References

1. J. Stewart, *Essential calculus* (2nd edn), Brooks/Cole (2013).
2. W. Dunham, *Euler: the master of us all*, Mathematical Association of America (1999).

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107.08 An interesting equivalent of squaring the circle

Consider the following task.

> Given the rectangular strip shown in Figure 1(a), construct the point P on the mid-line such that, when the circle with centre P which touches the horizontal edge of the strip is drawn, the area outside the circle at the top of the strip is equal to the area of the segment on the side.

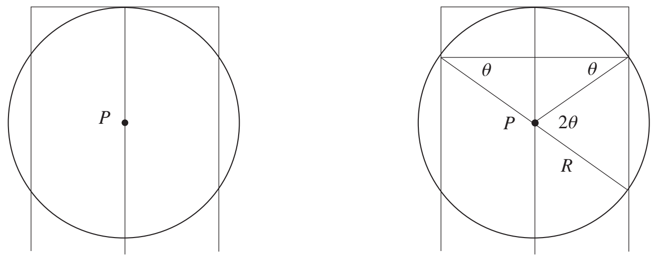

FIGURE 1

With notation added as in Figure 1(b), the area, A_s, of the segment on the side is given by

$$A_s = \tfrac{1}{2}R^2\left(2\theta - \sin 2\theta\right) = R^2\theta - \tfrac{1}{2}R^2 \sin 2\theta.$$

The area, A_t, of the area outside the circle at the top of the strip is given by

$$\begin{aligned} A_t &= \tfrac{1}{2}R^2 \sin(\pi - 2\theta) + 2R\cos\theta \cdot R\left(1 - \sin\theta\right) - \tfrac{1}{2}R^2(\pi - 2\theta) \\ &= \tfrac{1}{2}R^2 \sin 2\theta + 2R^2\cos\theta - R^2 \sin 2\theta - \tfrac{1}{2}R^2\pi + R^2\theta \\ &= 2R^2\cos\theta - \tfrac{1}{2}R^2\pi + R^2\theta - \tfrac{1}{2}R^2 \sin 2\theta. \end{aligned}$$

Equality of areas, $A_s = A_t$, thus necessitates $\cos\theta = \dfrac{\pi}{4}$.

If point P were constructible using ruler and compasses, we could continue the construction as shown in Figure 2 where chord ABC produced is parallel to the horizontal chord in Figure 1(b). Point D is on the line ABC such that $BD = R$. Point E is on the extension of the left-hand edge of the strip with $DE = DA$ and $CEFG$ is a square.

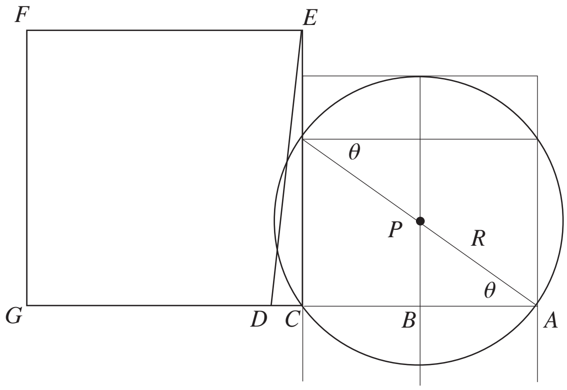

FIGURE 2

Then

$$\begin{aligned} EC^2 &= ED^2 - DC^2 \\ &= AD^2 - DC^2 \\ &= (R + R\cos\theta)^2 - (R - R\cos\theta)^2 \\ &= 4R^2\cos\theta \\ &= \pi R^2, \end{aligned}$$

so that the square *CEFG* has the same area as the circle, which is well known to be impossible. Hence the point *P* cannot be constructed.

(For a very readable and wide-ranging account of the classical unsolvable Greek construction problems, I thoroughly recommend the recently published book, [1]: it's a real gem!)

Acknowledgement

This Note arose from contemplating Catriona Agg's delightful problem from Day 1 of the 2021 on-line MA Conference which involved a configuration like Figure 1(b) with a value of θ very close to the $\cos^{-1}\frac{\pi}{4}$ of this Note.

Reference

1. D. S. Richeson, *Tales of impossibility*, Princeton University Press (2019).

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