Canad. Math. Bull. Vol. **57** (4), 2014 pp. 708–720
http://dx.doi.org/10.4153/CMB-2014-004-9

Strong Asymptotic Freeness for Free Orthogonal Quantum Groups

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Abstract. It is known that the normalized standard generators of the free orthogonal quantum group O_N^+ converge in distribution to a free semicircular system as $N \to \infty$. In this note, we substantially improve this convergence result by proving that, in addition to distributional convergence, the operator norm of any non-commutative polynomial in the normalized standard generators of O_N^+ converges as $N \to \infty$ to the operator norm of the corresponding non-commutative polynomial in a standard free semicircular system. Analogous strong convergence results are obtained for the generators of free unitary quantum groups. As applications of these results, we obtain a matrix-coefficient version of our strong convergence theorem, and we recover a well-known L^2-L^∞ norm equivalence for non-commutative polynomials in free semicircular systems.

1 Introduction

Given a closed subgroup G of the compact Lie group U_N of $N \times N$ unitary matrices over $\mathbb{C}$, a fundamental mathematical problem is the computation of *polynomial integrals* over G. That is, we consider the $*$-algebra $\mathrm{Pol}(G) \subseteq C(G)$ of polynomial functions on G generated by the N^2 standard coordinate functions

$$\{u_{ij}\}_{1\le i,j\le N} \subset C(G); \quad u_{ij}(g) = (i,j)\text{-th coordinate of the matrix } g \in G,$$

and seek to evaluate the integrals

$$h_G(f) = \int_G f(g)\,dg \quad \big(f \in \mathrm{Pol}(G)\big),$$

where dg denotes the Haar probability measure on G. Equivalently, this amounts to determining the joint distribution of the standard coordinates $\{u_{ij}\}_{1\le i,j\le N}$, viewed as bounded random variables in $L^\infty(G, dg)$.

The need to compute polynomial integrals for various examples of compact matrix groups arises in many areas of mathematics and physics, including group representation theory, statistical physics, random matrix theory and free probability. For most infinite subgroups $G \subseteq U_N$, the evaluation (or even the approximation) of arbitrary polynomial integrals is a non-trivial task. Even for the most natural examples, such as the orthogonal and unitary groups O_N and U_N, the computation of polynomial integrals remains an active area of research. See, for example, [1, 14, 16, 25].

Received by the editors October 9, 2013; revised January 3, 2014.
Published electronically May 7, 2014.
This research was supported by an NSERC Postdoctoral Fellowship.
AMS subject classification: **46L54**, 20G42, 46L65.
Keywords: quantum groups, free probability, asymptotic free independence, strong convergence, property of rapid decay.

In the context of polynomial integrals over O_N or U_N, one is often interested in their behavior in the large N limit. In this regime, the calculations are simplified by the fact that the normalized random variables $\{\sqrt{N}u_{ij}\}_{1\le i,j\le N} \subset L^\infty(O_N, dg)$ (respectively $L^\infty(U_N, dg)$) are asymptotically independent and identically distributed $N(0,1)$ real (respectively complex) Gaussian random variables; see [17]. For example (in the orthogonal case), if $\mathcal{G} = \{g_{ij}\}_{i,j\in\mathbb{N}}$ denotes an i.i.d. $N(0,1)$ real Gaussian family on some standard probability space (Ω, μ) with expectation $\mathbb{E} = \int_\Omega \cdot \, d\mu$, then for any (non-commutative) polynomial $P \in \mathbb{C}\langle X_{ij} : i, j \in \mathbb{N}\rangle$, we have

$$\lim_{N\to\infty} h_{O_N}\big(P(\{\sqrt{N}u_{ij}\}_{1\le i,j\le N})\big) = \mathbb{E}\big(P(\mathcal{G})\big).$$

The above equality essentially says that for large N, the value of a polynomial integral over O_N can be approximated by the corresponding (often simpler to compute) Gaussian integral. An analogous convergence statement holds for polynomial integrals over U_N.

In [32], S. Wang introduced natural non-commutative analogues of the orthogonal and unitary groups: the *free orthogonal and free unitary quantum groups* O_N^+ and U_N^+. These objects are compact quantum groups (in the sense of S. Woronowicz [35]) and are given in terms of the pairs

$$O_N^+ = \big(C(O_N^+), \Delta_o\big) \quad \text{and} \quad U_N^+ = \big(C(U_N^+), \Delta_u\big),$$

where $C(U_N^+)$ is the universal C^*-algebra generated by N^2 elements $\{v_{ij}\}_{1\le i,j\le N}$ subject to the relations which make the matrices $V = [v_{ij}]$ and $\bar{V} = [v_{ij}^*]$ unitary in $M_N\big(C(U_N^+)\big)$, and $C(O_N^+) = C(U_N^+)/\langle v_{ij} = v_{ij}^*\rangle$. That is, $C(O_N^+)$ is the universal C^*-algebra generated by N^2 self-adjoint elements $\{u_{ij}\}_{1\le i,j\le N}$ subject to the relations which make the matrix $U = [u_{ij}]$ unitary. The *coproduct* $\Delta_u \colon C(U_N^+) \to C(U_N^+) \otimes_{\min} C(U_N^+)$ is the unique unital C^*-algebra homomorphism defined by

$$\Delta_u(v_{ij}) = \sum_{k=1}^N v_{ik} \otimes v_{kj}, \quad 1 \le i, j \le N,$$

giving $C(U_N^+)$ a (bisimplifiable) C^*-bialgebra structure. The formula for the coproduct $\Delta_o \colon C(O_N^+) \to C(O_N^+) \otimes_{\min} C(O_N^+)$ is the obvious analogue. Note that the C^*-algebras $C(O_N^+)$ and $C(U_N^+)$ are free analogues of the commutative function algebras $C(O_N)$ and $C(U_N)$, respectively.

Since $\mathbb{G} = O_N^+, U_N^+$ is a compact quantum group, it admits a *Haar state* $h_{\mathbb{G}} \colon C(\mathbb{G}) \to \mathbb{C}$ [35], which is the non-commutative analogue of the left and right translation-invariant Haar measure on a compact group. More precisely, $h_{\mathbb{G}}$ is the unique state satisfying the following Δ-bi-invariance condition

$$(h_{\mathbb{G}} \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes h_{\mathbb{G}})\Delta(a) = h_{\mathbb{G}}(a)1_{C(\mathbb{G})}, \quad a \in C(\mathbb{G}).$$

Note that for O_N^+ and U_N^+, the Haar state is tracial [32]. As a consequence of the existence of the Haar state, one can also consider in this non-commutative context the problem of computing polynomial integrals over O_N^+ and U_N^+. More precisely, if

$\{u_{ij}\}_{1\le i,j\le N}$ and $\{v_{ij}\}_{1\le i,j\le N}$ denote the generators of $C(O_N^+)$ and $C(U_N^+)$, respectively, then we want to compute all joint $*$-moments

$$h_{O_N^+}(u_{i(1)j(1)}u_{i(2)j(2)}\cdots u_{i(k)j(k)}),\quad 1\le i(r),\, j(r)\le N, 1\le r\le k, k\in\mathbb{N},$$
$$h_{U_N^+}(v_{i(1)j(1)}^{\epsilon(1)}v_{i(2)j(2)}^{\epsilon(2)}\cdots v_{i(k)j(k)}^{\epsilon(k)}),\quad 1\le i(r),\, j(r)\le N, \epsilon(r)\in\{1,*\},$$
$$1\le r\le k,\; k\in\mathbb{N}.$$

As in the classical case, the precise computation of these moments is extremely difficult for a fixed dimension N. However, as $N\to\infty$, T. Banica and B. Collins [4] have shown that joint distributions of $\{\sqrt{N}u_{ij}\}_{1\le i,j\le N}$ and $\{\sqrt{N}v_{ij}\}_{1\le i,j\le N}$ are modeled by the free probability analogues of real (respectively complex) Gaussian systems. Before stating their result, we remind the reader of some basic facts about free probability. For more information on these and related concepts, we refer the reader to the monograph [26].

Definition 1.1

(i) A *non-commutative probability space* (NCPS) is a pair (A,φ), where A is a unital C^*-algebra and $\varphi\colon A\to\mathbb{C}$ is a state (*i.e.*, a linear functional such that $\varphi(1_A)=1$ and $\varphi(a^*a)\ge 0$ for all $a\in A$). Elements $a\in A$ are called *random variables*.

(ii) Let (A,φ) be a NCPS. A family of $*$-subalgebras $\{A_r\}_{r\in\Lambda}$ of A is said to be *freely independent (or free)* if the following condition holds: for any choice of indices $r(1)\ne r(2), r(2)\ne r(3),\dots,r(k-1)\ne r(k)\in\Lambda$ and any choice of centered random variables variables $x_{r(j)}\in A_{r(j)}$ (*i.e.*, $\varphi(x_{r(j)})=0$), we have the equality

$$\varphi(x_{r(1)}x_{r(2)}\cdots x_{r(k)})=0.$$

(iii) A family of random variables $\{x_r\}_{r\in\Lambda}\subset(A,\varphi)$ is *free* if the family of unital $*$-subalgebras

$$\{A_r\}_{r\in\Lambda};\quad A_r:=\operatorname{alg}(1,x_r,x_r^*),$$

is free in the above sense.

(iv) A family of random variables $S=\{s_r\}_{r\in\Lambda}\subset(A,\varphi)$ is called a *free semicircular system* if S is free and each $s_r\in S$ is self-adjoint and identically distributed with respect to φ according to Wigner's semicircle law. That is,

$$\varphi(s_r^k)=\frac{1}{2\pi}\int_{-2}^{2}t^k\sqrt{4-t^2}\,dt,\quad k\ge 0, r\in\Lambda.$$

(v) A family of random variables $C=\{c_r\}_{r\in\Lambda}\subset(A,\varphi)$ is called a *free circular system* if C is free and each $c_r\in C$ has the same distribution as the operator $\frac{s_1+is_2}{\sqrt{2}}$, where $\{s_1,s_2\}$ is a standard free semicircular system.

(vi) Let $S_N=\{x_r^{(N)}\}_{r\in\Lambda}\subset(A_N,\varphi_N)$ be a sequence of families of random variables and $S=\{x_r\}_{r\in\Lambda}\in(A,\varphi)$ be another family of random variables. We say that S_N *converges to S (or $S_N\to S$) in distribution as $N\to\infty$* if, for any non-commutative polynomial $P\in\mathbb{C}\langle X_r\colon r\in\Lambda\rangle$, we have

$$\lim_{N\to\infty}\varphi_N\big(P(S_N)\big)=\varphi\big(P(S)\big).$$

Remark 1.2.

(i) Note that, just like classical independence, the joint moments of a freely independent family $\{x_r\}_{r\in\Lambda} \subset (A,\varphi)$ can be computed from the individual moments of each variable x_r using the definition of freeness.

(ii) In the context of free probability theory, a free semicircular (circular) system is the free probability analogue of a standard real (complex) $N(0,1)$-Gaussian family.

We are now in a position to state the asymptotic free independence result of Banica and Collins.

Theorem 1.3 ([4, Theorems 6.1 and 9.4]) *Let $S = \{s_{ij} : i, j \in \mathbb{N}\}$ be a free semicircular system and let $C = \{c_{ij} : i, j \in \mathbb{N}\}$ be a free circular system in a NCPS (A,φ). For each $N \in \mathbb{N}$, let $S_N = \{\sqrt{N}u_{ij}\}_{1\le i,j\le N} \subset \big(C(O_N^+), h_{O_N^+}\big)$ and $C_N = \{\sqrt{N}v_{ij}\}_{1\le i,j\le N} \subset \big(C(U_N^+), h_{U_N^+}\big)$ be the normalized generators of $C(O_N^+)$ and $C(U_N^+)$, respectively. Then*

$$S_N \to S \quad \textit{and} \quad C_N \to C \quad \textit{in distribution as } N \to \infty,$$

i.e., *for any $k \in \mathbb{N}$, $1 \le i(r), j(r), \le N$, $\epsilon(r) \in \{1,*\}$ and $1 \le r \le k$, we have*

$$\lim_{N\to\infty} N^{k/2} h_{O_N^+}(u_{i(1)j(1)}u_{i(2)j(2)}\cdots u_{i(k)j(k)}) = \varphi(s_{i(1)j(1)}s_{i(2)j(2)}\cdots s_{i(k)j(k)})$$
$$\lim_{N\to\infty} N^{k/2} h_{U_N^+}(v^{\epsilon(1)}_{i(1)j(1)}v^{\epsilon(2)}_{i(2)j(2)}\cdots v^{\epsilon(k)}_{i(k)j(k)}) = \varphi(c^{\epsilon(1)}_{i(1)j(1)}c^{\epsilon(2)}_{i(2)j(2)}\cdots c^{\epsilon(k)}_{i(k)j(k)}).$$

Theorem 1.3 shows that, in an approximate sense, the generators of O_N^+ and U_N^+ are modeled by free (semi)circular random variables. In particular, if we denote by $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$ the von Neumann algebras generated by the GNS constructions associated to the corresponding Haar states, then Theorem 1.3 suggests that these von Neumann algebras may share some analytic properties with the free group factors $L(\mathbb{F}_k)$ ($k \ge 2$). Over the last decade this has indeed been shown to be the case. See for example [13, 18, 21, 30, 31]. Unfortunately, the approximation result of Theorem 1.3 has yet to find a direct application to the study of the analytic structures of $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$. One reason for this is that the combinatorial "Weingarten methods" used to establish this theorem provide very little information about the rate of approximation to a free (semi)circular system.

By exploiting a certain connection between O_N^+ and Woronowicz's deformed $\mathrm{SU}_{-q}(2)$ quantum group [34] (where $N = q + q^{-1}$), Banica, Collins and Zinn-Justin computed the spectral measure of each generator $u_{ij} \in C(O_N^+)$ relative to the Haar state in [6]. Using this fact, the authors were able to substantially improve the approximation result of Theorem 1.3 for a *single generator* u_{ij} of O_N^+ by showing that $\sqrt{N}u_{ij}$ *superconverges* (in the sense of H. Bercovici and D. Voiculescu [9]) to a semicircular variable $s \in (A,\varphi)$. In particular, this implies that for any polynomial $P \in \mathbb{C}[X]$, not only do we have that $P(\sqrt{N}u_{ij}) \to P(s)$ in distribution, but we also have the convergence of corresponding operator norms:

$$\lim_{N\to\infty} \|P(\sqrt{N}u_{ij})\|_{L^\infty(O_N^+)} = \|P(s)\|_{L^\infty(A,\varphi)}.$$

We note that this norm convergence result is not at all clear from the methods of Theorem 1.3.

The main result of this note is the non-commutative multivariate analogue of the above norm convergence result. Before stating our main theorem, we recall the notion of strong convergence for random variables, as defined by C. Male in [24]: Let (A,φ) and (A_N,φ_N) $(N\in\mathbb{N})$ be NCPS's and let $S=\{x_r\}_{r\in\Lambda}\in(A,\varphi)$ and $S_N=\{x_r^{(N)}\}_{r\in\Lambda}\subset(A_N,\varphi_N)$ be a sequence of families of random variables. We say that $S_N\to S$ *strongly in distribution* as $N\to\infty$ if $S_N\to S$ in distribution and

$$\lim_{N\to\infty}\|P(S_N)\|_{L^\infty(A_N,\varphi_N)}=\|P(S)\|_{L^\infty(A,\varphi)}$$

for all non-commutative polynomials $P\in\mathbb{C}\langle X_r : r\in\Lambda\rangle$.

Our main result follows.

Theorem 1.4 *Let $S=\{s_{ij} : i,j\in\mathbb{N}\}$ be a standard free semicircular system in a finite von Neumann algebra (M,τ) and let*

$$S_N=\{\sqrt{N}u_{ij}^{(N)} : 1\le i,j\le N\}\cup\{0 : i \text{ or } j>N\}\subset(C(O_N^+),h_{O_N^+})$$

be the (normalized) standard generators of O_N^+. Then

$$S_N\longrightarrow S\quad\text{strongly in distribution as } N\to\infty.$$

The above strong asymptotic freeness result answers a question posed in Section 6 of [7] on the mode of convergence to free independence of the joint distribution of the standard generators of O_N^+. This result should also be compared with other recent strong asymptotic freeness results for independent random matrix ensembles. See for example [15, 20, 24].

The key ingredient we require for our proof of Theorem 1.4 is R. Vergnioux's property of rapid decay (property (RD)) for the discrete dual quantum groups $\widehat{O_N^+}$ [31]. We show that property (RD) for the quantum groups $\{\widehat{O_N^+}\}_{N\ge3}$ allows us to approximate to any desired degree of accuracy (uniformly in N) the operator norm of a fixed non-commutative polynomial in the variables $\{\sqrt{N}u_{ij}^{(N)}\}_{1\le i,j\le N}$ by its non-commutative L^p-norm, for some sufficiently large even integer p. This transfer from L^∞ to L^p-norms then allows us to immediately deduce Theorem 1.4 from Theorem 1.3.

The remainder of the paper is organized as follows: In Section 2, we remind the reader of some basic facts on quantum groups, focusing mainly on the example of O_N^+. We then prove Theorem 1.4 in Section 3 and end with some applications of our result and concluding remarks in Section 4. In particular, we consider the analogous strong convergence result for U_N^+ (Corollary 4.1), strong convergence for polynomials with matrix coefficients (Corollary 4.2), and an application to L^2-L^∞ norm inequalities for polynomials over free semicircular systems (Corollary 4.3).

2 Preliminaries on O_N^+

Our main reference for the theory of compact quantum groups will be the book [29]. All unexplained terminology can be found there. For the remainder of the paper we write $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1 Peter–Weyl Decomposition of $L^2(O_N^+)$

Denote by $\mathrm{Pol}(O_N^+) \subset C(O_N^+)$ the dense $*$-subalgebra generated by the canonical generators $\{u_{ij}^{(N)}\}_{1 \le i,j \le N} \subset C(O_N^+)$. We recall that by general compact quantum group theory, the Haar state is always faithful on $\mathrm{Pol}(O_N^+)$. Denote by $L^2(O_N^+)$ the GNS Hilbert space obtained by completing $\mathrm{Pol}(O_N^+)$ with respect to the sesquilinear form $\langle x|y\rangle = h_{O_N^+}(y^*x)$, and let $L^\infty(O_N^+) \subset \mathcal{B}\big(L^2(O_N^+)\big)$ denote the von Neumann algebra generated by $\mathrm{Pol}(O_N^+)$ acting on $L^2(O_N^+)$ by (extending) left multiplication. In the following, we simultaneously identify $\mathrm{Pol}(O_N^+)$ with its image as a σ-weakly dense subalgebra of $L^\infty(O_N^+)$ and as a norm-dense subspace of $L^2(O_N^+)$ via the GNS construction.

The irreducible representations of the quantum group O_N^+ were first studied by Banica in [2], and this gives rise to a natural orthogonal decomposition of $L^2(O_N^+)$ in terms of finite dimensional subspaces spanned by matrix elements of these irreducible representations.

Rather than discussing representations of compact quantum groups, we choose to describe this "Peter–Weyl" decomposition of $L^2(O_N^+)$ intrinsically as follows. For each $k \in \mathbb{N}_0$, define a subspace $H_k(N) \subset \mathrm{Pol}(O_N^+)$ by setting

$$H_0(N) := \mathbb{C}1_{\mathrm{Pol}(O_N^+)}, \quad H_1(N) := \mathrm{span}\{u_{ij}^{(N)}\}_{1 \le i,j \le N},$$
$$H_k(N) := H_1(N)H_{k-1}(N) \ominus H_{k-2}(N), \quad k \ge 2,$$

where $H_1(N)H_{k-1}(N) := \mathrm{span}\{xy : x \in H_1(N), y \in H_{k-1}(N)\}$. The fact that the above recursive definition for $H_k(N)$ makes sense follows from the analysis of the representation theory of O_N^+ in [2]. Moreover, we have the following result.

Theorem 2.1 ([2]) *For each $k \in \mathbb{N}_0$, there is a unique (up to isomorphism) unitary representation U^k of O_N^+ whose matrix elements span $H_k(N)$. Conversely, every irreducible unitary representation of O_N^+ arises this way. Moreover, the representations $\{U^k\}_{k \in \mathbb{N}_0}$ satisfy the tensor product decomposition rules*

$$U^n \boxtimes U^k \cong \bigoplus_{0 \le r \le \min\{k,n\}} U^{n+k-2r}, \quad n, k \in \mathbb{N}_0.$$

As a consequence, we have the following orthogonal decomposition

$$L^2(O_N^+) = \ell^2 - \bigoplus_{k \in \mathbb{N}_0} H_k(N),$$

and the following multiplication "rule"

$$xy \in \bigoplus_{0 \le r \le \min\{k,n\}} H_{n+k-2r}(N), \quad x \in H_n(N), y \in H_k(N).$$

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2.2 Quantum Numbers

Fix $0 < q < 1$. Recall that the *q-numbers and q-factorials* (see for example [23]) are defined by the formulas

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}} = \frac{q^{-a+1}(1-q^{2a})}{1-q^2}, \quad [a]_q! = [a]_q[a-1]_q \cdots [1]_q, \quad a \in \mathbb{N}.$$

Note that as $q \to 1$, $[a]_q \to a$.

2.3 The Property of Rapid Decay

Using Theorem 2.1, we can define a length function $\ell\colon \operatorname{Pol}(O_N^+) = \bigoplus_{k\in\mathbb{N}_0} H_k(N) \to \mathbb{N}_0$ by setting

$$\ell(x) = \min\Big\{ n \in \mathbb{N}_0 : x \in \bigoplus_{0\le k\le n} H_k(N) \Big\}, \quad x \in \operatorname{Pol}(O_N^+).$$

The main tool we will use to establish the strong asymptotic freeness for the generators of $L^\infty(O_N^+)$ is Vergnioux's *property of rapid decay (RD)* [31], which provides a way to estimate the L^∞-norm of any $x \in \operatorname{Pol}(O_N^+)$ in terms of its length $\ell(x)$ and its (much easier to compute) L^2-norm. The main estimate we require is as follows, and should be compared to U. Haagerup's fundamental inequality for free groups (see [19, Lemma 1.3]).

Theorem 2.2 ([31, Theorem 4.9]) *For each $l \in \mathbb{N}_0$, let $P_l\colon L^2(O_N^+) \to H_l(N)$ be the orthogonal projection. There exists a constant $D_N > 1$ such that for any $n, k, l \in \mathbb{N}_0$, $x \in H_n(N)$ and $y \in H_k(N)$, we have*

$$\|P_l(xy)\|_{L^2(O_N^+)} \le D_N \|x\|_{L^2(O_N^+)} \|y\|_{L^2(O_N^+)}.$$

Combining Theorem 2.2 with the fusion rules from Theorem 2.1, we obtain an essentially equivalent statement of property (RD) that is in a form more suitable for our purposes.

Corollary 2.3 *For each $x \in \operatorname{Pol}(O_N^+)$, we have*

$$\|x\|_{L^2(O_N^+)} \le \|x\|_{L^\infty(O_N^+)} \le D_N \big(\ell(x)+1\big)^{3/2} \|x\|_{L^2(O_N^+)}.$$

Proof It suffices to prove that for each $n \in \mathbb{N}_0$, $x \in H_n(N)$, we have

$$\|x\|_{L^\infty(O_N^+)} \le D_N(n+1)\|x\|_{L^2(O_N^+)}.$$

Indeed, since any $x \in \operatorname{Pol}(O_N^+)$ can be written as $x = \sum_{0\le n\le \ell(x)} x_n$ where $x_n \in H_n(N)$, we then have

$$\|x\|_{L^\infty(O_N^+)} \le \sum_{0\le n\le \ell(x)} D_N(n+1)\|x_n\|_{L^2(O_N^+)} \le D_N\big(\ell(x)+1\big)^{3/2}\|x\|_{L^2(O_N^+)}.$$

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Now fix $x \in H_n(N)$, $\xi = \sum_{k\in\mathbb{N}_0} P_k\xi \in L^2(O_N^+)$ and write

$$\|x\xi\|^2_{L^2(O_N^+)} = \sum_{l\in\mathbb{N}_0} \|P_l(x\xi)\|^2_{L^2(O_N^+)} = \sum_{l\in\mathbb{N}_0} \Big\| \sum_{k\in\mathbb{N}_0 : P_l(xP_k\xi)\neq 0} P_l(xP_k\xi) \Big\|^2_{L^2(O_N^+)}$$
$$\leq \sum_{l\in\mathbb{N}} \Big(\sum_{k\in\mathbb{N}_0 : P_l(xP_k\xi)\neq 0} D_N \|x\|_{L^2(O_N^+)} \|P_k\xi\|_{L^2(O_N^+)} \Big)^2,$$

where in the last line we have used Theorem 2.2. Now observe that by Theorem 2.1, the sets $\{k \in \mathbb{N}_0 : P_l(xP_k\xi) \neq 0\}$ and $\{l \in \mathbb{N}_0 : P_l(xP_k\xi) \neq 0\}$ are both finite and have cardinality at most $n+1$. Applying these facts and the Cauchy–Schwarz inequality to the above inequality, we obtain

$$\|x\xi\|^2_{L^2(O_N^+)} \leq D_N^2 \|x\|^2_{L^2(O_N^+)} \sum_{l\in\mathbb{N}_0} (n+1) \Big(\sum_{k\in\mathbb{N}_0 : P_l(xP_k\xi)\neq 0} \|P_k\xi\|^2_{L^2(O_N^+)} \Big)$$
$$= (n+1) D_N^2 \|x\|^2_{L^2(O_N^+)} \sum_{k\in\mathbb{N}_0} \Big(\sum_{l\in\mathbb{N}_0 : H_l(N)\subset H_n(N)H_k(N)} \|P_k\xi\|^2_{L^2(O_N^+)} \Big)$$
$$\leq D_N^2 (n+1)^2 \|x\|^2_{L^2(O_N^+)} \sum_{k\in\mathbb{N}_0} \|P_k\xi\|^2_{L^2(O_N^+)} = D_N^2 (n+1)^2 \|x\|^2_{L^2(O_N^+)} \|\xi\|^2_{L^2(O_N^+)}. \blacksquare$$

3 Main Result

In this section, we will give a proof of Theorem 1.4. Our strategy will be to first use the property of rapid decay to show that for any fixed non-commutative polynomial $P \in \mathbb{C}\langle X_{ij} : i, j \in \mathbb{N}\rangle$ and any $\epsilon > 0$, there is a fixed $p = p(P, \epsilon) \in (2, \infty)$ such that the non-commutative L^p-norm of $P(S_N) \in \mathrm{Pol}(O_N^+)$ is within ϵ of its L^∞-norm for all N. It is an elementary fact that for each N, there is a $p = p(P, \epsilon, N)$ which obtains the required approximation. The key point here is that we can select a *single* $p \in (2, \infty)$ that works for all N.

Using this uniform L^p-L^∞-estimate, we are able to prove Theorem 1.4 by transferring the problem of norm convergence of polynomials to a question about convergence in distribution.

3.1 Uniform L^p-L^∞-estimates

We start with a lemma which contains the key computation of this article.

Lemma 3.1 *Let D_N be the constant appearing in Theorem 2.2. Then D_N can be chosen so that $\lim_{N\to\infty} D_N = 1$. In particular, $(D_N)_{N\geq 3}$ is uniformly bounded.*

To prove this lemma, we will actually revisit and refine the proof of Theorem 2.2 given in [31].

Proof Fix $n, k, l \in \mathbb{N}_0$, $x \in H_n(N)$ and $y \in H_k(N)$. If U^l is not equivalent to a subrepresentation of $U^n \boxtimes U^k$, then $P_l(xy) = 0$ and there is nothing to prove. Otherwise, there exists some $0 \leq r \leq \min\{k, n\}$ such that $l = n + k - 2r$. Let $\mathcal{V}_l$, $\mathcal{V}_k$ and $\mathcal{V}_n$ denote the Hilbert spaces on which the representations U^l, U^k and U^n

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act, respectively, and let $t_r\colon \mathbb{C} \cong \mathcal{V}_0 \to \mathcal{V}_r \otimes \mathcal{V}_r$ be the unique (up to multiplication by $\mathbb{T}$) O_N^+-invariant isometry. Uniqueness of t_r follows from Theorem 2.1. Now we identify each representation U^k with the "highest weight" subrepresentation of $U^{\boxtimes k}$ in the canonical way (see [2]), and let $p_k\colon (\mathbb{C}^N)^{\otimes k} \to \mathcal{V}_k$ the orthogonal (Jones–Wenzl) projection. Using the map t_r and the projections p_n, p_k, p_l, we can define the following canonical O_N^+-invariant contraction:

$$\phi_l^{n,k} := (p_n \otimes p_k)(\mathrm{id}_{(\mathbb{C}^N)^{\otimes(n-r)}} \otimes t_r \otimes \mathrm{id}_{(\mathbb{C}^N)^{\otimes(k-r)}})p_l : \mathcal{V}_l \subset (\mathbb{C}^N)^{\otimes l} \to \mathcal{V}_n \otimes \mathcal{V}_k.$$

According to Lemmas 4.6 and 4.8, and Theorem 4.9 of [31] (see also Section 4.2 of [12] for an analogous calculation for quantum permutation groups), the following norm inequality holds for $x \in H_n(N)$, $y \in H_k(N)$:

$$\|P_l(xy)\|_{L^\infty(O_N^+)} \leq \Big(\frac{\dim \mathcal{V}_k \dim \mathcal{V}_n}{\dim \mathcal{V}_l (\dim \mathcal{V}_r)^2} \Big)^{1/2} \frac{\|x\|_{L^2(O_N^+)} \|y\|_{L^2(O_N^+)}}{\|\phi_l^{n,k}\|^2}.$$

Fix $0 < q < 1$ such that $N = q + q^{-1}$. Note that $q \to 0$ as $N \to \infty$. In [2] it is shown that $\dim \mathcal{V}_k = [k+1]_q$ for all $k \in \mathbb{N}_0$, so we can take

$$D_N = \sup_{(n,k,l)\in\mathbb{N}_0^3} \Big(\frac{[k+1]_q[n+1]_q}{[l+1]_q[r+1]_q^2} \Big)^{1/2} \|\phi_l^{n,k}\|^{-2}$$

and analyze its large N ($\Leftrightarrow$ small q) behavior.

As a first observation, note that

$$(1-q^2)^3 \leq \frac{(1-q^2)(1-q^{2n+2})(1-q^{2k+2})}{(1-q^{2r+2})^2(1-q^{2l+2})} = \frac{[k+1]_q[n+1]_q}{[l+1]_q[r+1]_q^2} \leq (1-q^2)^{-2}.$$

So

$$\lim_{q\to 0} \sup_{(n,k,l)\in\mathbb{N}_0^3} \Big(\frac{[k+1]_q[n+1]_q}{[l+1]_q[r+1]_q^2} \Big)^{1/2} = 1$$

and it suffices to restrict our attention to the quantity $\|\phi_l^{n,k}\|^{-2}$.

In [2], Banica showed that the monoidal C^*-tensor category generated by the fundamental representation $U = [u_{ij}^{(N)}]$ of O_N^+ is isomorphic to the Temperley–Lieb category $\mathrm{TL}(q+q^{-1})$. Through this isomorphism, the linear map $[r+1]_q^{1/2}\phi_l^{n,k}$ defined above corresponds to a *three-vertex* with parameters (n,k,l) in $\mathrm{TL}(q+q^{-1})$. See [23] for the definition of $\mathrm{TL}(q+q^{-1})$ and three-vertices. In particular, the results of Section 9.9–9.10 and Lemma 8 in [23] give an explicit expression for the norm of a three-vertex with parameters (n,k,l). Translating this result back to O_N^+, we obtain

$$\begin{aligned} 1 \leq \|\phi_l^{n,k}\|^{-2} &= \frac{[r+1]_q[l+1]_q![n]_q![k]_q!}{[l+1+r]_q![n-r]_q![k-r]_q![r]_q!} \\ &= \prod_{s=1}^{r} \frac{[1+s]_q[n-r+s]_q[k-r+s]_q}{[l+1+s]_q[s]_q^2} \\ &= \prod_{s=1}^{r} \frac{(1-q^{2+2s})(1-q^{2n-2n+2s})(1-q^{2k-2r+2s})}{(1-q^{2l+2+2s})(1-q^{2s})^2} \\ &\leq \Big(\prod_{s=1}^{r} \frac{1}{1-q^{2s}} \Big)^3 \leq \Big(\prod_{s=1}^{\infty} \frac{1}{1-q^{2s}} \Big)^3. \end{aligned}$$

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Since this last infinite product is finite and converges to 1 as $q \to 0$, it follows that $\lim_{N\to\infty} D_N = 1$. ∎

We are now ready to state our result on uniform L^p-L^∞ estimates. In the following, recall that $S_N = \{\sqrt{N}u_{ij}^{(N)} : 1 \le i, j \le N\} \cup \{0 : i \text{ or } j > N\}$ denotes the family of standard normalized generators of $\mathrm{Pol}(O_N^+)$.

Proposition 3.2 *Let $P \in \mathbb{C}\langle X_{ij} : i, j \in \mathbb{N}\rangle$ and $\epsilon > 0$. Then there exists an even integer $p = p(P, \epsilon) \in 2\mathbb{N}$ such that*

$$\|P(S_N)\|_{L^\infty(O_N^+)} \le (1+\epsilon)\|P(S_N)\|_{L^p(O_N^+)} \quad (N \ge 3).$$

Proof Let $r = \deg P$ and $x_N = P(S_N) \in \mathrm{Pol}(O_N^+)$. Then we have $\ell(x_N) \le r$ and $\ell\big((x_N^* x_N)^m\big) \le 2rm$ for all $m \in \mathbb{N}$. Fixing $m \in \mathbb{N}$ and applying Corollary 2.3, we then have

$$\begin{aligned}
\|x_N\|_{L^\infty(O_N^+)} &= \|(x_N^* x_N)^m\|_{L^\infty(O_N^+)}^{1/2m} \\
&\le \left(D_N \Big(\ell\big((x_N^* x_N)^m\big) + 1 \Big)^{3/2} \|(x_N^* x_N)^m\|_{L^2(O_N^+)} \right)^{1/2m} \\
&= \left(D_N \Big(\ell\big((x_N^* x_N)^m\big) + 1 \Big)^{3/2} \right)^{1/2m} \|x_N\|_{L^{4m}(O_N^+)} \\
&\le \big(D_N (2rm+1)^{3/2} \big)^{1/2m} \|x_N\|_{L^{4m}(O_N^+)}.
\end{aligned}$$

Applying Lemma 3.1, we can find an $m = m(r, \epsilon) \in \mathbb{N}$ such that $D_N^{1/2m}(2rm+1)^{3/4m} \le 1+\epsilon$ for all $N \ge 3$. Taking $p = 4m$ will then do the job. ∎

The proof of Theorem 1.4 now follows easily.

Proof of Theorem 1.4 Let $x_N = P(S_N) \in \mathrm{Pol}(O_N^+)$, $x = P(S) \in (M, \tau)$, and $\epsilon > 0$. Since $\|x\|_{L^\infty(M)} = \lim_{p\to\infty} \|x\|_{L^p(M)}$, we can choose $m \in \mathbb{N}$ large enough so that $\|x\|_{L^{2m}(M)} \ge \|x\|_{L^\infty(M)} - \epsilon$. Applying Theorem 1.3, we have

$$\|x\|_{L^{2m}(M)} = \tau\big((x^*x)^m\big)^{1/2m} = \lim_{N\to\infty} h_{O_N^+}\big((x_N^* x_N)^m\big)^{1/2m} = \lim_{N\to\infty} \|x_N\|_{L^{2m}(O_N^+)},$$

which yields

$$\|x\|_{L^\infty(M)} - \epsilon \le \|x\|_{L^{2m}(M)} = \lim_{N\to\infty} \|x_N\|_{L^{2m}(O_N^+)} \le \liminf_{N\to\infty} \|x_N\|_{L^\infty(O_N^+)}.$$

On the other hand, by Proposition 3.2, there is a $p = p(P, \epsilon) \in 2\mathbb{N}$ such that

$$\|x_N\|_{L^\infty(O_N^+)} \le (1+\epsilon)\|x_N\|_{L^p(O_N^+)}, \quad N \ge 3.$$

Applying Theorem 1.3 once again, we obtain

$$\limsup_{N\to\infty} \|x_N\|_{L^\infty(O_N^+)} \le \limsup_{N\to\infty} (1+\epsilon)\|x_N\|_{L^p(O_N^+)} = (1+\epsilon)\|x\|_{L^p(M)} \le (1+\epsilon)\|x\|_{L^\infty(M)}.$$

As $\epsilon > 0$ was arbitrary, we conclude that $\lim_{N\to\infty} \|x_N\|_{L^\infty(O_N^+)} = \|x\|_{L^\infty(M)}$. ∎

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4 Some Applications and Consequences

4.1 Strong Asymptotic Freeness for U_N^+

Let $X^{(k)} = \{x_i^{(k)}\}_{i\in I}$ and $Y^{(k)} = \{y_j^{(k)}\}_{j\in I}$ be two sequences of families of non-commutative random variables and assume that $(X^{(k)})_{k\in\mathbb{N}}$ and $(Y^{(k)})_{k\in\mathbb{N}}$ converge strongly in distribution to families X and Y, respectively. It has recently been shown by P. Skoufranis [28] (see also [27] for an alternate proof) that if in addition we assume that the familes $X^{(k)}$ and $Y^{(k)}$ are free for each k, then $\{X^{(k)}, Y^{(k)}\} \longrightarrow \{X, Y\}$ strongly in distribution.

Let $\{v_{ij}^{(N)}\}_{1\leq i,j\leq N}$ denote the canonical generators of $\mathrm{Pol}(U_N^+)$. It was shown in [3] that the joint distribution $\{v_{ij}^{(N)}\}_{1\leq i,j\leq N}$ (with respect to the Haar state) can be modeled in terms of the "free complexification" of the generators $\{u_{ij}^{(N)}\}_{1\leq i,j\leq N}$ of $\mathrm{Pol}(O_N^+)$. More precisely, let z be a unitary in a NCPS (A,φ) whose spectral measure relative to φ is the Haar measure on $\mathbb{T}$ (*i.e.*, z is a *Haar unitary*), and assume z is free from $\{u_{ij}^{(N)}\}_{1\leq i,j\leq N}$. Then the families

$$\{v_{ij}^{(N)}\}_{1\leq i,j\leq N} \quad \text{and} \quad \{zu_{ij}^{(N)}\}_{1\leq i,j\leq N}$$

are identically distributed (*i.e.*, have the same joint $*$-moments).

Combining the preceding two paragraphs with Theorem 1.4 yields the following corollary.

Corollary 4.1 *Let $C = \{c_{ij} : i, j \in \mathbb{N}\}$ be a standard free circular system in a finite von Neumann algebra (M,τ) and let $\{v_{ij}^{(N)} : 1 \leq i, j \leq N\}$ be the standard generators of* $\mathrm{Pol}(U_N^+)$*. Then*

$$\{\sqrt{N}v_{ij}^{(N)} : 1 \leq i, j \leq N\} \cup \{0 : i \text{ or } j > N\} \longrightarrow C$$

strongly in distribution as $N \to \infty$.

4.2 Polynomials with Matrix Coefficients

We remark that by a standard ultraproduct technique (see for example Proposition 7.3 in [24]), the strong convergence of a sequence of families of random variables is equivalent to the (*a priori* stronger) condition of norm convergence for non-commutative polynomials with matrix coefficients.

Corollary 4.2 *Let $S = \{s_{ij} : i, j \in \mathbb{N}\}$ be a standard free semicircular system in a finite von Neumann algebra (M,τ) and let $S_N = \{\sqrt{N}u_{ij}^{(N)} : 1 \leq i, j \leq N\} \cup \{0 : i, j > N\}$ be the standard normalized generators of* $\mathrm{Pol}(O_N^+)$*. Then for any $k \in \mathbb{N}$ and any non-commutative polynomial $P \in M_k(\mathbb{C}) \otimes \mathbb{C}\langle X_{ij} : i, j \in \mathbb{N}\rangle$, we have*

$$\|P(S_N)\|_{M_k(\mathbb{C})\otimes L^\infty(O_N^+)} \to \|P(S)\|_{M_k(\mathbb{C})\otimes M}.$$

Of course, the analogous result also holds for the generators of U_N^+.

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4.3 L^2-L^∞ Norm-inequalities for Free Semicircular Systems

By taking limits in Corollary 2.3, and applying Lemma 3.1 and Theorem 1.4, we obtain the following L^2-L^∞ norm inequality for a free semicircular system. A similar inequality was was first obtained by M. Bozejko [11] and was re-proved by P. Biane and R. Speicher using combinatorial methods (see [10, Theorem 5.3.4]).

Corollary 4.3 *Let $S = \{s_i : i \in I\}$ be a standard free semicircular system in a finite von Neumann algebra (M, τ) and let $P \in \mathbb{C}\langle X_i : i \in I\rangle$. Then*

$$\|P(S)\|_{L^2(M)} \leq \|P(S)\|_{L^\infty(M)} \leq (\deg P + 1)^{3/2}\|P(S)\|_{L^2(M)}.$$

4.4 Concluding Remarks

It would be natural to try to adapt the arguments of this paper to establish the strong convergence of certain polynomial functions over other classes of quantum groups, such as Wang's quantum permutation groups S_N^+ [33] or the free easy quantum groups studied by Banica and Speicher in [8]. For the quantum permutation groups S_N^+, all of the tools are in place: a Weingarten calculus for polynomial integrals over S_N^+ was developed by Banica and Collins in [5], and the author proved a property (RD) result for S_N^+ in [12]. Unfortunately, the analogue of Proposition 3.2 that one would require still does not follow immediately. This is because the constant D_N derived in our property (RD) result for S_N^+ turns out to grow *quadratically* in N. It is, however, quite plausible that a more precise analysis in the proof of property (RD) for S_N^+ could still yield a uniformly bounded sequence of associated constants D_N.

Acknowledgements This research was supported by an NSERC Postdoctoral Fellowship. The author also wishes to thank Benoît Collins for stimulating discussions which improved an earlier version of this work.

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