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LIFTING IDEALS IN NONCOMMUTATIVE INTEGRAL EXTENSIONS

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This note is intended as a commentary on a part of the paper [1] by Gulliksen, Ribenboim, and Viswanathan. It takes its inspiration from a colloquium talk by P. Ribenboim.

Our aim is a partial generalization of the theorem of Cohen-Seidenberg (cf. [2], IX.1, Propositions 9 and 10) to noncommutative algebras.

DEFINITION. Let A be a ring with 1 in the center of another ring B (with the same 1). B is said to be *integral* over A, if each element $b \in B$ satisfies an integral equation

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0, \quad \text{with } a_i \in A.$$

Henceforth the situation described in this definition will be assumed. An example of it is given in [1]: $B = A[G]$, the group-algebra of a locally finite group.

"Ideals" will be two-sided, unless otherwise specified. We recall that an ideal P of B is *prime*, if and only if its complement $S = B - P$ has the following property: if $b, b' \in S$, then $b \times b' \in S$ for suitable $x \in B$.

Intersecting a (possibly one-sided) ideal J of B with A produces an ideal j of A. We say that j is the *restriction* of J or that J *lies above* j. Denoting by **P**, **M**, **MR** the sets of prime, maximal, maximal right ideals (respectively) of a ring, we have the following.

THEOREM. *Restriction to A induces surjections of*

(a) $\mathbf{P}(B)$ *onto* $\mathbf{P}(A)$

(b) $\mathbf{M}(B)$ *onto* $\mathbf{M}(A)$

(c) $\mathbf{MR}(B)$ *onto* $\mathbf{M}(A)$.

Proof. Clearly every prime of B lies above a prime of A. The first thing to check is the analogue of this in the cases (b) and (c).

LEMMA 1. *Let J be a right ideal of B, $a \in A$. If $ab = 1$ (mod J) with $b \in B$, then there is an $a' \in A$ such that $aa' = 1$ mod $(J \cap A)$.*

Proof of Lemma 1 (extracted from [1]).

$$b^n = a_{n-1}b^{n-1} + \cdots + a_0$$

Multiplying by a^{n-1}, we get

$$b(ab)^{n-1} = a_{n-1}(ab)^{n-1} + \cdots + a^{n-1}a_0.$$

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Since $ab \equiv 1 (\bmod J)$, we have

$$b \equiv a_{n-1} + aa_{n-2} + \cdots + a^{n-1}a_0 (\bmod J),$$

where the right hand side is the required a'.

REMARK (due to A. Dress). To see that maximal ideals of B lie above maximal ideals of A, it suffices to have the integrality over A for the center of B only. One then uses the commutative Cohen–Seidenberg and the simple fact that simple rings have simple centers

In the proof of the theorem, the next question is the "liftability" of ideals of A. Here the answer comes from an argument reminiscent of Nakayama's lemma.

LEMMA 2. *For any ideal j of A, jB is an ideal of B* (i.e. *proper*).

Proof of Lemma 2. Without loss of generality: $j=m$ maximal. Put $\bar{A}=A/m$, $\tilde{A}=A_m$ (localization), $\bar{B}=B/mB=B\otimes_A \bar{A}$, $\tilde{B}=B_m=B\otimes_A \tilde{A}$.

We want to show that $\bar{B}\neq 0$. But $\bar{B}$ can be also obtained from $\tilde{B}$ by reduction mod $\tilde{m}$. Hence we may take $A=\tilde{A}$ to be local with maximal ideal m.

Suppose $\bar{B}=0$, i.e. there is a relation

$$(*) \qquad a_1b_1 + \cdots + a_rb_r = 1 \quad \text{with } a_i \in m.$$

We assume that (*) is of *minimal length*. Then the B-module generated by $a_1, \ldots, a_{r-1}$ is an ideal J (i.e. proper). By Lemma 1, there is an $a' \in A$ such that $a_ra' \equiv 1 (\bmod J)$. Thus

$$1 - a_ra' = a_1b'_1 + \cdots + a_{r-1}b'_{r-1}$$

with suitable $b'_i \in B$. But here the left-hand side is an A-unit and therefore leads to a shorter relation, contradicting minimality of (*). Q.E.D.

It is immediate from Lemma 2 that every maximal ideal of A is the restriction of a maximal ideal (and of a maximal right ideal) of B. To conclude the proof of the theorem, we finally have to lift prime ideals p of A. This is done as follows: take a maximal hence prime ideal $\tilde{P}$ in $\tilde{B}=B_p$ lying above the maximal ideal $\tilde{p}$ of A_p; pull $\tilde{P}$ back to B along the canonical arrow $B \to B_p$ to obtain an ideal P of B; the abovementioned criterion for the complements of $\tilde{P}$ and P easily shows that P is prime.

REMARK. As a consequence of the theorem, we see that various radicals (prime, Brown-McCoy, Jacobson) of A are obtained by intersecting A with the corresponding radicals of B.

REFERENCES

1. T. Gulliksen, P. Ribenboim, T. M. Viswanathan, *An elementary note on group rings*, (to appear).

2. S. Lang, *Algebra*, Addison-Wesley, Reading, Mass. (1965).

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