(0)} = 0$ on B. Furthermore, since $u_j^{(0)}$ is fully even under $x_i \rightarrow -x_i$, we have $\int_{B+S_\infty} u_j^{(0)} u_j^{(0)}\, \mathrm{d}S_j = 0$, from which it follows that $I_1 = -e_j^* \mathcal{F}_j^{(1)}$. We then consider I_2, which in the limit of infinite sphere radius can be written as

$$I_2 = -\int_{S_\infty} r^{-2} (u_j^{(1)})_0\, (\sigma_{ij}^{*(0)})_{-2}\, \mathrm{d}S_i. \tag{4.21}$$

Here, $(\sigma_{ij}^{*(0)})_{-2}$ is odd under $x_i \rightarrow -x_i$, so that only the even part of $(u_j^{(1)})_0$ can contribute. This term is given by (4.17), and we thus find

$$I_2 = -J_{jk}\mathcal{F}_k^{(0)} \int_{S_\infty} \sigma_{ij}^{*(0)}\, \mathrm{d}S_i. \tag{4.22}$$

Using the zeroth-order version of (4.20), it holds for the zeroth-order Stokes force that $\mathcal{F}_i^{*(0)} = -\int_{S_\infty} \sigma_{ij}^{*(0)}\, \mathrm{d}S_j$, so that (4.22) turns into $I_2 = -J_{jk}\mathcal{F}_j^{*(0)}\mathcal{F}_k^{(0)}$. Finally, $u_i^{(0)}$ is fully even under $x_i \rightarrow -x_i$, making the integrand of I_3 fully odd under $x_i \rightarrow -x_i$, thus $I_3 = 0$. Combining the results for I_1 and I_2, (4.13) narrows down to

$$\mathcal{F}_j^{(1)} e_j^* = J_{jk}\mathcal{F}_j^{*(0)}\mathcal{F}_k^{(0)}. \tag{4.23}$$

We now consider two choices for e_i^* that enable us to extract drag and lift force, respectively. First, we take $e_i^* = e_i$. Using (4.18), (4.23) then reduces to the formula for

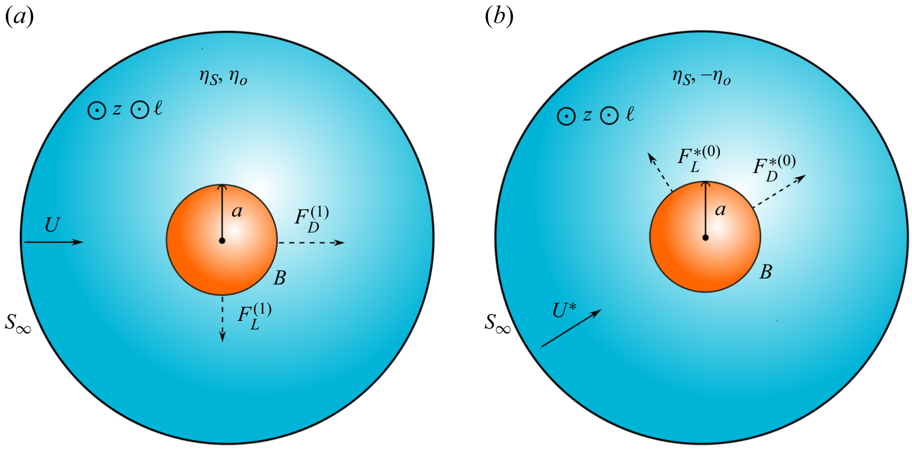

Figure 2. Graphical representation of the two fluid systems that are considered when using the Lorentz reciprocal theorem. These are (*a*) the first fluid system, which we wish to understand better, and (*b*) the auxiliary fluid system, which is used to gain understanding of the first one by means of the Lorentz reciprocal theorem.

the convective correction to drag force (Brenner & Cox 1963; Masoud & Stone 2019)

$$\mathcal{F}_j^{(1)} e_j = C_D^{(1)} = \frac{9\pi}{4} + O(\gamma_o^2). \tag{4.24a}$$

Let us now consider the case $e_i^* = \varepsilon_{ij} e_j$. We find the convective lift force correction

$$\varepsilon_{ij} \mathcal{F}_i^{(1)} e_j = C_L^{(1)} = \frac{9\pi}{16} \gamma_o + O(\gamma_o^3). \tag{4.24b}$$

Using (3.6) and restoring the dimensionality, we finally obtain the total drag force (Proudman & Pearson 1957)

$$F_D = F_i e_i = 6\pi a \, |U| \, \eta_s \left(1 + \tfrac{3}{8} \, Re\right) + O(\gamma_o^2, \log(Re) \, Re^2), \tag{4.25a}$$

and total lift force

$$F_L = \varepsilon_{ij} F_i e_j = 3\pi a \, |U| \, \eta_o \left(1 + \tfrac{3}{16} \, Re\right) + O(\gamma_o^3, \gamma_o \log(Re) \, Re^2). \tag{4.25b}$$

In Appendix B, we show an alternative and more direct way of computing (4.25).

5. Discussion

In this work, we studied the effect of convection for odd viscous flow past a sphere, and computed low Reynolds number corrections to the corresponding forces and torques. Since the low Reynolds number expansion is singular, we obtained inner and outer solutions that are matched using the method of matched asymptotic expansions. We considered two cases, namely the cases where the axis of chirality is parallel and orthogonal to the free-stream velocity. For the longitudinal case, we could use axial symmetry to fully apply the method of matched asymptotic expansions up to first order. Due to axial symmetry, lift force vanishes; however, we do find that the interplay between convection and odd viscosity can turn on torque for a translating sphere. Torque does not arise for Stokes flow, because in this case a coupling between torque and translation is precluded by the symmetry of Stokes flow under reflection along the direction of motion (Khain *et al.* 2023). This symmetry is also called fore–aft symmetry. Interestingly, the breaking of fore–aft symmetry exhibited by the flow is also found to generate torque on a sphere translating through an even viscous fluid subject to a background rotation. In this case, the torque arises when the rotation axis is aligned with the motion, similar to how in the odd viscous case the axis of chirality is aligned with the motion. In both cases, the stream-induced torque arises at first order in Reynolds number. Aurégan, Bonometti & Magnaudet (2023) have studied the force and torque for this rotating fluid system numerically, avoiding the need to assume certain characteristic numbers such as the Reynolds number to be small. It would be interesting to similarly numerically explore odd viscous flow past a sphere without assuming the Reynolds number or γ_o to be small. For Stokes flow, odd viscous flow was studied analytically and nonlinearly in γ_o in Everts & Cichocki (2024).

Because the stream-induced torque arises exclusively due to convection, measuring torque is an excellent avenue for experimentally studying the interplay between convection and odd viscosity. Specifically, one could measure the rotation of a spherical object that is dropped into an odd viscous fluid that has its axis of chirality aligned with the gravitational force. A good option for the experimental realization of a three-dimensional odd viscous fluid is a suspension of externally rotated spinners, as considered by Soni *et al.* (2019). However, in this work the spinners were confined to an interface, which prevents the odd viscosity from being three-dimensional. Another challenge in studying the interplay between convection and three-dimensional odd viscosity with a suspension of spinners is that such a fluid will also exhibit rotational viscosity, which can affect the torque on an obstacle.

For the transverse case, axial symmetry is broken, and the computation of convective effects on flow is more involved. We therefore computed only the first Reynolds number corrections to the forces with the help of the odd generalization of the Lorentz reciprocal theorem (Hosaka *et al.* 2021*b*). A relevant question is whether for the transverse case, stream-induced torque can also arise. Because axial symmetry breaking dramatically complicates this problem, an analytical computation of the first inner solution seems out of reach, and one instead must rely on an indirect method similar to the Lorentz reciprocal theorem that was employed to compute lift force. Because torque requires knowledge of a force dipole as opposed to a force monopole, finding such an approach is more involved.

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Author ORCIDs.
Ruben Lier https://orcid.org/0009-0001-1547-8256.

Appendix A

To derive the first outer solution, we start from (3.11*a*,*b*) and add a point force that approximates the sphere at distances far away from it. The corresponding point force solution will turn out to satisfy the inner boundary condition (3.12). Introducing the point force yields the equations

$$\tilde{\nabla}_i \tilde{u}_i'^{(1)} = 0, \tag{A1a}$$

$$(\delta_{ij} + \gamma_o \varepsilon_{ij})\, \tilde{\Delta} \tilde{u}_j'^{(1)} - \tilde{\nabla}_i \tilde{p}_m'^{(1)} = e_j\, \tilde{\nabla}_j \tilde{u}_i'^{(1)} + \mathcal{F}_i^{(0)}\, \tilde{\delta}, \tag{A1b}$$

where $\tilde{\delta}$ is the three-dimensional Dirac delta function located at the origin. Following Pozrikidis (1996), we take the divergence of (A1*b*) to obtain

$$\tilde{\Delta}\tilde{p}_m'^{(1)} = -\mathcal{F}_i^{(0)}\,\nabla_i\tilde{\delta} + \gamma_o\varepsilon_{ij}\,\tilde{\Delta}\tilde{\nabla}_i u_j'^{(1)}. \tag{A2}$$

Using the relation $\tilde{\delta} = -\tilde{\Delta}(1/4\pi\tilde{r})$, (A2) can be solved as

$$\tilde{p}_m'^{(1)} = \mathcal{F}_i^{(0)}\,\tilde{\nabla}_i\frac{1}{4\pi\tilde{r}} + \gamma_o\varepsilon_{ij}\,\tilde{\nabla}_i\tilde{u}_j'^{(1)}. \tag{A3}$$

Plugging this back into (A1*b*), we obtain

$$e_j\,\tilde{\nabla}_j\tilde{u}_i'^{(1)} = [\delta_{ij}\,\tilde{\Delta} + \gamma_o(\varepsilon_{ij}\,\tilde{\Delta} - \varepsilon_{kj}\,\tilde{\nabla}_i\tilde{\nabla}_k)]\tilde{u}_j'^{(1)} + \frac{\mathcal{F}_j^{(0)}}{4\pi}\,(\delta_{ij}\,\tilde{\Delta} - \tilde{\nabla}_i\tilde{\nabla}_j)\,\frac{1}{\tilde{r}}. \tag{A4}$$

At zeroth order in γ_o, (A4) can be solved with

$$\tilde{u}_i'^{(1,s)} = -\mathcal{F}_j^{(0,s)}(\delta_{ij}\,\tilde{\Delta} - \tilde{\nabla}_i\tilde{\nabla}_j)\,\mathbb{M}, \tag{A5}$$

where $\mathbb{M}$ is given by

$$(\tilde{\Delta} - e_j\,\tilde{\nabla}_j)\,\mathbb{M} = \frac{1}{4\pi\tilde{r}}. \tag{A6}$$

Note that the ansatz of (A5) is such that (A1*a*) is automatically satisfied. We can take the Laplacian of (A6) to find

$$(\tilde{\Delta} - e_j\,\tilde{\nabla}_j)\,\Delta\mathbb{M} = -\tilde{\delta}. \tag{A7}$$

We define

$$\tilde{\Delta}\mathbb{M} = \Phi\,\exp(\tilde{x}_i e_i/2), \tag{A8}$$

so that (A7) turns into the Helmholtz equation

$$\tilde{\Delta}\Phi - \tfrac{1}{4}\Phi = -\tilde{\delta}, \tag{A9}$$

whose solution is given by

$$\Phi = \frac{1}{4\pi\tilde{r}}\exp(-\tilde{r}/2). \tag{A10}$$

Equation (A8) then turns into

$$\tilde{\Delta}\mathbb{M} = \frac{\exp(-\tilde{\zeta})}{4\pi\tilde{r}}, \tag{A11}$$

with $\tilde{\zeta} = \frac{1}{2}(\tilde{r} - \tilde{x}_i e_i)$. Equation (A11) can be solved to find (Pozrikidis 1996)

$$\mathbb{M} = \frac{1}{4\pi}\int_0^{\tilde{\zeta}} \frac{1-\exp(-\xi)}{\xi}\,\mathrm{d}\xi. \tag{A12}$$

Having found $\tilde{u}_i'^{(1,s)}$, we move on to $\tilde{u}_i'^{(1,o)}$. For this, we take the ansatz

$$\tilde{u}_i'^{(1,o)} = -\mathcal{F}_j^{(0,s)}A_{ijkl}\,\tilde{\nabla}_k\tilde{\nabla}_l\mathbb{N} - \mathcal{F}_j^{(0,o)}(\delta_{ij}\,\tilde{\Delta} - \tilde{\nabla}_i\tilde{\nabla}_j)\,\mathbb{M}, \tag{A13}$$

where A_{ijkl} is some four-tensor. The second term on the right-hand side of (A13) cancels out the odd part of the point-force term in (A4). Plugging (A13) into (A4) and considering the contributions $O(\gamma_o)$, we find

$$\mathcal{F}_j^{(0,s)} A_{ijkl}\,\tilde{\nabla}_k\tilde{\nabla}_l(\tilde{\Delta}\mathbb{N} - e_m\,\tilde{\nabla}_m\mathbb{N}) = -\mathcal{F}_j^{(0,s)}(\varepsilon_{il}\,\tilde{\Delta} - \varepsilon_{kl}\,\tilde{\nabla}_i\tilde{\nabla}_k)(\delta_{lj}\tilde{\Delta} - \tilde{\nabla}_l\tilde{\nabla}_j)\,\mathbb{M}. \quad \text{(A14)}$$

Simplifying leads to

$$A_{ijkl}\,\tilde{\nabla}_k\tilde{\nabla}_l(\tilde{\Delta}\mathbb{N} - e_m\,\tilde{\nabla}_m\mathbb{N}) = -(\varepsilon_{ij}\delta_{kl} + \varepsilon_{jk}\delta_{il} - \varepsilon_{ik}\delta_{lj})\,\tilde{\nabla}_k\tilde{\nabla}_l\Delta\mathbb{M}. \quad \text{(A15)}$$

To solve (A15), we take

$$A_{ijkl} = \varepsilon_{ij}\delta_{kl} + \varepsilon_{jk}\delta_{il} - \varepsilon_{ik}\delta_{lj}, \quad \text{(A16)}$$

and also impose

$$\tilde{\Delta}\mathbb{N} - e_i\,\tilde{\nabla}_i\mathbb{N} = -\tilde{\Delta}\mathbb{M}. \quad \text{(A17)}$$

Note that we have

$$\begin{aligned}\tilde{\nabla}_i\tilde{u}_i^{\prime(1,o)} &= -\tilde{\nabla}_i[\mathcal{F}_j^{(0)}(\varepsilon_{ij}\delta_{kl} + \varepsilon_{jk}\delta_{il} - \varepsilon_{ik}\delta_{lj})\,\tilde{\nabla}_k\tilde{\nabla}_l\mathbb{N}] \\ &= -\frac{\mathcal{F}_j^{(0)}}{\eta_s}\,(\varepsilon_{kj}\,\tilde{\nabla}_k + \varepsilon_{jk}\,\tilde{\nabla}_k)\,\tilde{\Delta}\mathbb{N} = 0, \end{aligned} \quad \text{(A18)}$$

thus incompressibility is guaranteed for $\tilde{u}_i^{(1,o)}$. Proceeding, we find that with the help of (A11), (A17) can be rewritten as

$$\tilde{\Delta}\mathbb{N} - e_i\,\tilde{\nabla}_i\mathbb{N} = -\frac{\exp(-\tilde{\zeta})}{4\pi\tilde{r}}, \quad \text{(A19)}$$

which is solved when one takes

$$\mathbb{N} = \frac{\exp(-\tilde{\zeta})}{4\pi}. \quad \text{(A20)}$$

The combination of (A3), (A5) and (A13) fully specifies $\tilde{u}_i^{\prime(1,o)}$. It turns out that this point force solution satisfies (3.12), and therefore $\tilde{u}_i^{(1,o)} = \tilde{u}_i^{\prime(1,o)}$. We have thus obtained (3.14).

Appendix B

In this appendix, we show a more direct way to compute the convective corrections to the Stokes forces from the first outer solutions. First, let us define

$$v_i = -Re\,\tilde{u}_i^{(1)}, \quad \text{(B1)}$$

where $\tilde{u}_i^{(1)}$ is the first outer solution given in (3.14). We revert back to the coordinate r and to the frame where the fluid is stationary at $r \to \infty$ and the sphere is moving with a sphere velocity $V_i = -U_i$. We then find the fluid velocity

$$v_i = (H_{ij}^{(s)} + \gamma_o H_{ij}^{(o)})\mathcal{F}_j^{(0)} + O(\gamma_o^2), \quad \text{(B2)}$$

where we introduced the Green's functions $H_{ij}^{(s)}$ and $H_{ij}^{(o)}$ given by

$$H_{ij}^{(s)} = \frac{1}{4\pi\,Re}\,(\delta_{ij}\,\Delta - \nabla_j\nabla_i)\int_0^{Re\,\zeta}\frac{1-\exp(-\xi)}{\xi}\,\mathrm{d}\xi, \quad \text{(B3}a\text{)}$$

$$H_{ij}^{(o)} = \frac{1}{4\pi\,Re}\,(\varepsilon_{ij}\delta_{kl} + \delta_{li}\varepsilon_{jk} - \delta_{lj}\varepsilon_{ik})\,\nabla_k\nabla_l\exp(-Re\,\zeta), \quad \text{(B3}b\text{)}$$

with $\zeta = \frac{1}{2}(r - x_i e_i)$. Since we are interested in the transverse case, we take $\mathcal{F}_i^{(0)}\ell_i = 0$. To compute the inertial corrections, we will follow the method described in Pozrikidis (1996).

This method involves extracting the inertial correction to the Stokes force by looking at how, from the point of view of the first outer solution, the sphere is accelerated by convection. To see this, let us expand v_i for small r using the notation introduced in (4.15). It follows from the matching condition (3.12) that $(v_i)_{-1}$ is given by the zeroth inner solution, which represents Stokes flow. Therefore, $(v_i)_{-1}$ does not depend on Reynolds number. For the sub-leading contribution, we find from (B2) that

$$(v_i)_0 = -\frac{Re}{16\pi}\left[(\delta_{ij} - e_i e_j)\left(\frac{3}{2} - \frac{x_\parallel(4r^2 - x_\parallel^2 - z^2)}{2r^3}\right) + e_i e_j\left(1 - \frac{x_\parallel(x_\parallel^2 + r^2)}{2r^3}\right)\right.$$
$$\left. - \varepsilon_{ij}\gamma_o\left(1 - \frac{x_\parallel(r^2 - z^2)}{2r^3}\right)\right]\mathcal{F}_j^{(0)} + O(\gamma_o^2, Re^2), \tag{B4}$$

where $x_\parallel = e_l x_l$. The right-hand side of (B4) tells us the effective correction to the sphere velocity due to convective corrections. Only the part that is even under $x_i \to -x_i$ can be viewed as such an effective sphere velocity correction, whereas the part that is odd has its effective contribution to the sphere velocity cancelled out. Focusing on the part of (B4) that is even under $x_i \to -x_i$, we find

$$(v_i)_0 = -Re\, J_{ij}\mathcal{F}_j^{(0)} + \text{terms odd under } x_i \to -x_i + O(\gamma_o^2, Re^2), \tag{B5}$$

where J_{ij} was defined in (4.18). We then extract from (B5) the leading-order contribution to the effective sphere velocity V_i', which is given by

$$V_i' = V_i - Re\, J_{ij}\mathcal{F}_j^{(0)}. \tag{B6}$$

We then use the formula for the Stokes force of (3.6) to find the effective Stokes force induced by V_i'. We find

$$\begin{aligned}\mathcal{F}_i'^{(0)} &= -(C_D^{(0)}\delta_{ij} + C_L^{(0)}\varepsilon_{ij})V_j' \\ &= [(C_D^{(0)} + Re\, C_D^{(1)})\delta_{ij} + (C_L^{(0)} + Re\, C_L^{(1)})\varepsilon_{ij}]U_j,\end{aligned} \tag{B7}$$

where

$$C_D^{(1)} = \frac{9\pi}{4} + O(\gamma_o^2), \quad C_L^{(1)} = \frac{9\pi}{16}\gamma_o + O(\gamma_o^3). \tag{B8a,b}$$

Equations (B8*a*,*b*) coincide with (4.24).

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