# FINITE AND INFINITE CYCLIC EXTENSIONS OF FREE GROUPS

**Dedicated to the memory of Hanna Neumann**

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## 1. Introduction

Using Stalling's characterization [11] of finitely generated (f. g.) groups with infinitely many ends, and subgroup theorems for generalized free products and HNN groups (see [9], [5], and [7]), we give (in Theorem 1) a n.a.s.c. for a f.g. group to be a finite extension of a free group. Specifically (using the terminology and notation of [5]), a f.g. group $G$ is a finite extension of a free group if and only if $G$ is an HNN group

$$(1) \qquad \langle t_1, \cdots, t_n, K;\ \text{rel } K,\ t_1 L_1 t_1^{-1} = M_1, \cdots, t_n L_n t_n^{-1} = M_n \rangle,$$

where $K$ is a tree product of a finite number of finite groups (the vertices of $K$), and each (associated) subgroup $L_i$, $M_i$ is a subgroup of a vertex of $K$.

[We recall the definition of a tree product: Let $\{A_i\}$ be a collection of groups pictured as vertices of a tree, and with each edge connecting $A_i$ to $A_j$ let there be given a pair of isomorphic subgroups $U_{ij}$ and $U_{ji}$ of $A_i$ and $A_j$, respectively, and an isomorphism $\phi_{ij}$ between them. Then the group whose presentation is obtained by using all the generators of the $\{A_i\}$ as generators, and by using all the defining relators of the $\{A_i\}$ together with relators amalgamating $U_{ij}$ and $U_{ji}$ according to $\phi_{ij}$ as defining relators, is called the tree product of the $\{A_i\}$ with subgroups $U_{ij}$ and $U_{ji}$ amalgamated under $\phi_{ij}$.]

Stallings [12] independently obtained the special case where the commutator subgroup $G'$ of $G$ is of finite index (f.i.) in $G$ (this is equivalent to saying that there are no $t_i$). Moreover, [12] contains a formula which we use to derive (in Theorem 2) a generalized Schreier rank formula. Specifically, let $G$ be as in (1), and let $H$ be a free subgroup of $G$ of rank $r$ and index $j$. Then

$$(2) \quad r = j\left(\frac{1}{f_1} + \cdots + \frac{1}{f_n} + \frac{1}{e_1} + \cdots + \frac{1}{e_p} - \frac{1}{v_1} - \cdots - \frac{1}{v_{p+1}}\right) + 1,$$

where $f_i$ is the order of $L_i$, $p + 1$ is the number of vertices of $K$, and $e_i$, $v_j$ range over the orders of the edges (amalgamated subgroups) and vertices respectively, of the tree product $K$.

Theorem 1 allows us to determine (see Theorem 3) all groups with non-trivial center $Z$ which are infinite cyclic extensions of a f.g. free group: Any such group $G$ is an HNN group whose base is a tree product of infinite cyclic groups and whose associated subgroups are contained in vertices of the base. If further $G'Z$ is of f.i. in $G$, then $G$ is simply a tree product of infinite cyclic groups or free abelian of rank two.

In [10], the last result is used to determine the structure of one-relator groups with non-trivial center. The result also shows that knot groups with non-trivial center are tree products of infinite cyclic groups; but thus far we do not see how to obtain from this the characterization in [1], namely, that such knot groups are tree products with merely two infinite cyclic factors, i.e., groups of the type $\langle a, b; a^p = b^q \rangle$.

## 2. Finite extensions of free groups

THEOREM 1. *Let $G$ be a f.g. group. Then $G$ is a finite extension of a free group $H$ if and only if $G$ is an HNN group of the form* (1).

PROOF. We use induction on the rank (the minimum number of generators) of $H$. If $H$ has rank zero then $G$ is finite. If $H$ has rank one, then $G$ has two ends (see, e.g., [2]). Thus by [11] either $G = (A * B; F)$ where $F$ is finite and of index two in both $A$ and $B$, or $G = \langle t, F; tFt^{-1} = F \rangle$ where $F$ is finite. Hence, $G$ is of the form (1) when $H$ has rank zero or one.

Assume $H$ has rank greater than one. Then $H$ has infinitely many ends, and so too does $G$. Thus by [11] either

$$G = (A * B; F) \tag{3}$$

where $F$ is finite and $A \neq F \neq B$, or

$$G = \langle t, K;\ \text{rel } K,\ tFt^{-1} = \phi(F) \rangle \tag{4}$$

where $F$ is finite and $\phi$ maps $F$ isomorphically onto $\phi(F) \subseteq K$.

Now first suppose (3) holds. If $A$ and $B$ are finite the conclusion follows. Assume $A$ is infinite. Then since $H$ has trivial intersection with the conjugates of $F$ in $G$, by the subgroup theorem of [5] or [9], $H = (H \cap A) * (H \cap B) * \cdots$. Moreover, since $H$ is of finite index in $G$ and $A$ is of infinite index in $G$, $H \cap A \neq H$. Moreover, $H \cap A \neq \{1\}$; for otherwise $A \cong AH/H \subseteq G/H$, contrary to $A$ being infinite. Thus rank $(H \cap A)$, rank $(H \cap B) <$ rank $H$. Therefore $A$, $B$ are finitely generated finite extensions of the free groups $H \cap A$, $H \cap B$, respectively, each of which has rank less than that of $H$, and so the inductive hypothesis applies. Thus

$$A = \langle t_1, \cdots, t_\mu, K; \text{ rel } K, \cdots, t_i L_i t_i^{-1} = M_i, \cdots \rangle$$

and

$$B = \langle s_1, \cdots, s_v, N; \text{ rel } N, \cdots, s_j P_j s_j^{-1} = Q_j, \cdots \rangle,$$

where $K$, $N$ are tree products of finite groups and $L_i$, $M_i$ are in vertices of $K$ and $P_j$, $Q_j$ are in vertices of $N$. Since $F$ is finite, $F$ is contained in a conjugate $aKa^{-1}$ and a conjugate $bNb^{-1}$, where $a \in A$ and $b \in B$. Therefore since $aAa^{-1} = A$, $bBb^{-1} = B$, we may assume $F$ is in $K$ and in $N$. Moreover, $F$ is contained in some vertex of a conjugate $kKk^{-1}$ and a conjugate $nNn^{-1}$ where $k \in K$, $n \in N$. Hence $(A * B; F)$ is an HNN group with free part

$$\cdots, kt_ik^{-1}, \cdots, ns_jn^{-1}, \cdots$$

and whose base is the tree product of the vertices of $kKk^{-1}$ and $nNn^{-1}$, with the vertices of $kKk^{-1}$ joined as previously, the vertices of $nNn^{-1}$ joined as previously, and the vertex of $kKk^{-1}$ and of $nNn^{-1}$ which contain $F$ joined by an edge representing $F$. The associated subgroups are $kL_ik^{-1}$, $kM_ik^{-1}$, $nP_jn^{-1}$, and $nQ_jn^{-1}$. Thus $G$ has the desired form if (3) holds.

Next suppose (4) holds. If $K$ is finite the conclusion follows. Assume $K$ is infinite. Since $H$ is of finite index in $G$ and $K$ is of infinite index in $G$, it follows that $1 \neq K \cap H \neq H$. Moreover, since $H$ intersects the conjugates of $F$ in $G$ trivially, by the subgroup theorem of [7], $H = (H \cap K) * \cdots$. Therefore rank $(H \cap K) <$ rank $(H)$ and the inductive hypothesis applies to $K$. Thus

$$K = \langle s_1, \cdots, s_v, N; \text{rel } N, \cdots, s_j P_j s_j^{-1} = Q_j, \cdots \rangle$$

where $N$ is a tree product of finite groups. Since $F$ is of finite order, $F$ is contained in some conjugate $kNk^{-1}$ where $k \in K$; moreover, since $kNk^{-1}$ is a tree product, $F$ is contained in some vertex of the tree product $(knk^{-1})(kNk^{-1})(kn^{-1}k^{-1}) = knNn^{-1}k^{-1}$ where $n \in N$. Hence

$$G = \langle t, knKn^{-1}k^{-1}; \text{ rel}(knKn^{-1}k^{-1}), tFt^{-1} = \phi(F) \rangle$$

where $F$ is in a vertex of the HNN group $knKn^{-1}k^{-1}$. Since $\phi(F)$ is also finite, $\phi(F)$ is in some vertex of the tree product base of the HNN group $k_1knKn^{-1}k^{-1}k_1^{-1}$ where $k_1 \in K$. Hence

$$G = \langle k_1^{-1}t, knKn^{-1}k^{-1}; \text{ rel}(knKn^{-1}k^{-1}), (k_1^{-1}t)F(t^{-1}k_1) = k_1^{-1}(\phi(F))k_1 \rangle$$

is an HNN group of the desired type.

Consequently, a f.g. finite extension of a free group has the form (1).

Conversely, suppose $G$ has the form (1). We wish to show that $G$ is a finite extension of a free group. Clearly, to show this, it suffices to establish the following result: If $A$, $B$, $K$ are f.g. finite extensions of free groups and $F$ is a finite subgroup of $A$, $B$, and $K$, then both $(A * B; F)$ and $\langle t, K; \text{rel } K, tFt^{-1} = \phi(F) \rangle$ are finite extensions of free groups (where $\phi$ is an isomorphism of $F$ into $K$).

For the first case see [4].

In the second case we consider a homomorphism with free kernel, of $K$ onto a finite group $K_1$. This homomorphism is an isomorphism on $F$ and on $\phi(F)$; let $F_1$ be the image of $F$. The group

$$\langle t, K_1 \colon \text{rel } K_1,\ tF_1t^{-1} = \phi(F_1)\rangle$$

can be mapped homomorphically onto a finite group $G^*$ in such a way that on $K_1$ this homomorphism is an isomorphism (see, e.g., Theorem 2 of [6]). The composite homomorphism is an isomorphism on $F$; therefore the kernel $N$ is a free product of a free group and conjugates of $N \cap K$. But $N \cap K$ is just the kernel of the given homomorphism of $K$ onto $K_1$. Thus $N$ is again a free group of f.i. in $G$.

More generally, if $G$ is an HNN group whose base is a tree product of arbitrarily many finite groups of uniformly bounded orders, and whose associated subgroups are contained in vertices of the tree product base, then $G$ is a finite extension of a free group. We conjecture that the converse holds, i.e., every finite extension of a free group is an HNN group of this form.

COROLLARY. *If $G$ is a f.g. finite extension of a free group and $N$ is the subgroup generated by the elements of finite order in $G$, then $G/N$ is a free group.*

PROOF. It is easy to see that $N$ is just the normal subgroup of $G$ generated by its tree product base when $G$ is presented as an HNN group (1).

As an illustration of the corollary, we obtain a special case of a result in [3]. Let $G$ be a one-relator group

$$G = \langle a, b, c, \cdots; R^q\rangle$$

where $R$ is not a true power in the free group $D$ on $a, b, c, \cdots$. Then $G$ is a finite extension of a free group if and only if $R$ is primitive in $D$. This follows easily since in this case $G/N = \langle a, b, c, \cdots; R\rangle$, and this last group is a free group if and only if $R$ is primitive in $D$.

The rank of a free subgroup of f.i. can be computed from its index by using the following generalization of the Schreier rank formula.

THEOREM 2. *Let $G$ be as in* (1) *and let $H$ be a free subgroup of rank $r$ and index $j$. Then the rank $r$ of $H$ is given by the formula in* (2).

PROOF. Let $C$ be a f.g. group and let $D$ be a free subgroup of rank $r$ and of f.i. $j$ in $C$. The the number $\chi(C)$, which Stallings [12] refers to as "Wall's rational Euler characteristic" of $C$, is defined by the formula $\chi(C) = (1-r)/j$. Clearly, if $C$ is finite, $\chi(C)$ is just the reciprocal of the order of $C$. Now $\chi(C)$ is independent of the particular free subgroup of finite index used. Moreover, in [12] it is shown that $\chi(A * B;\ F) = \chi(A) + \chi(B) - \chi(A \cap B)$, where $F$ is finite and $\chi(A)$, $\chi(B)$ are defined. Moreover, in [8] it is shown that

$$\chi(\langle t, K;\ \text{rel } K,\ tFt^{-1} = \phi(F)\rangle) = \chi(K) - \chi(F),$$

where $F$ is finite and $\chi(K)$ is defined.

If we apply these formulas to the given group $G$, we obtain an expression for $\chi(G) = (1 - r)/j$ in terms of the characteristics of the component groups, and the generalized Schreier rank formula follows immediately.

## 3. Groups which are infinite cyclic extensions of free groups and have a non-trivial center

LEMMA: *Let $G = \pi^*(A_i \mid U_{jk} = U_{kj})$ be a tree product of finitely many finite cyclic groups $A = \{\cdots, A_i, \cdots\}$ where $|A_i| = \gamma_i$ and $U_{jk} \neq A_j$. Then $G$ has a presentation*

$$\text{(5)} \qquad \langle \cdots, a_i, \cdots a_j, \cdots ; \cdots, a_i^{\gamma_i} = 1, \cdots, a_j^{\alpha_{jk}} = a_k^{\alpha_{kj}}, \cdots \rangle$$

*where $a_i$ generates $A_i$, $a_j^{\alpha_{jk}}$ generates $U_{jk}$, and $\alpha_{jk} \mid \gamma_j, \gamma_j/\alpha_{jk} = \gamma_k/\alpha_{kj}$.*

PROOF. We shall prove the lemma by induction on the maximal level of the vertices of the tree product where we choose some fixed vertex to have level zero.

The lemma is obviously true for a tree product consisting of exactly one vertex. Let $G$ be a tree product in which the maximal level of its vertices is $n + 1$. Let $B$ consist of those vertices of level $n + 1$. Then $A - B$ determines a tree product $H$ in which $n$ is the maximal level of its vertices. Hence by our inductive hypothesis $H$ has a presentation (5), where $A_i \in A - B$. Thus a presentation for $G$ is obtained by adjoining to the presentation (5) generators $a_i$ for $A_i \in B$, and the corresponding relations $a_i^{\gamma_i} = 1$, as well as relations $\cdots, a_p^{\alpha_{pq}} = a_q^{\beta_{qp}}, \cdots$, for each amalgamation of subgroups $U_{pq}$ and $U_{qp}$ from vertices $A_p \in A$, $A_q \in B$ where $a_p^{\alpha_{pq}}, a_q^{\beta_{qp}}$ generate $U_{pq}$ and $U_{qp}$ respectively, and $\alpha_{pq} \mid \gamma_p$.

Define $\alpha_{qp} = gcd(\beta_{qp}, \gamma_q)$. We know by Dirichlet's theorem that the arithmetic sequence

$$\delta(n) = \beta_{qp}/\alpha_{qp} + (\gamma_q/\alpha_{qp})n$$

includes infinitely many primes. Choose $n$ such that $\delta = \delta(n)$ is a prime that does not divide $\gamma_q$. Now $a_q^\delta$ is a generator of $A_q$, since $gcd(\delta, \gamma_q) = 1$. Moreover,

$$(a_q^\delta)^{\alpha_{qp}} = a_q^{\beta_{qp} + \gamma_q n} = a_q^{\beta_{qp}}.$$

Hence by replacing each generator $a_q$ of $A_q \in B$ by $a_q^\delta$ we obtain a tree product with presentation of the form (5).

THEOREM 3. *If $G$ is an infinite cyclic extension of a free group $H$ of finite rank and if $G$ has a non-trivial center $Z$, then $G$ is an HNN group whose base is a tree product of infinite cyclic groups and whose associated subgroups are contained in vertices. Specifically, $G$ has a presentation*

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$$\langle t_1, \cdots, t_n, \cdots, a_i, \cdots, a_j, \cdots; \cdots, a_j^{\alpha_{jk}} = a_k^{\alpha_{kj}}, \cdots, t_m a_p^{\varepsilon_m} t_m^{-1} = a_q^{\varepsilon'_m}, \cdots\rangle$$

*where each $a_i$ generates a vertex of the tree, and for each $t_m$ there corresponds associated subgroups generated by $a_p^{\varepsilon_m}$ and $a_q^{\varepsilon'_m}$ respectively. Moreover, $Z$ is in the center of the tree product base, or $G$ is free abelian of rank two.*

PROOF. If $H$ has rank zero, the result is trivial. If $H$ has rank one, then $G = \langle a, b; aba^{-1} = b^{\varepsilon}\rangle$, $\varepsilon = \pm 1$, and again the result is obvious, where one writes $\langle a, b; aba^{-1} = b^{-1}\rangle$ as $\langle c, d; c^2 = d^2\rangle$.

Assume therefore that $H$ has rank at least two; then $H \cap Z = 1$, and $Z$ is infinite cyclic. Now $G/Z$ is a finite cyclic extension of the free group $HZ/Z$, since $(G/Z)/(HZ/Z) \simeq G/HZ \simeq (G/H)/(HZ/H)$ is a finite cyclic group. Moreover, any finite subgroup of $G/Z$ intersects $HZ/Z$ trivially and must therefore be cyclic. By Theorem 1 and the lemma, $G/Z$ has a presentation with generators $t_1, \cdots, t_n, \cdots, a_i, \cdots$ and defining relations

$$\cdots, a_i^{\gamma_i} = 1, \cdots \tag{6}$$

$$\cdots, a_j^{\alpha_{jk}} = a_k^{\alpha_{kj}}, \cdots \tag{7}$$

$$\cdots, t_m a_p^{\lambda_m} t_m^{-1} = a_q^{\mu_m}, \cdots, \tag{8}$$

where $\gamma_i$, $\alpha_{jk}$, are positive integers,

$$\alpha_{jk} \,|\, \gamma_j,\ \gamma_j/\alpha_{jk} = \gamma_k/\alpha_{kj}, \text{ and } \gamma_p/(\gamma_p, \lambda_m) = \gamma_q/(\gamma_q, \mu_m).$$

To obtain a presentation for $G$, we select pre-images $u_1, \cdots, u_n$ in $G$ of $t_1, \cdots, t_n$ in $G/Z$; moreover, let $z$ be a generator of $Z$. Since $G$ is torsion-free, it follows that the pre-image of a finite cyclic vertex $gp(a_i)$ is an infinite cyclic group $B_i$. Choose a generator $b_i$ of $B_i$ so that $z = b_i^{\eta_i}$ where $\eta_i$ is positive. Clearly, $G$ is generated by $u_1, \cdots, u_n, \cdots, b_i, \cdots$. To obtain a simple set of defining relations for $G$, we first present $G/Z$ on the images of these generators for $G$. This is obtained from the presentation with defining relations (6), (7), (8) by applying Tietze transformations which define a generator $b_i$ in $G/Z$ as $a_i^{\delta'_i}$ and then solving for $a_i = b_i^{\delta_i}$ where $\delta_i \delta'_i \equiv 1 \pmod{\gamma_i}$. The resulting presentation for $G/Z$ is

$$\langle t_1, \cdots, t_n, \cdots, b_i, \cdots;$$

$$\cdots, b_i^{\gamma_i} = 1, \cdots \tag{9}$$

$$\cdots, b_j^{\alpha_j \delta_j} = b_k^{\alpha_{kj}\delta_k}, \cdots \tag{10}$$

$$\cdots, t_m b_p^{(\lambda_m \delta_p)} t_m^{-1} = b_q^{(\mu_m \delta_q)}, \cdots\rangle. \tag{11}$$

Now the relations (11) may be replaced by

$$\cdots, t_m b_p^{\varepsilon_m} t_m^{-1} = b_q^{\sigma_m}, \cdots \tag{12}$$

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where $\varepsilon_m \mid \gamma_p$ and $\gamma_p / \varepsilon_m = \gamma_q / (\gamma_q, \sigma_m)$. Hence $G$ has a presentation on the generators $u_1, \cdots, u_n, \cdots, b_i, \cdots, z$ with defining relations

$$\cdots, b_i^{\gamma_i} = z^{\nu_i}, \cdots \tag{13}$$

$$\cdots, b_j^{\alpha_{jk}\delta} = b_k^{(\alpha_{kj}\delta_k)} z^{\tau_{jk}}, \cdots \tag{14}$$

$$\cdots, u_m b_p^{\varepsilon_m} u_m^{-1} = b_q^{\sigma_m} z^{\theta_m}, \cdots \tag{15}$$

$$\cdots, [b_i, z] = 1, \cdots \tag{16}$$

$$\cdots, [u_m, z] = 1, \cdots \tag{17}$$

for suitable integers $\nu_i$, $\tau_{jk}$, and $\theta_m$.

Using the fact that $z = b_i^{\eta_i}$ and that $b_i$ has order $\gamma_i$ in $G/Z$, it follows that $\gamma_i \mid \eta_i$, and therefore $z = (z^{\nu_i})^{\eta_i/\gamma_i}$, so that $\nu_i = 1$. Thus (13) becomes

$$\cdots, b_i^{\gamma_i} = z, \cdots \tag{18}$$

and (16) can be eliminated.

We now simplify the relations of (14) by determining $\tau_{jk}$. From (14) we obtain the following:

$$z^{\delta_j} = b_j^{\gamma_j\delta_j} = (b_j^{\alpha_{jk}\delta_j})^{(\gamma_j/\alpha_{jk})} = b_k^{\alpha_{kj}\delta_k(\gamma_k/\alpha_{kj})} z^{\tau_{jk}\gamma_j/\alpha_{jk}} = z^{\delta_k + \tau_{jk}(\gamma_j/\alpha_{jk})}.$$

Thus $\tau_{jk}\gamma_j / \alpha_{jk} = \delta_j - \delta_k = \tau_{jk}\gamma_k / \alpha_{kj}$. Hence

$$b_j^{\alpha_{jk}\delta_j} = b_k^{\alpha_{kj}\delta_k} z^{\tau_{jk}} = b_k^{\alpha_{kj}\delta_k + \gamma_k\tau_{jk}} = b_k^{\alpha_{kj}\delta_j}.$$

Thus by using (18), we can replace (14) by

$$\cdots, b_j^{\alpha_{jk}\delta_j} = b_k^{\alpha_{kj}\delta_j}, \cdots. \tag{19}$$

We next show in fact that the relations

$$\cdots, b_j^{\alpha_{jk}} = b_k^{\alpha_{kj}}, \cdots \tag{20}$$

are derivable from (18) and (19). Since $(\delta_j, \gamma_j / \alpha_{jk}) = 1$ it follows that

$$1 = x\delta_j + y(\gamma_j / \alpha_{jk}) = x\delta_j + y(\gamma_k / \alpha_{kj})$$

for some integers $x$, $y$. From (18) we obtain $b_j^{\gamma_j} = b_k^{\gamma_k}$. Hence,

$$b_j^{\alpha_{jk}} = b_j^{\alpha_{jk}(x\delta_j + y(\gamma_j/\alpha_{jk}))} = b_k^{\alpha_{kj}(x\delta_j + y(\gamma_k/\alpha_{kj}))} = b_k^{\alpha_{kj}}.$$

Consequently (20) is equivalent to (19) in the face of (18).

We may also simplify the relations of (15) by determining $\theta_m$. Now

$$\begin{aligned} z &= b_p^{\gamma_p} = (u_m b_p^{\varepsilon_m} u_m^{-1})^{(\gamma_p/\varepsilon_m)} = b_q^{\sigma_m\gamma_p/\varepsilon_m} z^{\theta_m\gamma_p/\varepsilon_m} \\ &= b_q^{\sigma_m\gamma_q/(\gamma_q,\sigma_m)} z^{\theta_m\gamma_p/\varepsilon_m} = z^{\sigma_m/(\gamma_q,\sigma_m)} z^{\theta_m\gamma_p/\varepsilon_m} \end{aligned}$$

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$$= z^{\sigma_m/(\gamma_q,\sigma_m)+\theta\gamma_m/\varepsilon_m}.$$

Therefore

$$1 = \sigma_m/(\gamma_q,\sigma_m) + \theta_m\gamma_p/\varepsilon_m = \sigma_m/(\gamma_q,\sigma_m) + \theta_m\gamma_q/(\gamma_q,\sigma_m).$$

Thus

$$b_q^{\sigma_m}z^{\theta_m} = b_q^{\sigma_m+\gamma\cdot\theta_m} = b_q^{(\gamma_q,\sigma_m)}.$$

Hence (15) can be replaced by

$$(21) \qquad \cdots, u_m b_p^{\varepsilon_m} u_m^{-1} = b_q^{(\gamma_q,\sigma_m)}, \cdots.$$

Let $\varepsilon'_m = (\gamma_q, \sigma_m)$. Clearly (17) is derivable from (21) because

$$u_m z u_m^{-1} = u_m b_p^{\gamma_p} u_m^{-1} = (u_m b_p^{\varepsilon_m} u_m^{-1})^{\gamma_p/\varepsilon_m} = (b_q^{\varepsilon_m})^{\gamma_q/\varepsilon_m} = b_q^{\gamma} = z.$$

Thus $G$ has presentation

$$\langle u_1, \cdots, u_n, \cdots, b_i, \cdots, z; \cdots, b_i^{\gamma_i} = z, \cdots, b_j^{\alpha_{jk}} = b_k^{\alpha_{kj}}, \cdots, u_m b_p^{\varepsilon_m} u_m^{-1} = b_q^{\varepsilon'_m}, \cdots\rangle$$

with $\gamma_j/\alpha_{jk} = \gamma_k/\alpha_{kj}$. Hence $G$ can be presented by

$$\langle u_1, \cdots, u_n, \cdots, b_i, \cdots; \cdots, b_j^{\alpha_{jk}} = b_k^{\alpha_{kj}}, \cdots, t_m b_p^{\varepsilon_m} t_m^{-1} = b_q^{\varepsilon'_m}, \cdots\rangle.$$

Consequently, $G$ is an HNN group with base a tree product of infinite cyclic groups and associated subgroups contained in the vertices of the base.

The following corollary is not difficult to prove:

COROLLARY: *Let $G$ be a f.g. group with non-trivial centre. Then $G$ is a tree product of infinite cyclic groups if and only if the following three conditions are satisfied*: (i) *$G$ has an infinite cyclic center $Z$*; (ii) *$G$ is an infinite cyclic extension of a f.g. free group*; *and* (iii) *$G'Z$ is of f.i. in $G$, where $G'$ is the commutator subgroup of $G$.*

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