ORBITS

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ABSTRACT. Orbits that respect at least three isolating integrals of motion have very special structures in phase space. The main characteristics of this structure are reviewed, and the concrete examples that are provided by orbits in Stäckel potentials, are discussed. Many orbits in general potentials admit three approximate isolating integrals and closely resemble orbits in Stäckel potentials. If the potential is that of an elliptical galaxy with negligible figure rotation, the overall orbital stucture of the potential differs from that of a Stäckel potential only by the presence of a few unimportant families of resonant orbits. However, this elegant picture is shattered by the introduction of non-negligible figure rotation: though substantial regions of phase space may still be occupied by orbits that individually resemble orbits in Stäckel potentials, the overall orbital structure is radically changed by figure rotation, and in a rotating potential significant portions of phase space are given over to chaotic orbits, quite unlike orbits in Stäckel potentials.

1. INTRODUCTION

The emergence a decade ago of the view that elliptical galaxies are normally triaxial bodies, obliged us to discover what kinds of orbits are possible in a given triaxial potential. In this review I shall concentrate exclusively on this question, neglecting for example the extensive work of Richstone and his collaborators (see Richstone 1984), of Kent (1983) and others on the orbital structures of axisymmetric potentials.

The last decade has yielded a good basic understanding of orbits in potentials with figures fixed in inertial space, and this understanding has formed the basis of much recent work on galaxy models. Most of this review is taken up with a summary of the key results obtained in this area. Many of the concepts that arise from this work, especially the concept of orbital tori, are widely applicable in stellar dynamics. However, we should not lose sight of the likelihood that many orbits in elliptical galaxies are significantly influenced by rotation of the potential's figure, and therefore that we shall not be able to construct fully satisfactory galaxy models until we have cracked the much harder problem of motion in a potential with non-negligible figure rotation. I shall mention some pioneering work on this complex problem at the end of the review, but the space allocated to this topic in no way reflects its likely importance.

T. de Zeeuw (ed.), Structure and Dynamics of Elliptical Galaxies, 229–239.

Galaxies are three-dimensional, but two-dimensional orbits are much easier to study (not least because the power of Poincaré's surfaces of section). In the interests of brevity, results that are equally valid for two- and three-dimensional orbits, will be described in terms of n dimensions.

2. REGULAR ORBITS

Simple numerical experiments show that few, if any, orbits in galaxy-like potentials explore the whole "energy" hypersurface $H(\mathbf{x},\mathbf{v}) = E$. The dimensionality of the phase-space subset to which a given orbit is confined can be elucidated by studying the range of velocity vectors $\mathbf{v}$ with which an orbiting particle passes by a particular place $\mathbf{x}$. On an *irregular* orbit, the range of velocities $\mathbf{v}$ at $\mathbf{x}$ is at least one-dimensional. The orbit is said to be *regular* if this range consists of a small number, typically 2–6, of isolated possibilities. In the case of two-dimensional orbits, it is immediately apparent from a simple tracing of the orbit, whether the orbit is regular or irregular; irregular orbits *look* messy [see, for example, Fig. 3 of Binney (1982)].

Since the value of $\mathbf{v}$ at a given point $\mathbf{x}$ on an n-dimensional regular orbit is determined up to a few-fold degeneracy, the $2n$ phase-space coordinates $(\mathbf{x},\mathbf{v})$ of points on the orbit must be constrained by n relations of the form $H \equiv I_1(\mathbf{x},\mathbf{v}) = i_1 \equiv E, \dots, I_n(\mathbf{x},\mathbf{v}) = i_n$, where the *isolating integrals* I_k are smooth single-valued functions of the phase-space coordinates. Conversely, along an irregular orbit, fewer functional relationships constrain the coordinates $(\mathbf{x},\mathbf{v})$; One usually has $H = E$ and one or more inequalities $i'_k < I_k < i_k$.

The phase-space structure of regular orbits is strongly constrained by the nature of Hamilton's equations of motion. From the mere existence of the n isolating integrals $I_1, \dots, I_n$, one may demonstrate the following (Arnold 1978):

(i) In $2n$-dimensional phase space the orbit lies on an n-dimensional surface which is topologically equivalent to an n-torus. In other words, a continuous one-to-one map exists of the orbital surface onto the unit cube of n-dimensional Euclidean space with opposite faces identified.

(ii) The *action integrals* $J_\gamma \equiv (2\pi)^{-1} \oint_\gamma \mathbf{v} \cdot \mathrm{d}\mathbf{x}$ around a given orbital torus are equal for any two closed paths γ on the torus that can be continuously deformed into one another by motions confined to the torus.

(iii) It is possible to incorporate n of these action integrals into a system of *angle-action* coordinates $(\boldsymbol{\theta},\mathbf{J})$ for that part of the phase space in which the I_k are integrals. In this portion of phase space, the action integrals label the orbital tori, while position on any torus is specified by the n angle variables $\theta_1, \dots, \theta_n$. The coordinate system $(\boldsymbol{\theta},\mathbf{J})$ is canonical. In particular, a small element of phase-space volume is $\mathrm{d}^n\mathbf{x}\mathrm{d}^n\mathbf{v} = \mathrm{d}^n\boldsymbol{\theta}\mathrm{d}^n\mathbf{J}$ and Hamilton's equations $\dot{\boldsymbol{\theta}} = [\boldsymbol{\theta}, H]$, $\dot{\mathbf{J}} = [\mathbf{J}, H]$ apply. The Hamiltonian H, being constant on orbital tori, is a function $H(\mathbf{J})$ of the actions only. Hence Hamilton's equations $\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}(\mathbf{J}) \equiv (\partial H/\partial \mathbf{J})$ integrate immediately to $\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \boldsymbol{\omega} t$.

The Cartesian coordinates $(\mathbf{x},\mathbf{v})$ are periodic functions of the θ_k with period 2π; $\mathbf{x}(\boldsymbol{\theta} + 2\pi\mathbf{m}, \mathbf{J}) = \mathbf{x}(\boldsymbol{\theta},\mathbf{J})$ for any integer vector $\mathbf{m}$. Hence we may expand $\mathbf{x}$ as a Fourier series $\mathbf{x} = \sum_{\mathbf{m}} \mathbf{X}_{\mathbf{m}}(\mathbf{J}) \exp[i\mathbf{m} \cdot \boldsymbol{\theta}]$. Substituting for $\boldsymbol{\theta}(t)$, we obtain the position $\mathbf{x}$ of an orbiting particle as a *quasi-periodic* function of time: $\mathbf{x}(t) = \sum_{\mathbf{m}} \mathbf{X}'_{\mathbf{m}}(\mathbf{J}) \exp[i\mathbf{m} \cdot \boldsymbol{\omega} t]$, where $\mathbf{X}'_{\mathbf{m}} \equiv \mathbf{X}_{\mathbf{m}} \exp[i\mathbf{m} \cdot \boldsymbol{\theta}(0)]$. Thus the Fourier decomposition $\widetilde{\mathbf{x}}(\omega)$ of the position vector $\mathbf{x}(t)$ along a regular orbit consists of a series of discrete lines. The frequencies $\mathbf{m}\cdot\boldsymbol{\omega}$ at which these lines occur are integer combinations of three fundamental frequencies ω_k, and by deducing the integer vector $\mathbf{m}$ associated with each line, one can reconstruct the angle representation $\mathbf{x}(\boldsymbol{\theta})$ from the time evolution $\mathbf{x}(t)$ (Binney & Spergel 1984). This reconstruction is useful, because $\mathbf{x}(\boldsymbol{\theta})$ contains much more information than $\mathbf{x}(t)$ (see also Ratcliff *et al.* 1984).

Regular orbits fall naturally into *families*. Each orbit family is parented by a sequence of stable closed orbits. In a realistic non-rotating triaxial potential there are three principal orbit families (Schwarzschild 1979): the box family, whose parents are the long-axis orbits; the short-axis tube family, which is parented by the closed short-axis loop orbits, and the long-axis tube family which has the closed long-axis loop orbits for its parents. (De Zeeuw (1985) additionally subdivides the long-axis tube family into inner- and outer-tubes.) At a given energy we may think of the orbital tori of each family as wrapped around the degenerate wire-like torus of the parent orbit of that energy, much as the insulator and sheath of a coaxial cable encircle the cable's central wire [see Fig. 1 of Lynden-Bell (1962)].

The orbits of each orbit family form a n-dimensional continuum. A useful graphical display of these continua is obtained by treating n independent action integrals over the orbits as Cartesian coordinates, and thus identifying the orbit that has actions $(J_1,\dots,J_n)$ with the corresponding point in an n-dimensional *action space*. The orbits associated with neighbouring points in action space, occupy adjacent regions of phase space. Furthermore, the volume of phase space that is occupied by the orbits whose representative points lie with a volume element $\mathrm{d}^n\mathbf{J}$ in action space, is $(2\pi)^n\mathrm{d}^n\mathbf{J}$. Consequently, action space gives a fair representation of the *a priori* probability of a group of orbits.

If the frequencies ω_k are nearly everywhere incommensurable (as will usually be the case) a strengthened Jeans theorem applies: the distribution function of a steady-state galaxy in which almost all orbits are regular with incommensurable frequencies, may be presumed to be a function $f(\mathbf{J})$ of the actions only. Furthermore, the number of stars with actions in the range $\mathrm{d}^n\mathbf{J}$ is $\mathrm{d}N=(2\pi)^n f(\mathbf{J})\mathrm{d}^n\mathbf{J}$, so f is up to a constant, simply the density of stars in action space.

In general, the orbits of different orbit families have to be accomodated in different action spaces. Schwarzschild's principal families of orbits in a non-rotating potential form an exception to this rule, however; it is possible to define the actions of orbits of the principal families in such a way that the continua of all three families may be fitted together into a single action space, the principal action space. Any additional orbit family gives rise to a zone of missing actions in the principal action space. The volume of this zone is proportional to the phase-space volume occupied by the subfamily's orbits, but actions cannot be assigned to the family's orbits in such a way that they occupy the zone of missing actions in the principal action space (Binney & Spergel 1984).

2.1 Resonances and Extra Integrals

If the fundamental frequencies ω_k of a regular orbit are rationally related, that is, if we have $\mathbf{m}\cdot\boldsymbol{\omega}=0$ for integer vector $\mathbf{m}$, then the orbit does not explore all of the torus $\{H=E,\dots,I_n=i_n\}$ to which it is confined. The restriction of the orbit to a subset of its torus may be attributed to an extra isolating integral, $I_{n+1}\equiv\mathbf{m}\cdot\boldsymbol{\theta}$. If $n=2$, the orbit is closed. If $n=3$ the orbit closes only if a second rational relationship holds, $\mathbf{m}'\cdot\boldsymbol{\omega}=0$ for $\mathbf{m}'\neq\mathbf{m}$. Familiar examples of these phenomena are furnished by motion in spherical potentials; the four isolating integrals (H, L_x, L_y and L_z) may be decomposed into three, say H, $|\mathbf{L}|$ and L_z that specify a torus, and a fourth, say L_x/L_y that arises because the orbit has only two independent frequencies, the radial frequency κ and the azimuthal frequency Ω. If κ and Ω happen to be rationally related, as in Kepler ($\kappa = \Omega$) or harmonic ($\kappa = 2\Omega$) motion, a fifth integral arises (the position angle of the apocentre), and the orbit closes.

2.2 Stäckel Potentials

Recently, de Zeeuw (1985) has shown that potentials studied a century ago by Jacobi and Stäckel provide analytic models of the most important features of the orbital structures of non-rotating elliptical-like potentials. In particular, (i) essentially all orbits in a Stäckel potential are confined to tori; (ii) motion on the tori is quasiperiodic; (iii) in realistic cases, all orbits belong to the same three orbit families as orbits earlier integrated numerically by Schwarzschild (1979). De Zeeuw's discovery of handy analytic models of orbits in non-rotating galaxy-like potentials, has opened up a rich vein of exploration. It is worth taking a little time to review the main features of Stäckel orbits.

Orbits in Stäckel potentials are intimately connected with systems of confocal ellipsoidal coordinates. In two dimensions these coordinates are most neatly expressed by writing ($x = \Delta_1 \sinh u \cos v$, $y = \Delta_1 \cosh u \sin v$), where (x, y) are the usual Cartesian coordinates, $\Delta_1 > 0$ is a constant, and u and v, which are constant on ellipses and hyperbolae respectively, are the new coordinates. In three dimensions, ellipsoidal coordinates ($\lambda \geq 0 \geq \mu \geq -{\Delta_1}^2 \geq \nu \geq -{\Delta_2}^2$) may be defined as the roots for τ of the cubic

$$\frac{x^2}{\tau} + \frac{y^2}{\tau + {\Delta_1}^2} + \frac{z^2}{\tau + {\Delta_2}^2} = 1 \tag{1}$$

where $0 \leq \Delta_1 \leq \Delta_2$ are constants. λ is constant on ellipsoids which at large $|\mathbf{x}|$ approximate spheres of radius $|\mathbf{x}| \simeq \sqrt{\lambda}$. In the (x, y) plane, $\lambda = {\Delta_1}^2 \sinh^2 u$, $\mu = -{\Delta_1}^2 \cos^2 v$. At large $|\mathbf{x}|$, μ and ν specify angular position, μ depending mainly on azimuth ϕ and ν depending most strongly on colatitude θ (de Zeeuw 1985, Appendix A).

Let p_τ be the momentum canonically conjugate to $\tau = \lambda$, μ or ν. Then the remarkable property of Stäckel potentials is that on an any orbit in one of these potentials (and these alone!), p_τ is a function of the corresponding coordinate alone. In fact

$$p_\tau^2(\tau) = 2\Big(E - \frac{i_2}{\tau} - \frac{i_3}{\tau + {\Delta_2}^2} + G(\tau)\Big)\Big/(\tau + {\Delta_1}^2), \tag{2}$$

where E, i_2 and i_3 are the values of the energy and two non-classical integrals on the orbit, and G defines the potential Φ through

$$\Phi(\mathbf{x}) = -\sum_{\lambda \to \mu \to \nu} \frac{\lambda(\lambda + {\Delta_2}^2)G(\lambda)}{(\lambda - \mu)(\lambda - \nu)}. \tag{3}$$

The real-space boundaries of the orbits are the curves on which one of the momenta vanishes; hence all orbits are bounded by coordinate surfaces. The number of

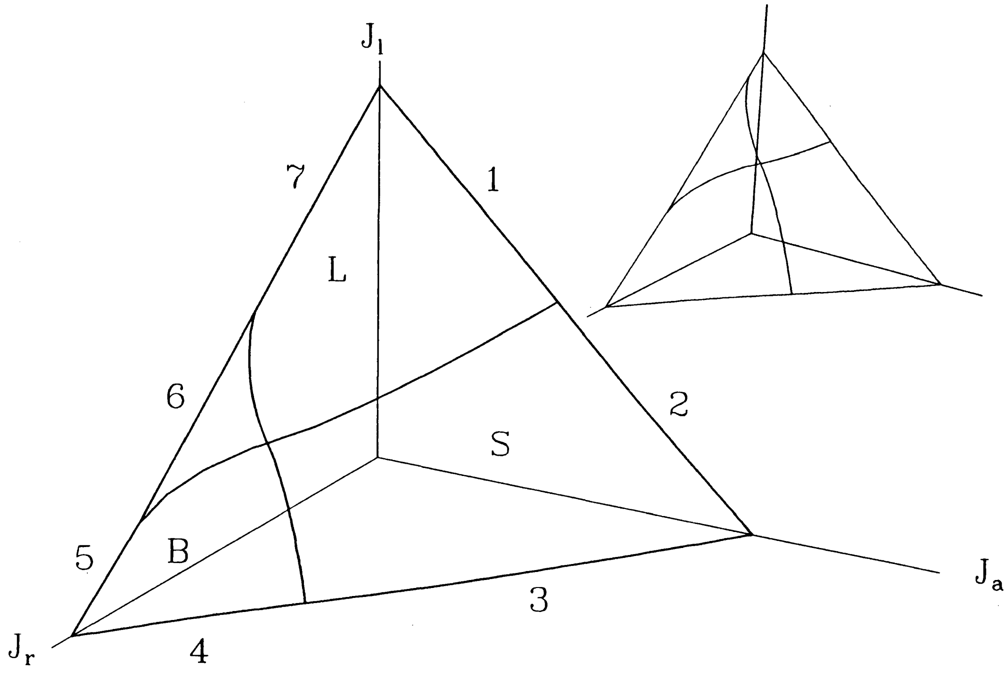

Figure 1. The partition of action space between orbit families in the potential of the perfect ellipsoid that has unit mass and axes in the ratios $b/a = 0.75$, $c/a = 0.5$. The small and large triangles are surfaces of constant energy; $E = -0.4$ and $E = -0.2$ respectively. They are plotted on the same scale and divided into the domains of the boxes (B), short-axis tubes (S) and the long-axis tubes (L). The quantity γ plotted in Fig. 2 is the ratio of the length of the portion of the edge of the triangle marked "3" to the sum of the lengths of "3" and "4".

possible momentum, and therefore velocity, vectors at any point $\mathbf{x}$ on the orbit ranges from 1–8 depending on how many of the p_τ change sign on the orbit. On box orbits, all momenta change sign, and eight velocity vectors are possible at any given point, while on tube orbits, only two momenta, p_λ and one other, change sign, so only four velocities occur at a point. The number of roots of the equations $p_\tau = 0$, and thus the family to which a given orbit belongs, depends on the values of the constants E, $\imath_2$ and $\imath_3$.

The action integrals which enable all orbits to be represented in a common action space, are

$$J_\tau(\mathbf{x},\mathbf{v}) = J_\tau(H, I_2, I_3) \equiv \frac{k}{\pi}\int_{\tau_{\min}}^{\tau_{\max}} |p_\tau(\tau)|\,\mathrm{d}\tau, \quad \text{where} \quad k = \begin{cases} 1 & \text{for } \tau = \lambda \\ 2 & \text{for } \tau = \mu \text{ or } \nu, \end{cases} \tag{4}$$

and $\tau_{\min}$ and $\tau_{\max}$ are the smallest and largest values of τ along the orbit. Unfortunately no comparably simple expressions are available for the angle coordinates θ_τ as functions of the phase-space coordinates.

While Stäckel potentials provide invaluable models of galaxy potentials, they are subject to significant limitations. The most important of these arise from the speed with which the isopotential surfaces become round at large $|\mathbf{x}|$. If the po-

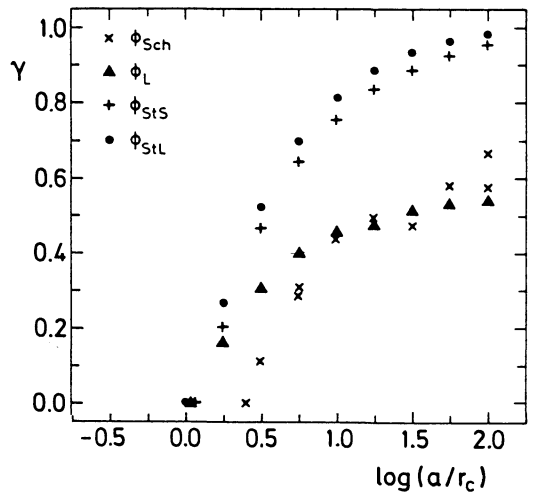

Figure 2. The quantity γ defined in the caption to Fig. 1 plotted for four potentials against $a(E)$, the distance along the potential's major axis at which the potential $\Phi = E$. r_c is the core radius of the body generating the potential. In their cores, the bodies generating the four potentials $\Phi_{\rm Sch}$, $\Phi_{\rm L}$, $\Phi_{\rm StS}$ and $\Phi_{\rm StL}$ all have axis ratios similar to those of the perfect ellipsoid for which Fig. 1 is plotted (see Gerhard & Binney 1985 for details). The potential $\Phi_{\rm Sch}$ is an approximation to Schwarzschild's (1979) potential ($\rho \propto r^{-3}$ for $r \gg r_c$), and $\Phi_{\rm StS}$ is the Stäckel potential that most closely approximates it. Similarly, $\Phi_{\rm StL}$ is a Stäckel approximation to the potential $\Phi_{\rm L}$ generated by a body with roughly constant axis ratios and density profile $\rho \propto r^{-2}$ at $r \gg r_c$.

tential were axisymmetric, all orbits would belong to one of the tube families. (Short-axis tubes if the potential were oblate, and long-axis tubes in the prolate case.) Hence the fraction of orbits of a given energy E that are boxes is a natural measure of the importance of the potential's triaxiality at radii characteristic of E. Figure 1 shows the action-space boundaries between the different orbit families for two values of E in a particular Stäckel potential. Notice (i) that the box-orbit fraction decreases quite rapidly with increasing energy, and (ii) that the shape of the box domain is almost independent of energy. Consequently, at any energy the box-orbit fraction is roughly proportional to $(1-\gamma)^2$, where $\gamma(E)$ is the fraction of orbits in the potential's equatorial plane that are loops. (These orbits fall in the (J_r, J_a) plane in action space.) In Figure 2 γ is plotted against a measure a of orbital energy for two Stäckel and two general potentials. One sees that with increasing energy, $(1-\gamma)^2 \to 0$ rather rapidly in the case of the Stäckel potentials, but slowly, if at all, for the other potentials plotted.

The concentration of the effects of triaxiality to the centres of Stäckel potentials is mirrored in conditions which the density profile $\rho(\mathbf{x})$ of a body must satisfy if it is to generate, via Poisson's equation, a Stäckel potential. In fact, if $\rho(\mathbf{x})$ does generate a Stäckel potential, then (i) then $\rho(\mathbf{x})$ must have a homogeneous core, *i.e.* $\rho(\mathbf{x}) \simeq$ constant for $|\mathbf{x}| < a$, a constant; and (ii) at $|\mathbf{x}| \gg a$, the non-spherical part of ρ, $\rho_2 \equiv \sqrt{\sum_m \left| \int \mathbf{Y}_2^m(\mathbf{\Omega})\rho(\mathbf{x})d^2\mathbf{\Omega}\right|^2}$, must fall off as $\rho_2 \propto |\mathbf{x}|^{-4}$. In particular, the ellipticity of a surface of constant ρ is independent of $|\mathbf{x}|$ only if $\rho \propto |\mathbf{x}|^{-4}$ for

$|\mathbf{x}| \gg a$, as in de Zeeuw's perfect ellipsoids. If we require that the density along the body's minor axis falls off as $\rho \propto |\mathbf{x}|^{-n}$, with $n \approx 3$ as is suggested by Hubble's law, or $n \approx 2$ as would be required to generate a flat rotation curve, then the body can generate a Stäckel potential only if it rapidly becomes round for $|\mathbf{x}| > a$. Thus bodies that generate Stäckel potentials are characterized by cores in which stars move nearly harmonically, and outer envelopes, in which the potential is either near Keplerian (if $n \geq 3$) or dominated by a locally spherical density distribution (if $n < 3$). Unfortunately there are many problems in galactic dynamics for which it is essential to consider models which have either singular central densities (see Gerhard, this meeting) or massive, strongly aspherical outer envelopes (*e.g.* Binney *et al.* 1986), and the application of Stäckel models to such problems can be frustrating.

3. ORBITS IN SLOWLY-ROTATING POTENTIALS

Rotation of the figure of a triaxial potential has far-reaching consequences for both regular and irregular orbits. Observationally, the most important effect of figure rotation is to imbue the vital triaxiality-supporting box orbits with a definite sense of circulation about the potential's rotation axis (usually assumed to coincide with the galaxy's shortest axis) (Schwarzschild 1982). The work of Merritt (1980) and Vietri (1985) has shown that strongly triaxial systems cannot achieve significant ratios v/σ of rotational and random velocities unless rotation of the potential has thus enabled the boxes to contribute to the overall stellar circulation.

Rotation affects the work of the galaxy modeler in two further ways: (i) in the presence of rotation the action spaces of Schwarzschild's principal orbit families no longer fit neatly together to form a single principal action space (Binney & Spergel 1984); (ii) some distance from the centre, figure rotation eliminates the majority of orbital tori in favour of a sea of stochastic orbits.

4. CHAOS

Since the frequencies ω_k are continuous functions of the phase-space coordinates, a general potential supports an infinite number of closed orbits. Consideration of the stability characteristics of these orbits gives some insight into the way in which the regular orbital structure exemplified by Stäckel potentials can dissolve into chaos.

If one linearizes the equations of motion around a closed orbit of period T, one obtains a set of coupled linear differential equations with coefficients that are periodic functions of time. By Floquet's theorem (*e.g* Margenau & Murphy 1956), any solution of these linear equations can be written as a sum of solutions of the form $Xe^{\mu t}P(t)$, where X and μ are constants, and P has period T: $P(t+T) = P(t)$. There are generally $(n-2)$ possible non-zero values of μ. The orbit is unstable if the real part of any of the μ's is positive. The orbit is stable if at least one μ is non-zero, and every such μ is pure imaginary. A stable closed orbit, is always a parent of an orbit family.

There are infinitely many closed orbits of any given energy E in a Stäckel potential, but *no* subsidiary orbit families, because all but 3–6 of these closed orbits are neutrally stable (every $\mu = 0$). This is a *very* special circumstance. In less special potentials, there are infinitely many of both stable and unstable closed orbits. In many cases of astrophysical interest, a few subsidiary families are able to push the principal tori back enough to gain a significant foothold in phase space, but the families parented by all other stable closed orbits are too small to be seen in a quick survey of possible orbits; these subsidiary families live like lice squeezed flat between the barely ruffled surfaces of the principal families. Fig. 7 of Binney (1982) shows an example of this phenomenon.

Now, just as stable closed orbits on the tori of a principal family give birth to subsidiary families, so stable closed orbits can arise on the tori of a subsidiary family, and call into being orbit families of the third generation. Closed orbits on tori of the third-generation spawn fourth-generation orbit familes, and so on *ad infinitum*— Gustavson (1966) shows a concrete example of this hierarchy. Little of the original regular phase-space structure survives the endless formation of orbit families unless the proportion of phase space that is claimed by families of any generation m is significantly less than that claimed by families of generation $(m-1)$.

At present there is no inexpensive way to estimate how much of phase space a particular resonance will seize. If a moderately triaxial potential is stationary in inertial space, resonances are unimportant, but resonances rapidly create havoc in phase space once we set the potential rotating. A partial explanation of this phenomenon is as follows. By setting the potential rotating at angular frequency Ω, we shift some of the orbital frequencies ω_k by $\pm\Omega$ while leaving others unchanged, and thus call into being a whole new set of resonances. Contopoulos & Mertzanides (1977), Athanassoula *et al.* (1983), Teuben & Sanders (1985) and others have studied these resonances in planar bars, while Binney (1981), Magnenat (1982), Mulder & Hooimeyer (1983) and Pfenniger (1984) have studied resonances that involve the motion perpendicular to the potential's equatorial plane. Evidently, these resonances of rotating potentials are much more effective at breaking up tori of the principal families than are the resonances of non-rotating potentials, but no simple mechanism has been identified to date.

The study of the generation of chaos by resonances, provides endless entertainment for analysts and computer enthusiast alike. In fact, the detailed structure of phase space at the onset of chaos is so fascinating, that before plunging deeply into its study, it is well to decide what we *need* to know about the chaotic regions of phase space before we can build serviceable galaxy models. My list would include:

(i) What are the characteristics of a potential Φ that determine whether its phase space is largely regular or largely chaotic? One possible answer to this question is "the distance of Φ from the nearest integrable potential Φ_I". Unfortunately, along any given orbit, *any* Hamiltonian lies arbitrarily close to an integrable Hamiltonian (Contopoulos 1963), so this is not a very satisfactory answer as it stands. Evidently the regular orbital structures of some integrable Hamiltonians are more easily disrupted than those of others. Also the effectiveness of a potential perturbation $\delta\Phi$ in introducing chaos depends on more than just the magnitude of $\delta\Phi$ (Gerhard 1985). Figure rotation seems to be an especially potent form of perturbation.

(ii) How should we incorporate stochastic orbits into our galaxy models? On closer examination, stochastic orbits often prove to be nearly quasi-periodic for several orbital periods at a time, and seem chaotic only because they switch abruptly from one quasi-periodic structure to another in an apparently random way. Are numerical experiments trustworthy here? What are the statistical characteristics of this switching? Can stochastic orbits be treated as linear combinations of regular orbits?

(iii) How many fundamentally different stochastic orbits are there at each energy? Goodman & Schwarzschild (1981) found that a single stochastic sea in Schwarzschild's (1979) model galaxy actually contains orbits that differ from one another on time-scales of a Hubble time. As Petrou (1984) and Pfenniger (1985) have pointed out, partial barriers, or "cantori" in phase-space can contain stochastic orbits for many orbital periods before suddenly releasing them into a wider volume of phase space. In the long run, this process gives rise to "Arnold diffusion" through phase space. How should we describe this sort of process mathematically, and what are its astronomical consequences?

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DISCUSSION

Toomre: What is the best evidence you know that the shapes of triaxial galaxies (which I have finally gotten used to!) actually *revolve* appreciably in space?

Binney: In a non–rotating potential, the box orbits, which form the backbone of a triaxial galaxy, do not contribute to the circulation. Here one can get the stars to circulate only by weighting tube orbits, which are fundamentally opposed to the bar. The degree of their opposition grows with the strength of the bar. Merritt (1980, *Astrophys. J. Suppl.*, **63**, 435) found that for Schwarzschild's axis ratios $a : b : c = 1 : 0.625 : 0.5$, only modest rotation could be found in this way. Thus if rapidly–rotating galaxies are strongly triaxial near their core, their figures must rotate. I suspect that there are two main classes of rapidly rotating galaxies: (a) those which are strongly triaxial at small radii and have rapidly rotating figures which become axisymmetric well in advance of the Inner Lindblad Resonance; (b) those which are axisymmetric at small radii and become triaxial with low pattern speed far from the center.

Ostriker: Am I correct in thinking that the limitation (for non–rotating figures) to small regions of substantial triaxiality for Stäckel potentials is not fundamental. Real galaxies could have large (in units of core radius) triaxial parts but one simply could not compute the equilibrium analytically.

Binney: Yes. The logarithmic potential investigated by many people has a beautifully regular orbital structure to very large radii. But at large radii this orbit structure diverges from that characteristic of Stäckel potentials in that it supports a host of subsidiary families in the box domain. We still haven't figured out the details of galaxy building with potentials of this sort, but I see no fundamental problem.

Illingworth: My question relates to Alar Toomre's question. (a) What general constraints can be placed on the amount of figure rotation for a given (e.g., observed) level of rotation? (b) Could figures counter–rotate?

Binney: As far as I am aware, the literature contains no satisfactory study of the problem you raise in (a). Vietri's study of triaxial spheroids for our Galaxy is the best reference I know. As to (b), in principle figures can counter–rotate (Freeman's analytic models do) but never by very much and I rate the probability of such models very low—see Vietri for details.

White: We know that the bars in barred spirals are rapidly rotating triaxial systems. If nature can do the trick in this case, how confident can we be that it is not possible for more elliptical–like systems?

Binney: The density profiles of bars and ellipticals are quite different. Bars in disks have fairly constant surface density inside pretty sharp edges, while ellipticals have smooth, steep density profiles. Thus in a bar, the orbital frequencies can all be comparable with the pattern speed Ω, while in an elliptical the range in frequencies is very great. Furthermore, the sharp edges of bars can be identified

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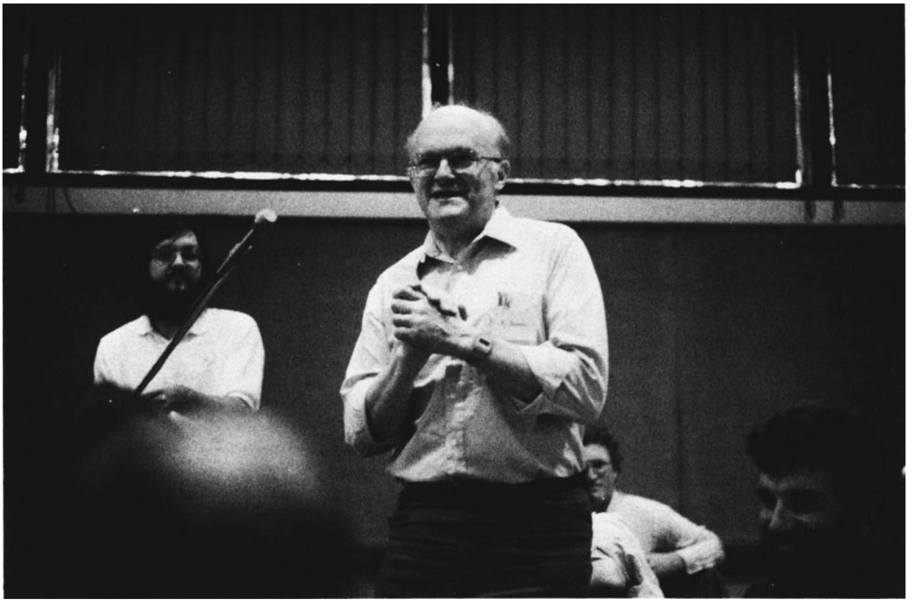
with a resonance, perhaps corotation or the Inner Lindblad Resonance. Ellipticals seem to have no comparable characteristic radius.

Statler: Levison & Richstone find in the non–rotating logarithmic potential, strange orbits that look like boxes but do not line up with the symmetry planes. Are these resonances, or nearly–stochastic orbits, or something else?

Binney: Many resonances arise in the logarithmic potential when the semi–axes of the zero–velocity surfaces are much larger than the potential's core radius (Levison and Richstone *do* use a finite core contrary to what is said in their poster). Their weird orbits are almost certainly trapped, or partially trapped, around such resonances.

Sellwood: N–body simulations by Wilkinson & James, Gerhard etc. took a remarkably long time to settle to a steadily rotating potential. Is it likely that real elliptical galaxies have settled yet to uniform figure rotation?

Binney: No. In studying steady–state models we are surely doing no more than getting zeroth–order approximations to reality. As yet we know little of how a steady–state system will respond to perturbations. I expect many will be stable, and others will display only slowly–damped long–period oscillations.



Toomre asking a question, in his usual style.

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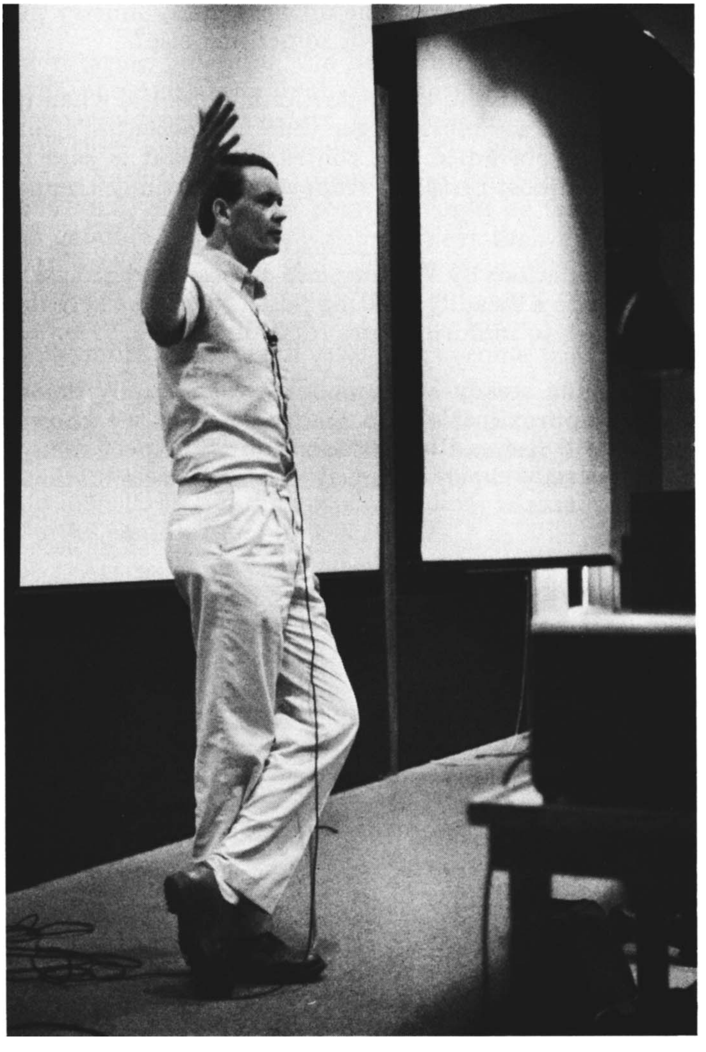

James Binney explains that orbits are simple.

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