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Abstract

Let Y be a homology sphere which contains an incompressible torus. We show that Y cannot be an L-space, i.e. the rank of $\widehat{\mathrm{HF}}(Y)$ is greater than 1. In fact, if the homology sphere Y is an irreducible L-space, then Y is S^3, the Poincaré sphere $\Sigma(2,3,5)$ or hyperbolic.

1. Introduction

1.1 Background and main results

Heegaard Floer theory, defined by Ozsváth and Szabó [OS04a], has been powerful in extracting topological properties of three-manifolds. Surprisingly, in rare cases homology spheres have the Heegaard Floer homology of S^3. The Poincaré sphere $\Sigma(2,3,5)$ is an example of an irreducible homology sphere with $\widehat{\mathrm{HF}}(\Sigma(2,3,5)) = \widehat{\mathrm{HF}}(S^3) = \mathbb{Z}$. It is thus not true in general that Heegaard Floer homology is capable of distinguishing S^3 from other homology spheres. However, a conjecture of Ozsváth and Szabó predicts that $\Sigma(2,3,5)$ is the only non-trivial example of an irreducible homology sphere with trivial Heegaard Floer homology. In this paper, we address the case of a three-manifold which contains an incompressible torus. Throughout the paper, we let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Theorem 1.1. *If a homology sphere Y contains an incompressible torus, then*

$$\widehat{\mathrm{HF}}(Y;\mathbb{F}) \neq \mathbb{F} = \widehat{\mathrm{HF}}(S^3;\mathbb{F}).$$

By Thurston's geometrization conjecture/Perelman's theorem (see [Thu82, Per03], see also [MT07, MT14]), Theorem 1.1 reduces the Ozsváth–Szabó conjecture to the homology spheres which are either Seifert fibered or hyperbolic. It is shown that $\Sigma(2,3,5)$ and S^3 are the only Seifert fibered homology spheres with trivial Heegaard Floer homology [Rus04, Eft09]. The Ozsváth–Szabó conjecture is thus reduced to the following.

Conjecture 1.2. If the homology sphere Y is hyperbolic, then $\widehat{\mathrm{HF}}(Y;\mathbb{F}) \neq \mathbb{F}$.

Besides Seifert fibered homology spheres, the Ozsváth–Szabó conjecture was also proved for graph manifolds by Boileau and Boyer [BB15].

If the homology sphere Y includes an incompressible torus, it is obtained by splicing the complements of a pair of non-trivial knots K_1 and K_2 in the homology spheres Y_1 and Y_2, respectively. In this case, we write $Y = Y(K_1, K_2)$. Theorem 1.1 may then be re-stated as the following.

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Theorem 1.3. *If K_1 and K_2 are non-trivial, then* $\widehat{\mathrm{HF}}(Y(K_1,K_2);\mathbb{F})\neq\mathbb{F}$.

When both Y_1 and Y_2 are L-spaces, Theorem 1.3 is the main result of [HL16]. When $Y_1=S^3$ and K_1 is the trefoil, Theorem 1.3 is [Eft15, Corollary 1.3].

The reduced Khovanov homology of a knot $K\subset S^3$ is related to the Heegaard Floer homology of the double cover of S^3 branched over K [OS05]. The Ozsváth and Szabó Conjecture 1.2 thus implies that the reduced Khovanov homology (and thus Khovanov homology) detects the unknot; a theorem of Kronheimer and Mrowka [KM11]. The result of this paper re-proves a few special cases of the aforementioned theorem. A knot $K\subset S^3$ is *π-hyperbolic* if S^3-K admits a Riemannian metric with constant negative curvature which becomes singular folding with angle π around K.

Corollary 1.4. *Suppose that $K\subset S^3$ is either a prime satellite knot or it is not π-hyperbolic. Then the rank of the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ is greater than* 1.

1.2 Bordered Floer homology for a knot complement

Let K be an oriented knot inside the homology sphere Y and $Y(K)$ denote the bordered manifold determined from the knot complement $Y-\mathrm{nd}(K)$ by parametrizing its boundary using a meridian μ and a zero-framed longitude λ for K. The proof of Theorem 1.1 rests heavily on a construction of the bordered Floer module $\widehat{\mathrm{CFD}}(Y(K))$ using the knot Floer complex $\mathrm{CFK}^\bullet(Y,K)$, which we will now describe. Consider a doubly pointed Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,z)$ for K, and let $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ denote the totally real tori in the symmetric product $\mathrm{Sym}^g(\Sigma)$ which correspond to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. The markings u and z determine the map

$$\mathfrak{s}=\mathfrak{s}_{u,z}:\mathbb{T}_\alpha\cap\mathbb{T}_\beta\longrightarrow\underline{\mathrm{Spin}}^c(Y,K),$$

where $\mathfrak{s}(\mathbf{x})$ denotes the relative Spin^c class assigned to $\mathbf{x}$ in the sense of [Ni09], which is defined by assigning a nowhere-vanishing vector field on $Y-\mathrm{nd}(K)$ to $\mathbf{x}$ which is tangent to the boundary. Multiplying the vector fields by -1 gives an involution map J on $\underline{\mathrm{Spin}}^c(Y,K)$, and the map

$$\mathfrak{s}\mapsto\frac{1}{2}(c_1(\mathfrak{s})-\mathrm{PD}[\mu])=\frac{\mathfrak{s}-J(\mathfrak{s})-\mathrm{PD}[\mu]}{2}\in\mathrm{H}^2(Y,K;\mathbb{Z})$$

and the evaluation of cohomology classes over a Seifert surface for the knot K give an identification of $\underline{\mathrm{Spin}}^c(Y,K)$ with $\mathbb{Z}$, which will be implicit in this paper. In this convention, although the set of relative Spin^c structures only depends on the knot complement, the identification with $\mathbb{Z}$ depends on the meridian of K. If $\mathfrak{s}\in\underline{\mathrm{Spin}}^c(Y,K)$ is a relative Spin^c structure and $i\in\mathbb{Z}$ is an integer, we will abuse the notation and denote $\mathfrak{s}+i\mathrm{PD}[\mu]$ by $\mathfrak{s}+i$. Under the aforementioned identification of relative Spin^c structures with $\mathbb{Z}$, this is of course a compatible (abuse of) notation. Moreover, we will write $\mathfrak{s}=i$ if $\mathfrak{s}\in\underline{\mathrm{Spin}}^c(Y,K)$ is identified with $i\in\mathbb{Z}$ under the above correspondence.

If $\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ are given, and $\phi\in\pi_2(\mathbf{x},\mathbf{y})$ is a Whitney disc connecting them, we have

$$\mathfrak{s}(\mathbf{x})-\mathfrak{s}(\mathbf{y})=n_z(\phi)-n_u(\phi).$$

Consider the $\mathbb{Z}\oplus\mathbb{Z}$ filtered chain complex

$$C=\langle[\mathbf{x},i,j]\mid\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta,\quad\mathfrak{s}(\mathbf{x})-i+j=0\rangle_{\mathbb{Z}}$$

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associated with K. The differential d of C is defined by

$$d[\mathbf{x}, i, j] = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R})[\mathbf{y}, i-n_z(\phi), j-n_u(\phi)].$$

Following [OS08], we may consider the submodules

$$C\{i=a, j=b\}, \quad C\{i=a, j\leqslant b\} \quad \text{and} \quad C\{i\leqslant a, j=b\}, \quad a,b\in\mathbb{Z}\cup\{\infty\}$$

with the induced structure as a chain complex. Set

$$C\{i=a\} = C\{i=a, j\leqslant\infty\} \quad \text{and} \quad C\{j=b\} = C\{i\leqslant\infty, j=b\}.$$

Note that the order of i and j filtrations used in this paper is the opposite of the ordered one used in [OS04b]. For every relative Spin^c class $\mathfrak{s}\in\underline{\mathrm{Spin}}^c(Y,K)$, define

$$\begin{gathered} i_n^{\mathfrak{s}} = i_n^{\mathfrak{s}}(K) : C\{i\leqslant\mathfrak{s}, j=0\}\oplus C\{i=0, j\leqslant n-\mathfrak{s}-1\} \longrightarrow C\{j=0\},\\ i_n^{\mathfrak{s}}([\mathbf{x},i,0],[\mathbf{y},0,j]) := [\mathbf{x},i,0]+\Xi[\mathbf{y},0,j], \end{gathered}$$

where $\Xi : C\{i=0\}\to C\{j=0\}$ is the chain homotopy equivalence corresponding to the Heegaard moves which change $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};z)$ to $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$. It may be interesting to note that

$$C\{i=0, j\leqslant n-\mathfrak{s}-1\} = C\{i=0, j\leqslant J(\mathfrak{s})+n\}.$$

Let $Y_n(K)$ denote the three-manifold obtained from Y by n-surgery on K and let K_n denote the corresponding knot inside $Y_n(K)$, determined by the aforementioned surgery.

PROPOSITION 1.5. *For every $\mathfrak{s}\in\underline{\mathrm{Spin}}^c(Y_n(K),K_n)=\underline{\mathrm{Spin}}^c(Y,K)$, the homology of the mapping cone $M(i_n^{\mathfrak{s}})$ gives*

$$\mathbb{H}_n(K;\mathfrak{s}) = \widehat{\mathrm{HFK}}(Y_n(K),K_n;\mathfrak{s}).$$

A few warnings are necessary here. The set of relative Spin^c classes associated with $(Y_n(K), K_n)$, as defined here, is naturally identified with $\underline{\mathrm{Spin}}^c(Y,K)$. However, the identification of this latter space with $\mathbb{Z}$, which was fixed above, can be different from the identification induced by K_n. In particular, we do not see the symmetry in knot Floer homology groups corresponding to the integers $s,-s$ (or to $s,-s\in\frac{1}{2}+\mathbb{Z}$), unless an appropriate shift in the Spin^c grading is used. The resulting 'symmetric' grading, which we may call the Alexander grading, takes its values in $\mathbb{Z}$ or $\frac{1}{2}+\mathbb{Z}$, depending on whether n is odd or even. If $\mathbb{H}'_n(K;s)$ denotes the Heegaard Floer homology group $\widehat{\mathrm{HFK}}(Y_n(K),K_n)$ in Alexander grading s, we will have

$$\mathbb{H}'_n(K;s) = \mathbb{H}_n\left(K; s+\frac{n-1}{2}\right) \simeq H_*(M(i_n^{s+(n-1)/2})).$$

It is then easier to understand the symmetry of Heegaard Floer homology from the surgery formula of Proposition 1.5. In particular, the difference between our convention for relative Spin^c structures in this paper and the convention used in [Eft05] and [Eft15] is a result of these two different points of view.

Note that $M(i_0^{\mathfrak{s}})$ is a subcomplex of both $M(i_1^{\mathfrak{s}+1})$ and $M(i_1^{\mathfrak{s}})$. We denote the embedding maps by $F_\infty^{\mathfrak{s}+1} = F_\infty^{\mathfrak{s}+1}(K)$ and $\overline{F}_\infty^{\mathfrak{s}} = \overline{F}_\infty^{\mathfrak{s}}(K)$, respectively. The quotient of $M(i_1^{\mathfrak{s}})$ by $F_\infty^{\mathfrak{s}}(M(i_0^{\mathfrak{s}-1}))$ is isomorphic to $\widehat{\mathrm{CFK}}(K;\mathfrak{s}) \simeq C\{i=\mathfrak{s}, j=0\}$. Denote the quotient map by $F_0^{\mathfrak{s}} = F_0^{\mathfrak{s}}(K)$. We thus obtain a short exact sequence

$$0 \longrightarrow M(i_0^{\mathfrak{s}-1}) \xrightarrow{F_\infty^{\mathfrak{s}}} M(i_1^{\mathfrak{s}}) \xrightarrow{F_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K;\mathfrak{s}) \longrightarrow 0.$$

Similarly, the quotient map $\overline{F}_0^{\mathfrak{s}} = \overline{F}_0^{\mathfrak{s}}(K)$ from $M(i_1^{\mathfrak{s}})$ to $M(i_1^{\mathfrak{s}})/\mathrm{Im}(\overline{F}_\infty^{\mathfrak{s}})$ sits in the short exact sequence

$$0 \longrightarrow M(i_0^{\mathfrak{s}}) \xrightarrow{\overline{F}_\infty^{\mathfrak{s}}} M(i_1^{\mathfrak{s}}) \xrightarrow{\overline{F}_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K;\mathfrak{s}) \longrightarrow 0.$$

Let $C_\bullet(K) = \bigoplus_{\mathfrak{s}\in\mathbb{Z}} C_\bullet(K;\mathfrak{s})$, where $C_\bullet(K;\mathfrak{s}) = M(i_\bullet^{\mathfrak{s}})$ for $\bullet = 0,1$ and $C_\infty(K;\mathfrak{s}) = C\{i=\mathfrak{s}, j=0\}$. Denote the differential of $C_\bullet(K)$ by $d_\bullet$ for $\bullet \in \{0,1,\infty\}$. Set $M(K) = C_0(K) \oplus C_1(K)$ and $L(K) = C_1(K) \oplus C_\infty(K)$. The maps $F_\bullet = F_\bullet(K)$ and $\overline{F}_\bullet = \overline{F}_\bullet(K)$, obtained by putting all $F_\bullet^{\mathfrak{s}}$ and $\overline{F}_\bullet^{\mathfrak{s}}$ together, will be called the *bypass homomorphisms*.

A differential graded algebra $\mathcal{A}(T^2,0)$ is associated with the torus boundary of $Y - \mathrm{nd}(K)$, which will be denoted by $-T^2$. The bordered Floer module $\widehat{\mathrm{CFD}}(Y(K))$ is then a module over $\mathcal{A}(T^2,0)$. Following the notation of § 4.2 from [LOT14], $\mathcal{A}(T^2,0)$ is generated, as a module over $\mathbb{F}$, by the idempotents $\imath_0$ and $\imath_1$ and the chords $\rho_1,\rho_2,\rho_3,\rho_{12}=\rho_1\rho_2,\rho_{23}=\rho_2\rho_3$ and $\rho_{123}=\rho_1\rho_2\rho_3$.

Theorem 1.6. *The bordered Floer complex $\widehat{\mathrm{CFD}}(Y(K))$ is quasi-isomorphic to the left module over $\mathcal{A}(T^2,0)$, which is generated by $\imath_0.L(K)$ and $\imath_1.M(K)$ and is equipped with the differential $\partial\colon \widehat{\mathrm{CFD}}(Y(K)) \to \widehat{\mathrm{CFD}}(Y(K))$ defined by*

$$\partial\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \begin{cases} \begin{pmatrix} d_0(\mathbf{x}) \\ \overline{F}_\infty(\mathbf{x}) + d_1(\mathbf{y})\end{pmatrix} + \begin{pmatrix}\rho_1 F_\infty(\mathbf{x}) \\ \rho_3\overline{F}_0(\mathbf{y}) + \rho_{123}\overline{F}_0(F_\infty(\mathbf{x}))\end{pmatrix} & \text{if } \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \in M(K), \\ \begin{pmatrix} d_1(\mathbf{x}) \\ F_0(\mathbf{x}) + d_\infty(\mathbf{y})\end{pmatrix} + \rho_2\begin{pmatrix}0\\ \mathbf{x}\end{pmatrix} & \text{if } \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \in L(K). \end{cases} \tag{1}$$

This theorem should be compared with Theorem 11.26 from [LOT08], which addresses the case where $Y = S^3$.

The paper is organized as follows. In § 2, we study the surgery formulas for Heegaard Floer homology and prove Proposition 1.5. In [Eft15], a splicing formula for a pair of knots K_1 and K_2 is presented in terms of the groups $\mathbb{H}_\bullet(K_i), \bullet \in \{0,1,\infty\}$, $i=1,2$, and a number of homomorphisms between them (the *bypass homomorphisms*). In § 3, we study the bypass homomorphisms and obtain explicit formulas for them which are compatible with the surgery formula of Proposition 1.5. Together with the results proved in [Eft15], this gives a very explicit splicing formula in terms of knot Floer homology and, in particular, proves Theorem 1.6. Sections 2 and 3 contain the main technical arguments of the paper. With the aforementioned splicing formula in place, we prove a number of basic linear algebra properties of bypass homomorphisms in § 4 and use these properties in § 5 to obtain strong restrictions on the bypass homomorphisms associated with the pairs (K_1,K_2) such that $Y(K_1,K_2)$ are L-spaces. These restrictions are further studied in § 6 to complete the proof of Theorem 1.1. A number of applications are also discussed in § 6.

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2. Surgery on null-homologous knots

2.1 The Heegaard diagram

By a Heegaard n-tuple we mean the data

$$(\Sigma, \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n; u_1, \dots, u_r),$$

where Σ is a Riemann surface of genus g, each $\boldsymbol{\alpha}_i$ is g-tuples of disjoint simple closed curves for $i = 1, \dots, n$ and u_j are markings in $\Sigma - \bigsqcup_{i=1}^n \boldsymbol{\alpha}_i$. Let $\mathbb{T}_i \subset \mathrm{Sym}^g(\Sigma)$ denote the torus associated with $\boldsymbol{\alpha}_i$. Choose $\mathbf{x}_i \in \mathbb{T}_i \cap \mathbb{T}_{i+1}$ for $i = 1, \dots, n-1$ and $\mathbf{x}_n \in \mathbb{T}_1 \cap \mathbb{T}_n$. Let $\pi_2(\mathbf{x}_1, \dots, \mathbf{x}_n)$ denote the set of homotopy classes of n-gons connecting $\mathbf{x}_1, \dots, \mathbf{x}_n$ and

$$\pi_2^j(\mathbf{x}_1, \dots, \mathbf{x}_n; u_1, \dots, u_r) \subset \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_n)$$

denote the subset of classes with Maslov index j which have zero intersection number with the codimension-2 subvarieties $L_{u_1}, \dots, L_{u_r}$ of $\mathrm{Sym}^g(\Sigma)$ corresponding to the markings $u_1, \dots, u_r$. Associated with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j; u_1, \dots, u_r)$ we obtain a *hat* Heegaard Floer complex, which will be denoted by

$$\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j; u_1, \dots, u_r).$$

Let $K \subset Y$ be an oriented knot inside a homology sphere Y. Consider a Heegaard diagram

$$H = (\Sigma, \boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}, \boldsymbol{\beta} = \{\beta_1, \dots, \beta_{g-1}, \beta_g = \lambda_\infty\})$$

for Y so that $(\Sigma, \boldsymbol{\alpha}, \widehat{\boldsymbol{\beta}} = \boldsymbol{\beta} - \{\lambda_\infty\})$ is a Heegaard diagram for $Y - \mathrm{nd}(K)$. Below, we will describe a Heegaard 5-tuple, together with several markings on it, which will be used throughout this section. Figure 1 illustrates a tubular neighbourhood of λ_∞ in this Heegaard diagram, which is called the *winding region*, and contains the markings. In Figure 1, the winding region is the cylinder which is obtained by identifying the upper and lower edges of the illustrated rectangle using a reflection.

We assume that α_g is the only curve in $\boldsymbol{\alpha}$ which cuts λ_∞, and that the rest of the curves in $\boldsymbol{\alpha}$ do not enter the winding region. Suppose that the oriented curve $\lambda \subset \Sigma - \widehat{\boldsymbol{\beta}}$ represents a zero-framed longitude for K, which cuts λ_∞ transversely in a single point. Choose the orientation on λ_∞ so that $\lambda \cdot \lambda_\infty = 1$. Let λ_n be a small perturbation of the juxtaposition $\lambda + n\lambda_\infty$ and β_i^n denote a small Hamiltonian isotope of β_i for $i = 1, \dots, g-1$. The Heegaard diagram

$$H_n = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n = \{\beta_1^n, \dots, \beta_{g-1}^n, \lambda_n\}, p_n)$$

gives a marked diagram for $(Y_n(K), K_n)$, where $p_n \in \lambda_n$ is a marking at the intersection of λ_n and λ_∞ which distinguishes λ_n from other curves in $\boldsymbol{\beta}_n$. With the integers $m > 0$ and n fixed and $\boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}$ constructed as above, we assume that λ_n and λ_{n+m} intersect each other in m transverse points and that, for an intersection point q of these latter curves, the points q, p_n and p_{m+n} are the vertices of a triangle Δ_a, which is one of the connected components in $\Sigma - (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \{\lambda_n, \lambda_{n+m}\})$. We place our first marking a in Δ_a. From the four quadrants which have q as a corner, two of them belong to the neighbours of Δ_a. Place a pair of markings u and v in these two quadrants. Let b denote another marking which is placed in the remaining quadrant around q which is opposite to the quadrant containing a. Label u and v so that the four quadrants surrounding q which contain the markings a, u, b and v appear in clockwise order. We may assume that when we follow the orientation of λ_n and λ_{n+m}, the marked point v is on the right and the marked point u is on the left.

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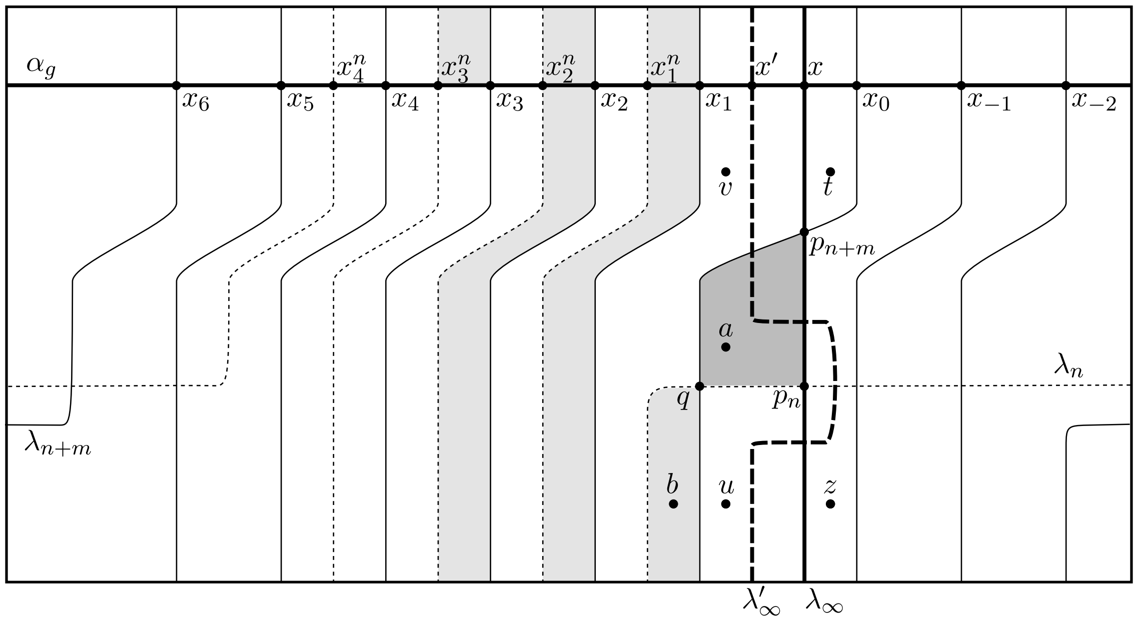

Figure 1. The winding region and the arrangement of the curves $\lambda_n, \lambda_{n+m}, \lambda_\infty, \lambda'_\infty$ and α_g in this region, as well as the markings a, b, t, u, v and z, are illustrated. The winding region is obtained after we identify the upper edge of the rectangle with its lower edge using a reflection. The intersection points of α_g with each one of the curves $\lambda_n, \lambda_{n+m}, \lambda_\infty, \lambda'_\infty$ is labelled. A triangle with vertices p_n, q and p_{n+m} containing the marked point a, and another triangle with vertices q, x_i and x_i^n missing the marked points a, u and v, are shaded.

Let $\boldsymbol{\beta}' = \{\beta'_1, \ldots, \beta'_{g-1}, \lambda'_\infty\}$ denote a set of g simple closed curves which are obtained from $\boldsymbol{\beta}$ by a very small Hamiltonian isotopy in the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}; u, v)$. Thus, β_i and β'_i intersect each other is a pair of cancelling intersection points. We assume that the small area bounded between the two curves λ_∞ and λ'_∞ is a union of two bigons; a small bigon which is a subset of one of the connected components of $\Sigma^\circ = \Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta} - \boldsymbol{\beta}_{n+m}$ and is cut into two triangles by λ_n, and a long and thin bigon which is stretched along λ_∞. For small Hamiltonian perturbations, the chain complex $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'; u)$ may be identified with $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u)$. Choose the marking z in the winding region so that there is an arc connecting z to u on Σ which cuts each one of the curves λ_∞ and λ'_∞ in a single transverse point and stays disjoint from all other curves in $\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{\beta}' \cup \boldsymbol{\beta}_n \cup \boldsymbol{\beta}_{n+m}$. Similarly, choose the marking t so that there is an arc connecting v to t on Σ which cuts each one of λ_∞ and λ'_∞ in a single transverse point and stays disjoint from all other curves in $\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{\beta}' \cup \boldsymbol{\beta}_n \cup \boldsymbol{\beta}_{n+m}$; see Figure 1. In the forthcoming discussions, we will use u and v as the main marked points (punctures) in the Heegaard diagram, while the rest of the markings help us keep track of the coefficients of holomorphic polygons in the domains containing them.

Let us denote the intersection point of λ_∞ with α_g by x. Every generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is then forced to include $x \in \alpha_g \cap \lambda_\infty$. Associated with every generator $\mathbf{x}$ for the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u)$, which in turn is a generator of $\widehat{\mathrm{CF}}(Y)$, we obtain $n+m$ generators for $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v)$. These $n+m$ generators will be denoted by

$$\mathbf{x}_{1-l}, \mathbf{x}_{2-l}, \ldots, \mathbf{x}_{m+n-l}, \quad \text{where } l = \left\lceil \frac{m}{2} \right\rceil.$$

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The generator $\mathbf{x}_i$ is obtained from $\mathbf{x}$ by replacing x with $x_i \in \alpha_g \cap \lambda_{m+n}$. The points

$$x_{1-l}, x_{2-l}, \dots, x_{m+n-l}$$

are all the intersection points between α_g and λ_{n+m} in the winding region, and appear with this order on α_g. We may assume that x_i is on the right of λ_∞ if $i \leqslant 0$ and is on the left of λ_∞ otherwise. The rest of the generators for the complex $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v)$ are in correspondence with the generators $\mathbf{y}$ of $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_0; u, v)$. Every such generator will be denoted by $\widehat{\mathbf{y}}$. Similarly, associated with a generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we obtain the generators

$$\mathbf{x}_i^{(n)} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_n}, \quad i = 1, \dots, n,$$

where $\mathbf{x}_i^{(n)}$ is obtained from $\mathbf{x}$ by replacing x with $x_i^n \in \alpha_g \cap \lambda_n$ in the winding region. We assume that $x_1^n, \dots, x_n^n$ appear on the left of λ_∞. Note that there is a triangle with small area in the Heegaard diagram with vertices x_i^n, q and x_i, which misses the markings u, v and a, provided that $i = 1, \dots, n$.

Lemma 2.1. *With the above notation fixed, we have*

$$\mathfrak{s}(\mathbf{x}_i) = \begin{cases} \mathfrak{s}(\mathbf{x}) - i & \text{if } i \leqslant 0, \\ \mathfrak{s}(\mathbf{x}) + n + m - i & \text{if } i > 0 \end{cases} \quad \text{and} \quad \mathfrak{s}(\widehat{\mathbf{y}}) = \mathfrak{s}(\mathbf{y}) + \left\lceil \frac{m}{2} \right\rceil.$$

Proof. The doubly pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v)$ is obtained from the marked Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; p_{n+m})$ by replacing $p_{n+m} \in \lambda_{n+m}$ with the two markings u, v on its two sides.

Choose a self-indexing Morse function $h : Y \to [0, 3]$ with unique critical points of indices 0 and 3, and compatible with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v)$. Modify the Morse function so that its critical points remain unchanged, but so that the value of h over the index-2 critical point corresponding to λ_{n+m} becomes $8/3$. The pre-images of $3/2$ and $7/3$ under h are then the surface Σ and a torus T, respectively. If we use the flow of a gradient-like vector field ζ for h, from the curve $\lambda_{n+m} \subset \Sigma$ we obtain a curve $\lambda_{n+m}^\circ \subset T$ which corresponds to a meridian for the knot K_{n+m}. Moreover, the knot complement $Y \backslash \mathrm{nd}(K) = Y_{n+m} \backslash \mathrm{nd}(K_{m+n})$ may be identified with $U = h^{-1}[0, 7/3]$. For such Heegaard diagrams, the map

$$s_{u,v} : \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_{n+m}} \to \underline{\mathrm{Spin}}^c(Y_{n+m}, K_{n+m}) = \underline{\mathrm{Spin}}^c(Y, K)$$

may be defined using the Morse function h as follows.

For a generator $\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_{n+m}}$, we may write $\mathbf{z} = \{z_1, \dots, z_g\}$, where $z_i \in \beta_i^{n+m}$ for $i = 1, \dots, g$. The points z_i determine a set of g flow lines for ζ connecting the critical points of index 1 to the critical points of index 2. One of these flow lines leaves U and the rest stay inside U. Modify ζ in a neighbourhood of the latter $g - 1$ flow lines so that the zeros of ζ at the end-point critical points are cancelled against each other. After these modifications, the new vector field ζ' will have two zeros in U. One of these zeros is at the unique critical point of index 0 and the other one is at a critical point of index 1, which corresponds to the α-curve containing z_g, i.e. α_g. The flow line of ζ which passes through z_g starts from the latter index-1 critical point and connects it to $z_g \in \lambda_{n+m}$ to give an oriented arc δ_1. There is an oriented arc δ_1 on λ_{n+m} which connects z_g to p_{n+m} in the direction of λ_{n+m}. Finally, the flow line of $-\zeta$ which passes through p_{n+m} determines another oriented arc δ_3 from $p_{n+m} \in \lambda_{n+m}$ to the critical point of index 0. Putting the three arcs δ_1, δ_2 and δ_3 together, we obtain a path δ in the interior of U which connects the remaining two critical points. These two critical points have indices 0 and 1 which are of different parity. Thus, the vector field ζ' may further be modified in a neighbourhood of δ to arrive at a nowhere-vanishing vector field ζ'' which restricts to the outward normal of $T = \partial(Y\backslash \mathrm{nd}(K))$. The restriction of this nowhere-vanishing vector field to the boundary is naturally isotopic to the translation-invariant vector field, and we thus obtain an element $\mathfrak{s}_{u,v}(\mathbf{z})$ of $\underline{\mathrm{Spin}}^c(Y, K)$.

Let us now assume that $\mathbf{z} = \mathbf{x}_0$ for some $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Then $\mathfrak{s}(\mathbf{x})$ is defined by modifying ζ' in a neighbourhood of a path δ', which is obtained by putting the arcs δ'_1, δ'_2 and δ_3 together. Here, δ'_1 is the flow line from x to the critical point corresponding to α_g and δ'_2 is an arc connecting x to p_{n+m} on λ_∞. The domain containing the marking t gives an isotopy between δ and δ' which is supported away from the other arcs, and the vector fields representing $\mathfrak{s}(\mathbf{x})$ and $\mathfrak{s}(\mathbf{x}_0)$ are thus isotopic relative to the boundary.

For an intersection point $\mathbf{x}_i$ with $i < 0$, the path δ'' which corresponds to $\mathbf{x}_i$ connects the same critical points as δ does, while it differs from δ in a closed curve isotopic to i times the curve λ_∞, which represents the meridian μ_K of the knot K. It follows that

$$\mathfrak{s}(\mathbf{x}_i) = \mathfrak{s}(\mathbf{x}_0) - i\mathrm{PD}[\mu_K] = \mathfrak{s}(\mathbf{x}) - i.$$

For $\mathbf{x}_i$ with $i > 0$, note that δ''_1 first goes out from the winding region and then returns to it from the right-hand side. It follows that the difference between δ'' and δ is $-\mathrm{PD}[\lambda_{n+m}] + i\mathrm{PD}[\mu_K]$, which gives

$$\mathfrak{s}(\mathbf{x}_i) = \mathfrak{s}(\mathbf{x}_0) + \mathrm{PD}[\lambda_{n+m}] - i\mathrm{PD}[\mu_K] = \mathfrak{s}(\mathbf{x}) + n + m - i.$$

Finally, if we choose p_0 and p_{n+m} close to each other, the difference between the path δ_{n+m} associated with the intersection point $\widehat{\mathbf{y}} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_{n+m}}$ and the path associated with $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_0}$ is in $\lceil m/2 \rceil \mathrm{PD}[\mu_K]$, implying the last claim. □

2.2 A triangle of chain maps

The complex associated with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$ is denoted by $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$. Note that the punctures u and v are in the same domain in the complement of the curves in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The corresponding chain complex may thus be denoted by $\widehat{\mathrm{CF}}(Y)$ when there is no confusion. The complex associated with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v)$ and a given relative Spin^c class $\mathfrak{s}$ is denoted by $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s})$ (and when there is no confusion by $\widehat{\mathrm{CF}}(K_n; \mathfrak{s})$), while the complex associated with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v)$ and the classes $\mathfrak{s}$ and $\mathfrak{s} + m$ is denoted by

$$C_{n,m}(\mathfrak{s}) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v; \mathfrak{s}) \oplus \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v; \mathfrak{s} + m).$$

Let Θ_f denote the top generator associated with $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}_{n+m}, \boldsymbol{\beta}; u, v)$. Consider the holomorphic triangle map

$$f^{\mathfrak{s}} : C_{n,m}(\mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$$

which is defined by

$$f^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\Delta \in \pi_2^0(\mathbf{x}, \Theta_f, \mathbf{z}; u, v)} \#(\widehat{\mathcal{M}}(\Delta)) \cdot \mathbf{z}. \tag{2}$$

The diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}; u, v)$ determines a cobordism from $Y_n(K) \coprod L$ to $Y_{n+m}(K)$, where $L = L(m, 1)\#(\#^{g-1} S^1 \times S^2)$. The intersection point q determines a canonical Spin^c class

$\mathfrak{s}_q \in \mathrm{Spin}^c(L)$ in the sense of [OS08, Definition 3.2]. Let Θ_g denote the top generator of $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}; u, v)$ which corresponds to $\mathfrak{s}_q$ or equivalently to the intersection point q. Define

$$g^{\mathfrak{s}} : \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v)$$

by

$$g^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_{n+m}}} \sum_{\Delta \in \pi_2^0(\mathbf{x}, \Theta_g, \mathbf{z}; u, v)} \#(\widehat{\mathcal{M}}(\Delta)) \cdot \mathbf{z}.$$

If $\Delta \in \pi_2^0(\mathbf{x}, \Theta_g, \mathbf{z}; u, v)$ is a triangle class which contributes to the coefficient of $\mathbf{z}$ in $g^{\mathfrak{s}}(\mathbf{x})$, the equalities $n_u(\Delta) = n_v(\Delta) = 0$ imply that

$$n_a(\Delta) + n_b(\Delta) = 1 \ \Rightarrow \begin{cases} \text{(i) } n_a(\Delta) = 1 \quad \text{and} \quad n_b(\Delta) = 0, \\ \text{(ii) } n_a(\Delta) = 0 \quad \text{and} \quad n_b(\Delta) = 1. \end{cases}$$

We may thus write $g^{\mathfrak{s}}(\mathbf{x}) = g_a^{\mathfrak{s}}(\mathbf{x}) + g_b^{\mathfrak{s}}(\mathbf{x})$, where $g_a^{\mathfrak{s}}(\mathbf{x})$ and $g_b^{\mathfrak{s}}(\mathbf{x})$ correspond to the contributions from holomorphic triangles of types (i) and (ii), respectively.

Lemma 2.2. *If $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}$ for a generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_n}$, then*

$$g_a^{\mathfrak{s}}(\mathbf{x}) \in \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v; \mathfrak{s}) \quad \textit{and} \quad g_b^{\mathfrak{s}}(\mathbf{x}) \in \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v; \mathfrak{s} + m).$$

Proof. Let us first assume that $\mathbf{x} = \mathbf{z}_i^{(n)}$ for some $\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and some $i = 1, \ldots, n$. Then there is a small triangle class missing u, v and a connecting $\mathbf{x} = \mathbf{z}_i^{(n)}, \Theta_g$ and $\mathbf{z}_i \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_{n+m}}$. The intersection of the domain $\mathcal{D}^i$ of this triangle with the winding region is the small triangle with vertices x_i^n, q and x_i, which is shaded in Figure 1 (for $i = 3$). If $\mathbf{y}$ is a generator in the image of $g_b^{\mathfrak{s}}(\mathbf{x})$, then there is a domain $\mathcal{D}$ (corresponding to the difference between $\mathcal{D}^i$ and the domain of a holomorphic triangle contributing to $g_b^{\mathfrak{s}}$) connecting $\mathbf{y}$ and $\mathbf{z}_i$ with boundary on $\boldsymbol{\alpha}, \boldsymbol{\beta}_n$ and $\boldsymbol{\beta}_{n+m}$ and missing the marked points u, v and a. Subtracting an appropriate multiple of the periodic domain bounded by β_i^n and β_i^{n+m} (for $i = 1, \ldots, g-1$), we may assume that the boundary of $\mathcal{D}$ is on $\alpha \cup \boldsymbol{\beta}^{n+m} \cup \lambda_n$. Moreover, since we know that

$$n_a(\mathcal{D}) = n_u(\mathcal{D}) = n_v(\mathcal{D}) = 0,$$

it follows that $n_b(\mathcal{D}) = 0$. In particular, $\mathcal{D}$ does not have any boundary on λ_n and is thus the domain of a Whitney disc in the Heegaard diagram H_{n+m} which misses u and v, and connects $\mathbf{y}$ and $\mathbf{z}_i$. In particular,

$$\begin{aligned} \mathfrak{s}(\mathbf{y}) &= \mathfrak{s}(\mathbf{z}_i) \\ &= \mathfrak{s}(\mathbf{z}) + m + n - i \\ &= \mathfrak{s}(\mathbf{z}_i^{(n)}) + m \\ &= \mathfrak{s} + m. \end{aligned}$$

Let us now assume that $\mathbf{x}$ is an arbitrary generator and that $\mathbf{y}$ appears in $g_b^{\mathfrak{s}}(\mathbf{x})$. Let $\mathcal{D}_1$ denote the domain of a Whitney disc for the Heegaard diagram H_n connecting $\mathbf{x}$ to a generator of the form $\mathbf{z}_i^{(n)}$ with $i \in \{1, \ldots, n\}$ and missing the markings u, v, and thus the markings a, b. Let $\mathcal{D}_2$ denote the domain of a Whitney disc for H_{n+m} connecting $\mathbf{y}$ to a generator of the form $\mathbf{z}_j$ and missing u, v (and thus a, b). If $\mathcal{D}_0$ is the domain of the triangle which corresponds to the contribution of $\mathbf{y}$ in $g_b^{\mathfrak{s}}(\mathbf{x})$, and $\mathcal{D}_3$ denotes the domain of a triangle class connecting $\mathbf{z}_i^{(n)}$ to $\mathbf{z}_i$, the domain

$$\mathcal{D} = \mathcal{D}_0 - \mathcal{D}_1 + \mathcal{D}_2 - \mathcal{D}_3$$

is a domain connecting $\mathbf{z}_i$ to $\mathbf{z}_j$ which misses u, v and a, and has its boundary on $\boldsymbol{\alpha} \cup \boldsymbol{\beta}^n \cup \boldsymbol{\beta}_{n+m}$. Subtracting appropriate multiples of the periodic domains bounded by β_i^n and β_i^{n+m} (for $i = 1, \ldots, g-1$), we may assume that the boundary of $\mathcal{D}$ is on $\alpha \cup \boldsymbol{\beta}^{n+m} \cup \lambda_n$. Again, since $n_a(\mathcal{D}) = n_u(\mathcal{D}) = n_v(\mathcal{D}) = 0$, we conclude that $n_b(\mathcal{D}) = 0$ and $\mathcal{D}$ has no boundary on λ_n. It follows that $\mathcal{D}$ is the domain of a Whitney disc which corresponds to the punctured Heegaard diagram H_{n+m}. It is implied that $\mathfrak{s}(\mathbf{z}_i) = \mathfrak{s}(\mathbf{z}_j)$ and, consequently, $i = j$. We thus find that

$$\begin{aligned}\mathfrak{s}(\mathbf{y}) &= \mathfrak{s}(\mathbf{z}_j^{(n)}) \\ &= \mathfrak{s}(\mathbf{z}) + m + n - i \\ &= \mathfrak{s}(\mathbf{z}_i^{(n)}) + m \\ &= \mathfrak{s} + m,\end{aligned}$$

and the proof of the second claim is complete. For the first claim, let $\mathbf{y}_a$ denote a generator which appears in $g_a^{\mathfrak{s}}(\mathbf{x})$ while $\mathbf{y}_b$ denotes a generator which appears in $g_b^{\mathfrak{s}}(\mathbf{x})$. Let $\mathcal{D}_a$ and $\mathcal{D}_b$ denote the domains of the corresponding triangle classes. Then $\mathcal{D} = \mathcal{D}_a - \mathcal{D}_b$ is a domain with coefficient 0 at u, v, which connects $\mathbf{y}_b$ to $\mathbf{y}_a$ and has boundary on $\boldsymbol{\alpha} \cup \boldsymbol{\beta}_n \cup \boldsymbol{\beta}_{n+m}$. Once again, subtracting appropriate multiples of the periodic domains bounded by β_i^n and β_i^{n+m} (for $i = 1, \ldots, g-1$), we may assume that the boundary of $\mathcal{D}$ is on $\alpha \cup \boldsymbol{\beta}^{n+m} \cup \lambda_n$. Furthermore, $n_b(\mathcal{D}) = -n_a(\mathcal{D}) = 1$. It follows that

$$\begin{aligned}\mathfrak{s}(\mathbf{y}_a) &= \mathfrak{s}(\mathbf{y}_b) + (n_v(\mathcal{D}) - n_b(\mathcal{D}))\mathrm{PD}[\lambda_{n+m}] + (n_b(\mathcal{D}) - n_u(\mathcal{D}))\mathrm{PD}[\lambda_n] \\ &= \mathfrak{s} + m - (n+m) + n = \mathfrak{s}.\end{aligned}$$

This completes the proof of the lemma. □

The top generator $\Theta_h \in \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}_n; u, v)$ and the triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_n; u, v)$ determine the holomorphic triangle map

$$h^{\mathfrak{s}} : \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s})$$

which is defined by

$$h^{\mathfrak{s}}(\mathbf{x}) := \sum_{\substack{\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_n} \\ \mathfrak{s}(\mathbf{z}) = \mathfrak{s}}} \sum_{\Delta \in \pi_2^0(\mathbf{x}, \Theta_h, \mathbf{z}; u, v)} \#(\widehat{\mathcal{M}}(\Delta)) \cdot \mathbf{z}.$$

We thus arrive at the following triangle of chain maps:

$$\begin{tikzcd} \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v) \arrow[rr, "h^{\mathfrak{s}} = h^{\mathfrak{s}}_{n,m}"] & & \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s}) \arrow[dl, "g^{\mathfrak{s}} = g^{\mathfrak{s}}_{n,m}"] \\ & C_{n,m}(\mathfrak{s}) \arrow[ul, "f^{\mathfrak{s}} = f^{\mathfrak{s}}_{n,m}"] & \end{tikzcd} \tag{3}$$

2.3 Exactness of triangle

Let $M(f^{\mathfrak{s}})$ denote the mapping cone of $f^{\mathfrak{s}}=f^{\mathfrak{s}}_{n,m}$.

Theorem 2.3. *For $m\gg 1$, there is a map*

$$H^{\mathfrak{s}}_{h_{n,m}}:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_n;u,v;\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,v)$$

which satisfies the relation

$$d\circ H^{\mathfrak{s}}_{h_{n,m}}+H^{\mathfrak{s}}_{h_{n,m}}\circ d=f^{\mathfrak{s}}_{n,m}\circ g^{\mathfrak{s}}_{n,m}.$$

Moreover, the map

$$\jmath^{\mathfrak{s}}_{n,m}:\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}_n;u,v;\mathfrak{s})\to M(f^{\mathfrak{s}}_{n,m}),$$

which is defined by $\jmath^{\mathfrak{s}}_{n,m}(\mathbf{x}):=(g^{\mathfrak{s}}_{n,m}(\mathbf{x}),H^{\mathfrak{s}}_{h_{n,m}}(\mathbf{x}))$ for $\mathbf{x}\in\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_n;u,v;\mathfrak{s})$, is a quasi-isomorphism.

Proof. The proof is almost identical to the proof used in [AE15, § 8]. We outline the proof to set up the notation. Define the map

$$H^{\mathfrak{s}}_{f}:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,v)\to C_{n,m}(\mathfrak{s})$$

by setting

$$H^{\mathfrak{s}}_f(\mathbf{x}):=\sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_{n+m}}\\ \mathfrak{s}(\mathbf{y})\equiv\mathfrak{s}\pmod m}}\ \sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta_h,\Theta_g,\mathbf{y};u,v)}\#(\mathcal{M}(\square))\cdot\mathbf{y}.$$

The condition $\mathfrak{s}(\mathbf{y})\equiv\mathfrak{s}\pmod m$ implies that $\mathfrak{s}(\mathbf{y})\in\{\mathfrak{s};\mathfrak{s}+m\}$, since m is large. Considering all possible boundary degenerations of the one-dimensional moduli space corresponding to the class $\square\in\pi_2^0(\mathbf{x},\Theta_h,\Theta_g,\mathbf{y};u,v)$, we find that

$$d\circ H^{\mathfrak{s}}_f+H^{\mathfrak{s}}_f\circ d=h^{\mathfrak{s}}\circ g^{\mathfrak{s}}.$$

For this, one should note that the contributing boundary degenerations of the form $\square=\Delta\star\Delta'$ with

$$\Delta\in\pi_2^0(\Theta_h,\Theta_g,\Theta;u,v),\quad \Delta'\in\pi_2^0(\mathbf{x},\Theta,\mathbf{y};u,v)$$

and $\Theta\in\mathbb{T}_{\beta_{n+m}}\cap\mathbb{T}_\beta$ come in cancelling pairs. In fact, corresponding to each Δ' the corresponding triangles Δ come in cancelling pairs, where the difference between the coefficients of every cancelling pair at the marking z is always a multiple of m. Similarly, define

$$H^{\mathfrak{s}}_g:C_{n,m}(\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_n;u,v)$$

by setting

$$H^{\mathfrak{s}}_g(\mathbf{x}):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n}}\ \sum_{\substack{\square\in\pi_2^{-1}(\mathbf{x},\Theta_f,\Theta_h,\mathbf{y};u,v)\\ n_z(\square)\equiv 0\pmod m}}\#(\mathcal{M}(\square))\cdot\mathbf{y}.$$

The contributing holomorphic triangles for $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta},\boldsymbol{\beta}_n;u,v)$ which correspond to the closed top generators Θ_f,Θ_h and $\Theta\in\mathbb{T}_{\beta_{n+m}}\cap\mathbb{T}_{\beta_n}$ come in cancelling pairs. The condition $n_z(\square)\equiv 0\pmod m$ implies that if $\mathbf{y}$ appears in $H^{\mathfrak{s}}_g(\mathbf{x})$, then $\mathfrak{s}(\mathbf{y})-\mathfrak{s}(\mathbf{x})$ is a multiple of m. If m

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is sufficiently large, it follows from $\mathfrak{s}(\mathbf{x}) \in \{\mathfrak{s}, \mathfrak{s}+m\}$ that $\mathfrak{s}(\mathbf{y}) = \mathfrak{s}$. Thus, the image of $H_g^{\mathfrak{s}}$ is in $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s})$. Consider the degenerations of a square class

$$\square \in \pi_2^0(\mathbf{x}, \Theta_f, \Theta_h, \mathbf{y}; u, v) \quad \text{with } n_z(\square) \equiv 0 \ (\mathrm{mod}\ m).$$

In degenerations of the form $\square = \Delta \star \Delta'$ with

$$\Delta \in \pi_2^0(\Theta_f, \Theta_h, \Theta; u, v) \quad \text{and} \quad \Delta' \in \pi_2^0(\mathbf{x}, \Theta, \mathbf{y}; u, v),$$

the corresponding triangles Δ come in cancelling pairs. From these observations, we find that

$$d \circ H_g^{\mathfrak{s}} + H_g^{\mathfrak{s}} \circ d = h^{\mathfrak{s}} \circ f^{\mathfrak{s}}.$$

Finally, define the homotopy map

$$H_h^{\mathfrak{s}} : \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$$

by setting

$$H_h^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \ \sum_{\square \in \pi_2^{-1}(\mathbf{x}, \Theta_g, \Theta_f, \mathbf{y}; u, v)} \#(\mathcal{M}(\square)) \cdot \mathbf{y}.$$

Employ the same argument again to show that $d \circ H_h^{\mathfrak{s}} + H_h^{\mathfrak{s}} \circ d = f^{\mathfrak{s}} \circ g^{\mathfrak{s}}$.

We next introduce the pentagon maps. Let $\boldsymbol{\beta}'_n$ denote a g-tuple of simple closed curves which are small Hamiltonian isotopes of the curves in $\boldsymbol{\beta}_n$. Choosing the Hamiltonian isotopy sufficiently small, we may assume that the chain complex $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'_n; u, v; \mathfrak{s})$ associated with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'_n; u, v)$ and the Spin^c class $\mathfrak{s}$ may be identified with $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s})$. We assume that the top intersection point q_n between λ_n and λ'_n is in the winding region, and appears on the common edge between Δ_a and the domain containing the marked point u; see Figure 2. Corresponding to the intersection point p_n we obtain $p'_n \in \lambda_\infty \cap \lambda'_n$. There is a top generator Θ'_h for $\widehat{\mathrm{CF}}(\boldsymbol{\beta}, \boldsymbol{\beta}'_n; u, v)$ which uses p'_n and is in correspondence with Θ_h.

Define the map

$$P_f^{\mathfrak{s}} : \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'_n; u, v; \mathfrak{s}) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v; \mathfrak{s})$$

by setting

$$P_f^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_n}} \ \sum_{\substack{\pentagon \in \pi_2^{-2}(\mathbf{x}, \Theta_g, \Theta_f, \Theta'_h, \mathbf{y}; u, v) \\ n_z(\pentagon) \equiv 0 \ (\mathrm{mod}\ m)}} \#(\mathcal{M}(\pentagon)) \cdot \mathbf{y}.$$

The condition $n_z(\pentagon) \equiv 0 \ (\mathrm{mod}\ m)$ implies that the image of $P_f^{\mathfrak{s}}$ is supported in relative Spin^c class $\mathfrak{s}$. Five types of the 10 possible degenerations in the boundary of the one-dimensional moduli space associated with a class $\pentagon \in \pi_2^{-1}(\mathbf{x}, \Theta_g, \Theta_f, \Theta'_h, \mathbf{y}; u, v)$ with $n_z(\pentagon)$ a multiple of m, which correspond to a degeneration to a bigon and a pentagon, contribute to the coefficient of $\mathbf{y}$ in $(d \circ P_f^{\mathfrak{s}} + P_f^{\mathfrak{s}} \circ d)(\mathbf{x})$. The remaining five types correspond to the degenerations of $\pentagon$ into a square $\square$ and a triangle Δ. Note that:

– there is a unique contributing class $\square \in \pi_2^{-1}(\Theta_g, \Theta_f, \Theta'_h, \Theta; u, v)$ which corresponds to the quadruple $(\Sigma, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}, \boldsymbol{\beta}, \boldsymbol{\beta}'_n; u, v)$. Moreover, $\Theta = \Theta_n$ is the top generator for the diagram $(\Sigma, \boldsymbol{\beta}_n, \boldsymbol{\beta}'_n; u, v)$. The intersection of the domain of this class with the winding region is a rectangle with vertices q, p_{n+m}, p'_n and q_n, which is illustrated in Figure 2. This class has a unique holomorphic representative;

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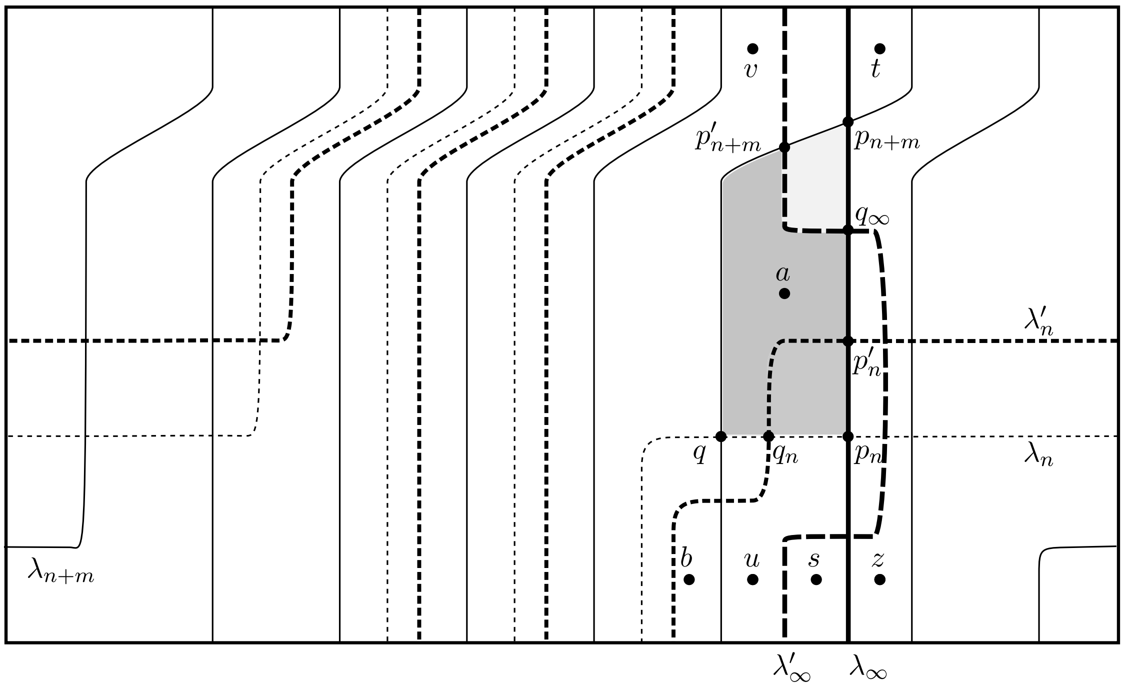

FIGURE 2. The curves λ'_∞ and λ'_n are Hamiltonian isotopes of the curves λ_∞ and λ_n, respectively. The triangle Δ_a is decomposed to two small triangles $[p_n q_n p'_n]$ and $[q_\infty p'_{n+m} p_{n+m}]$ and a pentagon. The union of the pentagon with the first triangle is a rectangle with vertices q, p_{n+m}, p'_n and q_n, which is lightly shaded, and its union with the second triangle is a rectangle with vertices $p_n, q, p'_{n+m}, q_\infty$, which is strongly shaded.

– the contributing triangle classes

$$\Delta \in \pi_2^0(\Theta_g, \Theta_f, \Theta; u, v) \quad \text{and} \quad \Delta' \in \pi_2^0(\Theta_f, \Theta'_h, \Theta; u, v)$$

corresponding to the triples $(\Sigma, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}, \boldsymbol{\beta}; u, v)$ and $(\Sigma, \boldsymbol{\beta}_{n+m}, \boldsymbol{\beta}, \boldsymbol{\beta}'_n; u, v)$ come in cancelling pairs.

These observations imply that $d \circ P_f^{\mathfrak{s}} + P_f^{\mathfrak{s}} \circ d + J_f^{\mathfrak{s}} = h^{\mathfrak{s}} \circ H_h^{\mathfrak{s}} + H_g^{\mathfrak{s}} \circ g^{\mathfrak{s}}$, where

$$J_f^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'_n}} \sum_{\Delta \in \pi_2^0(\mathbf{x}, \Theta_n, \mathbf{y}; u, v)} \#(\mathcal{M}(\Delta)) \cdot \mathbf{y}.$$

Similarly, define the holomorphic pentagon map

$$P_g^{\mathfrak{s}} : \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'; u)$$

by setting

$$P_g^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}} \sum_{\substack{\pentagon \in \pi_2^{-2}(\mathbf{x}, \Theta_h, \Theta_g, \Theta'_f, \mathbf{y}; u, v) \\ n_z(\pentagon) + \mathfrak{s}(\mathbf{x}) \equiv \mathfrak{s} \pmod m}} \#(\mathcal{M}(\pentagon)) \cdot \mathbf{y}.$$

Five types of the 10 possible degenerations in the boundary of the one-dimensional moduli space associated with a pentagon class $\pentagon \in \pi_2^{-1}(\mathbf{x}, \Theta_h, \Theta_g^q, \Theta'_f, \mathbf{y}; u, v)$ contribute to the coefficient of $\mathbf{y}$ in $(d \circ P_g^{\mathfrak{s}} + P_g^{\mathfrak{s}} \circ d)(\mathbf{x})$. The remaining five types correspond to the degenerations of $\pentagon$ into a square and a triangle. The choice of the markings implies that two of these degeneration types contribute to the coefficient of $\mathbf{y}$ in $(f^{\mathfrak{s}}\circ H^{\mathfrak{s}}_f + H^{\mathfrak{s}}_h\circ h^{\mathfrak{s}})(\mathbf{x})$. There is a unique contributing square class, corresponding to $(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}';u,v)$ and the intersection points $\Theta_h,\Theta_g,\Theta'_f,\Theta_\infty$, where Θ_∞ denotes the top generator for $(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}';u,v)$, which includes the top intersection point q_∞ of λ_∞ and λ'_∞. The intersection of the domain of this class with the winding region is the rectangle with vertices p_n,q,p'_{n+m} and q_∞, which is illustrated in Figure 2. Moreover, the triangles which contribute in $\pi_2(\Theta_h,\Theta_g,\Theta_f)$ and $\pi_2(\Theta_g,\Theta'_f,\Theta'_h)$ come in cancelling pairs. Thus, we obtain

$$d\circ P^{\mathfrak{s}}_g + P^{\mathfrak{s}}_g\circ d + J^{\mathfrak{s}}_g = f^{\mathfrak{s}}\circ H^{\mathfrak{s}}_f + H^{\mathfrak{s}}_h\circ h^{\mathfrak{s}},$$

where the map $J^{\mathfrak{s}}_g$ is defined by

$$J^{\mathfrak{s}}_g(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'}}\ \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_\infty,\mathbf{y};u,v)} \#(\mathcal{M}(\Delta))\cdot\mathbf{y}.$$

Let $\boldsymbol{\beta}'_{n+m}$ denote a g-tuple of simple closed curves which are small Hamiltonian isotopes of the curves in $\boldsymbol{\beta}_{n+m}$. Once again, we assume that the Hamiltonian isotope λ'_{n+m} of λ_{n+m} intersects it in a pair of cancelling intersection points, and that the top intersection point is located on the common edge of Δ_a with the domain containing the marking v. Again, we assume that the chain complex associated with $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}'_{n+m};u,v)$ and the Spin^c classes $\mathfrak{s},\mathfrak{s}+m$ is identified with $C_{n,m}(\mathfrak{s})$. There is a top generator Θ'_g for $(\boldsymbol{\beta}_n,\boldsymbol{\beta}'_{n+m})$ which is in correspondence with Θ_g. Define $P^{\mathfrak{s}}_h : C_{n,m}(\mathfrak{s})\to C_{n,m}(\mathfrak{s})$ by

$$P^{\mathfrak{s}}_h(\mathbf{x}) := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'_{n+m}}}\ \sum_{\substack{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_f,\Theta_h,\Theta'_g,\mathbf{y};u,v)\\ n_z(\pentagon)\equiv 0 \pmod m}} \#(\mathcal{M}(\pentagon))\cdot\mathbf{y}.$$

A similar argument implies that $d\circ P^{\mathfrak{s}}_h + P^{\mathfrak{s}}_h\circ d + J^{\mathfrak{s}}_h = g^{\mathfrak{s}}\circ H^{\mathfrak{s}}_g + H^{\mathfrak{s}}_f\circ f^{\mathfrak{s}}$, where

$$J^{\mathfrak{s}}_h(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'_{n+m}}}\ \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_{n+m},\mathbf{y};u,v)} \#(\mathcal{M}(\Delta))\cdot\mathbf{y},$$

and Θ_{n+m} is the top generator of $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}'_{n+m};u,v)$. Since $J^{\mathfrak{s}}_f,J^{\mathfrak{s}}_g,J^{\mathfrak{s}}_h$ are quasi-isomorphisms, [AE15, Lemma 3.3] completes the proof. ◻

Let $\Xi:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};z)\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$ denote the chain homotopy equivalence given by the Heegaard moves which change $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};z)$ to $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$. Define

$$\overline{f}^{\mathfrak{s}} : C_{n,m}(\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};z)$$

by setting

$$\overline{f}^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\Delta\in\pi_2^0(\mathbf{x},\mathbf{y};z,t)} \#(\mathcal{M}(\Delta))\cdot\mathbf{y}.$$

Lemma 2.4. *The chain maps $f^{\mathfrak{s}}$ and $\Xi\circ\overline{f}^{\mathfrak{s}}$ are chain homotopic.*

Proof. Note that the aforementioned Heegaard moves consist of $2g-2$ handle slides (composed with isotopies) on $\boldsymbol{\beta}$, supported away from the markings z,t. Denote the corresponding g-tuples of curves by $\boldsymbol{\beta}^0=\boldsymbol{\beta},\boldsymbol{\beta}^1,\dots,\boldsymbol{\beta}^{2g-2}$, where $\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^{2g-2};z)$ may be identified with $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$.

The triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;z)$ and the top generator Θ^i of the diagram $(\Sigma,\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;z,t)$ determine a chain map $\Xi^i:\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^{i-1};z)\to\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^i;z)$. The triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^i;z,t)$ and the top generator Θ^i_f of $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^i;z,t)$ determine $f^i:\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};z,t)\to\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^i;z)$. Finally, the quadruple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;z,t)$ together with Θ^{i-1}_f and Θ^i determines a homomorphism $H^i:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};z,t)\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^i;z)$. Considering different boundary degenerations of the one-dimensional moduli space associated with a square class of index 0, we find that

$$d\circ H^i+H^i\circ d=f^i+\Xi^i\circ f^{i-1},\quad i=1,\dots,2g-2. \tag{4}$$

Let us define $\Xi=\Xi^{2g-2}\circ\cdots\circ\Xi^1$ and set

$$H=H^{2g-2}+\Xi^{2g-2}\circ H^{2g-3}+\Xi^{2g-2}\circ\Xi^{2g-3}H^{2g-4}+\cdots+(\Xi^{2g-2}\circ\cdots\circ\Xi^2)\circ H^1.$$

Using (4), $d\circ H+H\circ d=f^{2g-2}+\Xi\circ f^0$. To complete the proof, note that $f^{\mathfrak{s}}$ and $\overline{f}^{\mathfrak{s}}$ are the restrictions of f^{2g-2} and f^0 to $C_{n,m}(\mathfrak{s})$, respectively. □

2.4 Surgery formulas

Theorem 2.3 implies that the chain complex

$$\widehat{\mathrm{CFK}}(K_n;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_n;u,v;\mathfrak{s})$$

is quasi-isomorphic, for m sufficiently large, to the mapping cone of

$$f^{\mathfrak{s}}=f^{\mathfrak{s}}_{n,m}:C_{n,m}(\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u).$$

Lemma 2.1 tells us that, with the notation fixed at the beginning of this section, we have

$$\mathfrak{s}(\mathbf{x}_i)=\begin{cases}\mathfrak{s}(\mathbf{x})-i & \text{if } i\leqslant 0,\\ \mathfrak{s}(\mathbf{x})+n+m-i & \text{if } i>0,\end{cases}\quad\text{and}\quad \mathfrak{s}(\widehat{\mathbf{y}})=\mathfrak{s}(\mathbf{y})+\left\lceil\frac{m}{2}\right\rceil.$$

Restricting our attention to the relative Spin^c classes $\mathfrak{s}$ and $\mathfrak{s}+m$, we find that

$$\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};u,v;\mathfrak{s})=\langle\mathbf{x}_{\mathfrak{s}-\mathfrak{s}(\mathbf{x})}\mid\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\text{ and }\mathfrak{s}(\mathbf{x})\leqslant\mathfrak{s}\rangle,$$
$$\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};u,v;\mathfrak{s}+m)=\langle\mathbf{x}_{\mathfrak{s}-\mathfrak{s}(\mathbf{x})-n}\mid\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\text{ and }\mathfrak{s}(\mathbf{x})>\mathfrak{s}-n\rangle.$$

If the curve λ_{n+m} is sufficiently close to the juxtaposition $\lambda\star(m+n)\lambda_\infty$, the first complex is identified with the subcomplex

$$\langle\mathbf{x}\mid\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\text{ and }\mathfrak{s}(\mathbf{x})\leqslant\mathfrak{s}\rangle$$

of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$, while the restriction of the map $f^{\mathfrak{s}}$ to $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};u,v;\mathfrak{s})$ is identified with the inclusion of the above subcomplex in $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$ (cf. proof of Theorem 4.4 in [OS04b]). Similarly, the second complex is identified with the subcomplex

$$\langle\mathbf{x}\mid\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\text{ and }\mathfrak{s}(\mathbf{x})>\mathfrak{s}-n\rangle$$

of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};z)$, while the restriction of the map $\overline{f}^{\mathfrak{s}}$ to $\widehat{\mathrm{CFK}}(K_{n+m};\mathfrak{s}+m)$ is identified with the inclusion of the aforementioned subcomplex in $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};z)$.

Let $C = C_K$ denote the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex generated by triples $[\mathbf{x}, i, j]$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ i, j \in \mathbb{Z}$ and $\mathfrak{s}(\mathbf{x}) - i + j = 0$. The differential of C is defined by

$$\begin{aligned} d[\mathbf{x}, i, j] &= \sum_{\substack{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \phi \in \pi_2^1(\mathbf{x}, \mathbf{y})}} \#(\widehat{\mathcal{M}}(\phi))[\mathbf{y}, i - n_z(\phi), j - n_u(\phi)] \\ &=: \sum_{a,b=0}^{\infty} [d^{a,b}(\mathbf{x}), i - a, j - b]. \end{aligned}$$

Since $d \circ d = 0$, we conclude that $d^{0,0} \circ d^{0,0} = 0$, while

$$\begin{gathered} d^{0,1} \circ d^{0,0} + d^{0,0} \circ d^{0,1} = 0, \quad d^{1,0} \circ d^{0,0} + d^{0,0} \circ d^{1,0} = 0, \\ d^{1,1} \circ d^{0,0} + d^{0,0} \circ d^{1,1} + d^{0,1} \circ d^{1,0} + d^{1,0} \circ d^{0,1} = 0. \end{gathered} \tag{5}$$

Following [OS08] (or the notation of the introduction), $\widehat{\mathrm{CF}}(Y) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u)$ is identified as $C\{j = 0\}$, while

$$\widehat{\mathrm{CFK}}(K_{n+m}; \mathfrak{s}) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v; \mathfrak{s})$$

and

$$\widehat{\mathrm{CF}}(K_{n+m}; \mathfrak{s} + m) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v; \mathfrak{s} + m)$$

are identified with $C\{i \leqslant \mathfrak{s}, j = 0\}$ and $C\{i = 0, j \leqslant n - \mathfrak{s} - 1\}$, respectively. There is a chain homotopy equivalence Ξ from $C\{i = 0\}$ to $C\{j = 0\}$. The following is thus a re-statement of Theorem 2.3.

Theorem 2.5. *For every $\mathfrak{s} \in \mathbb{Z} = \underline{\mathrm{Spin}}^c(Y, K)$ and every $n \in \mathbb{Z}$, the chain complex $\widehat{\mathrm{CFK}}(K_n; \mathfrak{s})$ is quasi-isomorphic to the mapping cone $M(i_n^{\mathfrak{s}})$ of*

$$\begin{gathered} i_n^{\mathfrak{s}} : C\{i \leqslant \mathfrak{s}, j = 0\} \oplus C\{i = 0, j \leqslant n - \mathfrak{s} - 1\} \longrightarrow C\{j = 0\}, \\ i_n^{\mathfrak{s}}([\mathbf{x}, i, 0], [\mathbf{y}, 0, j]) := [\mathbf{x}, i, 0] + \Xi[\mathbf{y}, 0, j]. \end{gathered}$$

3. The splicing formula for knot complements

3.1 The Heegaard diagram

Throughout this section, we will use a Heegaard diagram which is closely related to the Heegaard diagram used in the previous section, and is in fact constructed from it when $n = 0$. In particular, the Heegaard surface Σ and the sets $\boldsymbol{\beta}, \boldsymbol{\beta}_0, \boldsymbol{\beta}_m, \boldsymbol{\beta}'$ and $\boldsymbol{\alpha}$ of simple closed curves are chosen as before. Moreover, we include $\boldsymbol{\beta}_1$, which consists of the Hamiltonian isotopes β_i^1 of β_i for $i = 1, \ldots, g - 1$ and the 1-framed longitude λ_1 for K in the Heegaard diagram. We thus obtain a Heegaard 6-tuple

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_m, \boldsymbol{\beta}, \boldsymbol{\beta}'),$$

which will be studied through this section. As before, we assume that there is a winding region on Σ, which is a subsurface of Σ that is a tubular neighbourhood of $\lambda_\infty = \beta_g$. This winding region is a cylinder which is obtained from the rectangle illustrated in Figure 3 after we identify the upper edge with the lower edge using a reflection. The only curves which enter this cylinder are $\lambda_0, \lambda_1, \lambda_m, \lambda_\infty, \lambda'_\infty$ and α_g. Furthermore, the intersection pattern and the markings a, b, c, e, u, v, w and z on the Heegaard diagram are chosen following the pattern illustrated in Figure 3. Note that the Hamiltonian isotopy which changes $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$ now crosses the marked point u.

We assume that there is a triangle in the Heegaard diagram, with vertices

$$q \in \lambda_0 \cap \lambda_m, \quad q_0 \in \lambda_0 \cap \lambda_1 \quad \text{and} \quad q_m \in \lambda_1 \cap \lambda_n$$

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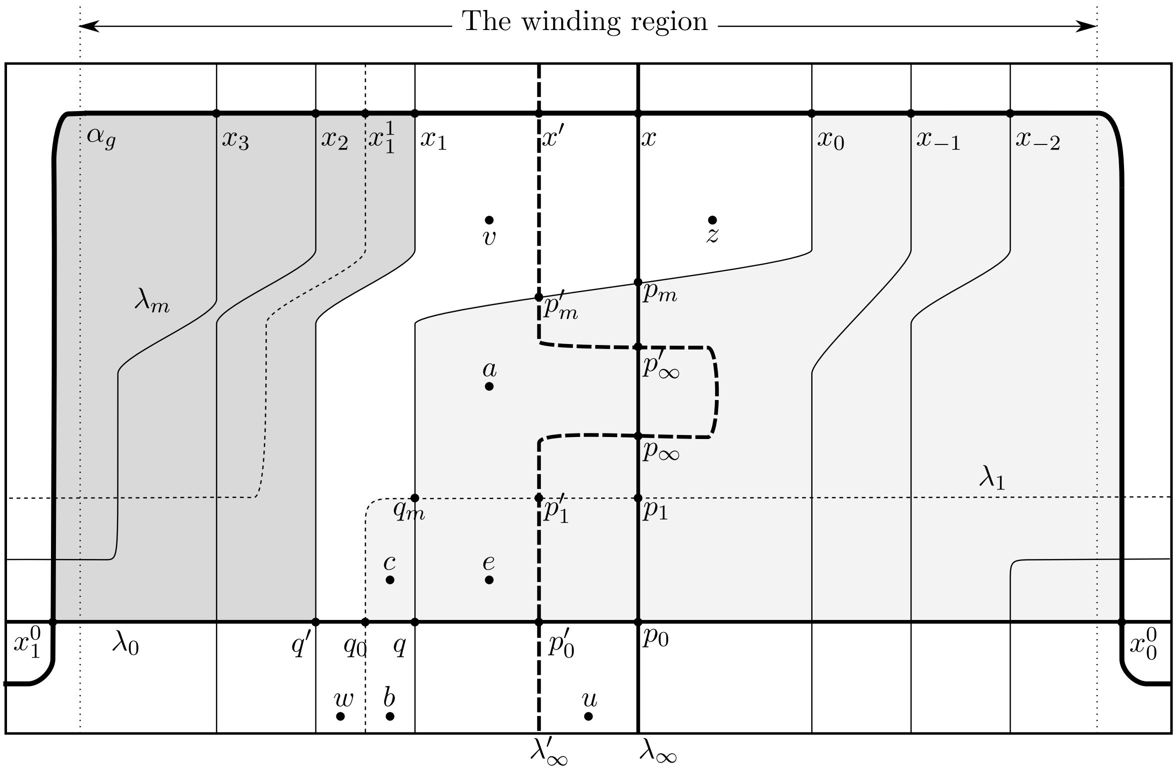

FIGURE 3. The arrangement of the curves in the winding region. The upper edge and the lower edge of the rectangle are identified by a reflection to form the winding region. The shaded area on the right is the intersection of the domain of a square class $\square_0$, which is in $\pi_2^0(\mathbf{w}_0^{(0)},\Theta_{0,1},\Theta_{g_1},\mathbf{w}_0;u,v,w,b)$, with the winding region. It is a 4-gon with vertices x_0^0,q_0,q_m and x_0. The shaded area on the left is the intersection of the domain of a triangle class $\Delta_0\in\pi_2^0(\mathbf{w}_1^{(0)},\Theta'_{g_0},\mathbf{w}_1;u,v,w)$ with the winding region, which is a triangle with vertices x_1^0,q' and x_1.

containing the marked point c, which is one of the connected components in

$$\Sigma^\circ=\Sigma\backslash(\boldsymbol{\alpha}\cup\boldsymbol{\beta}\cup\boldsymbol{\beta}'\cup\boldsymbol{\beta}_0\cup\boldsymbol{\beta}_1\cup\boldsymbol{\beta}_m).$$

The curves λ_0 and λ_m intersect each other in m points, which are all located in the winding region. One of these intersection points, which is next to q and is denoted by q', is characterized with the property that q_0 is the only intersection of the interval $(qq')\subset\lambda_0$ with the curves (other than λ_0) in the Heegaard 6-tuple. We will assume that $\lambda_\bullet$ cuts λ_∞ in $p_\bullet$ and cuts λ'_∞ in $p'_\bullet$ for $\bullet\in\{0,1,m\}$. We denote the top intersection point in $\lambda_\infty\cap\lambda'_\infty$ by p'_∞ and the other intersection point by p_∞.

As before, we assume that α_g cuts λ_∞ and λ'_∞ in unique transverse points, which are denoted by x and x', respectively. We label the intersection points between λ_m and α_g in the winding region by

$$x_i\in\alpha_g\cap\lambda_m,\quad i=1-\left\lceil\frac{m}{2}\right\rceil,2-\left\lceil\frac{m}{2}\right\rceil,\dots,m-\left\lceil\frac{m}{2}\right\rceil=\left\lfloor\frac{m}{2}\right\rfloor,$$

while the intersection of λ_1 with α_g inside the winding region is in a unique point denoted by x_1^1. Finally, we assume that α_g cuts λ_0 right outside the winding region at x_0^0 and x_1^0, as illustrated in Figure 3, although they may have other intersection points away from the winding region. Our convention is that x_i and $x_i^\bullet$ are on the left-hand side of λ_∞ if $i>0$, and are on its right-hand side otherwise. Corresponding to each generator $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ we may thus consider the generators

$$\mathbf{x}'\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\quad \mathbf{x}_0^{(0)},\mathbf{x}_1^{(0)}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_0},\quad \mathbf{x}_1^{(1)}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_1}$$

and

$$\mathbf{x}_i\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_m},\quad i=1-\left\lceil\frac{m}{2}\right\rceil,2-\left\lceil\frac{m}{2}\right\rceil,\dots,m-\left\lceil\frac{m}{2}\right\rceil.$$

Moreover, associated with every generator $\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_0}$, we have a generator $\mathbf{y}^{(1)}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_1}$ and a generator $\mathbf{y}^{(m)}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_m}$ which are not *supported in the winding region*. Lemma 2.1 tells us how to compute the relative Spin^c class associated with any of these generators. An argument similar to the argument used to prove Lemma 2.1 implies that

$$\mathfrak{s}(\mathbf{x}_0^{(0)})=\mathfrak{s}(\mathbf{x}_1^{(0)})=\mathfrak{s}(\mathbf{x}).$$

The chain complex associated with the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_\bullet;u,v,w)$ is the same as the chain complex associated with $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_\bullet;u,v)$, since w is always in the same domain as one of u and v in $\Sigma\backslash(\boldsymbol{\alpha}\cup\boldsymbol{\beta}_\bullet)$. We will sometimes abuse the notation and write

$$\widehat{\mathrm{CF}}(K_\bullet;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_\bullet;u,v,w;\mathfrak{s}).$$

It is important to note that the markings u and v are used to define the map from the set of generators to the set of relative Spin^c classes, and w is only used to define some of the chain maps and homomorphisms in the upcoming discussions.

The intersection point $q_0\in\lambda_0\cap\lambda_1$ (together with the top intersection points between β_i^0 and β_i^1 for $i=1,\dots,g-1$) determines a top generator $\Theta_{0,1}$ for the chain complex $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\beta}_0,\boldsymbol{\beta}_1;u,v,w)$. Similarly, q_m and p_m determine the top generators Θ_{g_1} and Θ_{f_1} which belong to the chain complexes $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_m;u,v,w)$ and $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\beta}_m,\boldsymbol{\beta};u,v,w)$, respectively.

For compatibility of the notation, let us further assume that m is a large even integer. In Figure 4, we consider a subdiagram of our main diagram, namely the Heegaard 4-tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}_0,\boldsymbol{\beta}_m;u,v,w)$. The shaded area, which contains the marking w, may be removed from the surface Σ, and the left-hand-side boundary in the resulting surface may be shifted to the right, so that the two boundary components are identified. The complex structure on Σ induces a complex structure on the resulting closed surface, provided that the winding region carries the quotient complex structure induced from the complex structure on the rectangle, and the curves λ_0 and λ_m are straight lines. Let us denote the resulting Riemann surface by Σ'. Clearly, Σ' is diffeomorphically identified with Σ, while the complex structure it carries slightly differs from the complex structure on Σ in the winding region. We may slightly abuse the notation and denote the resulting Heegaard diagram by $(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}_0,\boldsymbol{\beta}_{m-1};u,v)$. By choosing the shaded area thin enough, we may assume that $\widehat{\mathrm{CF}}(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta}_\bullet;u,v)$ is identified with $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_\bullet;u,v)$ for $\bullet\in\{0,\infty\}$. Correspondingly, we obtain the top generators $\Theta_{f_0},\Theta_{g_0}$ and Θ_{h_0}. The top generators Θ_{f_0} and Θ_{h_0} are in correspondence with top generators for the complexes

$$\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\beta}_m,\boldsymbol{\beta};u,v,w)\quad\text{and}\quad\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}_0;u,v,w),$$

respectively, which will also be denoted by Θ_{f_0} and Θ_{h_0}. Corresponding to Θ_{g_0}, we have a pair of top generators in $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\beta}_0,\boldsymbol{\beta}_m;u,v,w)$ which use the intersection points q and q', respectively. We will denote these generators with Θ_{g_0} and Θ'_{g_0}, respectively.

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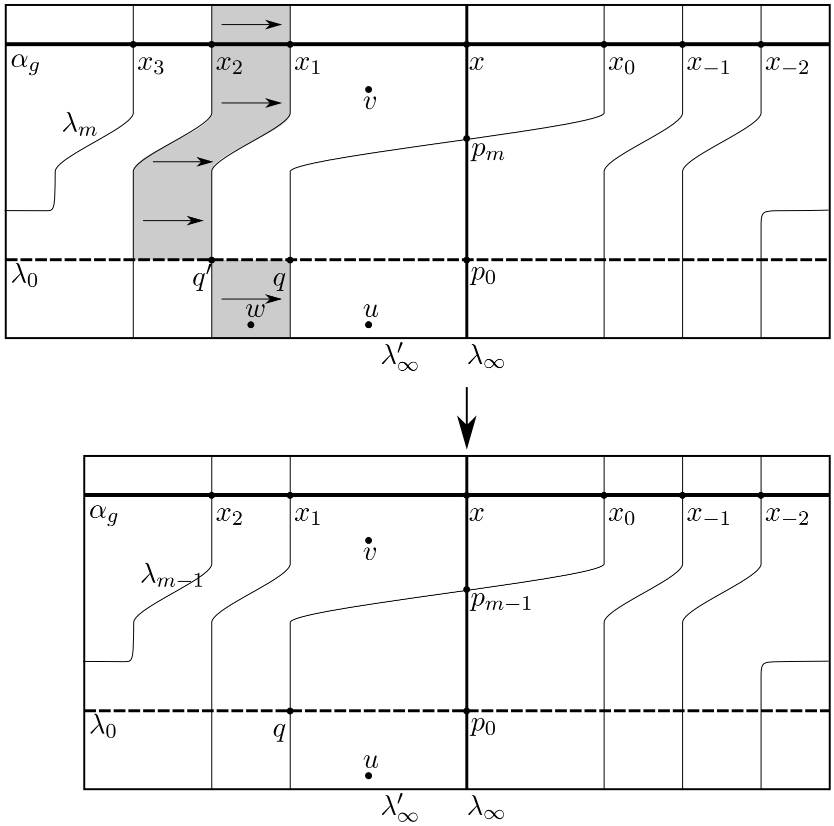

FIGURE 4. In the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_0, \boldsymbol{\beta}_m; u, v, w)$, we may take out the shaded area from the winding region and glue back the boundaries after shifting the left-hand-side part to the right to obtain a Heegaard diagram which may be identified with $(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_0, \boldsymbol{\beta}_{m-1}; u, v)$, where Σ' is the surface Σ but with a different complex structure.

From the Heegaard 4-tuple $(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_0, \boldsymbol{\beta}_{m-1}; u, v)$, we obtain the exact triangle

$$\begin{tikzcd}
\widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v) \arrow[rr, "h_0^{\mathfrak{s}} = h_{0,m-1}^{\mathfrak{s}}"] & & \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_0; u, v; \mathfrak{s}) \arrow[dl, "g_0^{\mathfrak{s}} = g_{0,m-1}^{\mathfrak{s}}"] \\
& C_{0,m-1}(\mathfrak{s}) \arrow[ul, "f_0^{\mathfrak{s}} = f_{0,m-1}^{\mathfrak{s}}"] &
\end{tikzcd}$$

as well as the square maps $H_{f_0}^{\mathfrak{s}}, H_{g_0}^{\mathfrak{s}}$ and $H_{h_0}^{\mathfrak{s}}$, as discussed in § 2. Similarly, from the Heegaard 4-tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_m; u, v)$, we obtain the exact triangle

$$\begin{tikzcd}
\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v) \arrow[rr, "h_1^{\mathfrak{s}} = h_{1,m-1}^{\mathfrak{s}}"] & & \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_1; u, v; \mathfrak{s}) \arrow[dl, "g_1^{\mathfrak{s}} = g_{1,m-1}^{\mathfrak{s}}"] \\
& C_{1,m-1}(\mathfrak{s}) \arrow[ul, "f_1^{\mathfrak{s}} = f_{1,m-1}^{\mathfrak{s}}"] &
\end{tikzcd}$$

as well as the square maps $H_{f_1}^{\mathfrak{s}}, H_{g_1}^{\mathfrak{s}}$ and $H_{h_1}^{\mathfrak{s}}$.

Moreover, we have $g_0^{\mathfrak{s}} = (g_{0,a}^{\mathfrak{s}}, g_{0,b}^{\mathfrak{s}})$ and $g_1^{\mathfrak{s}} = (g_{1,a}^{\mathfrak{s}}, g_{1,b}^{\mathfrak{s}})$, where

$$\begin{aligned} g_{0,a}^{\mathfrak{s}} &: \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_0; u, v; \mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_{m-1}; u, v; \mathfrak{s}),\\ g_{0,b}^{\mathfrak{s}} &: \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_0; u, v; \mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_{m-1}; u, v; \mathfrak{s}+m-1),\\ g_{1,a}^{\mathfrak{s}} &: \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_1; u, v; \mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_m; u, v; \mathfrak{s}),\\ g_{1,b}^{\mathfrak{s}} &: \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_1; u, v; \mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_m; u, v; \mathfrak{s}+m-1). \end{aligned}$$

Other than $g_\bullet^{\mathfrak{s}}$, the square maps $H_{f_\bullet}^{\mathfrak{s}}$ and $H_{h_\bullet}^{\mathfrak{s}}$, which are defined using the top generator $\Theta_{g_\bullet}$, may also be decomposed into two summands:

$$H_{f_\bullet}^{\mathfrak{s}} = H_{f_\bullet,a}^{\mathfrak{s}} + H_{f_\bullet,b}^{\mathfrak{s}} \quad \text{and} \quad H_{h_\bullet}^{\mathfrak{s}} = H_{h_\bullet,a}^{\mathfrak{s}} + H_{h_\bullet,b}^{\mathfrak{s}} \quad \text{for } \bullet = 0, 1.$$

Here, $H_{f_\bullet,a}^{\mathfrak{s}}$ and $H_{f_\bullet,b}^{\mathfrak{s}}$ record the contributions to $H_{f_\bullet}^{\mathfrak{s}}$ from the square classes which miss the markings b and a, respectively. Similarly, $H_{h_\bullet,a}^{\mathfrak{s}}$ and $H_{h_\bullet,b}^{\mathfrak{s}}$ record the contributions to $H_{h_\bullet}^{\mathfrak{s}}$ from the square classes which miss the markings b and a, respectively.

Having fixed the relative Spin^c class $\mathfrak{s}$, by choosing m sufficiently large we may identify $\widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_{m-1}; u, v; \mathfrak{s})$ with $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_m; u, v; \mathfrak{s})$, as they both have the same set of generators (all supported in the winding region), the corresponding Whitney discs are in correspondence and their domains are supported in subsurfaces of Σ' and Σ where the complex structures match. Moreover, for such sufficiently large values of m, the complex

$$\widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_{m-1}; u, v; \mathfrak{s}+m-1)$$

is identified with the subcomplex of $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_m; u, v; \mathfrak{s}+m-1)$ which is generated by $\mathbf{x}_i$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $i \geqslant 2$ and $\mathfrak{s}(\mathbf{x}_i) = \mathfrak{s}+m-1$. We will denote the inclusion of this subcomplex by

$$J^{\mathfrak{s}} : \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_{m-1}; u, v; \mathfrak{s}+m-1) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_m; u, v; \mathfrak{s}+m-1).$$

3.2 The homomorphisms in the surgery triangle

The top generators $\Theta_{0,1}$, Θ_{g_1} and Θ_{f_1} determine the holomorphic pentagon map

$$P^{\mathfrak{s}} : \widehat{\mathrm{CFK}}(K_0; \mathfrak{s}) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_0; u, v, w; \mathfrak{s}) \to \widehat{\mathrm{CF}}(Y) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v, w),$$

which is defined by setting

$$P^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \; \sum_{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\Theta_{f_1},\mathbf{y};u,v,w)} \#(\mathcal{M}(\pentagon)) \cdot \mathbf{y}.$$

Every pentagon class $\pentagon \in \pi_2^{-1}(\mathbf{x}, \Theta_{0,1}, \Theta_{g_1}, \Theta_{f_1}, \mathbf{y}; u, v, w)$ corresponds to a one-dimensional moduli space with boundary. The boundary points are in correspondence with the degeneration of the domain of $\pentagon$ into two parts. Since the generators $\Theta_{0,1}, \Theta_{g_1}$ and Θ_{f_1} are closed, the degenerations into a bigon and a pentagon correspond to the coefficient of $\mathbf{y}$ in $(d \circ P^{\mathfrak{s}} + P^{\mathfrak{s}} \circ d)(\mathbf{x})$. The remaining degenerations are the degenerations $\pentagon = \square \star \Delta$ to a triangle Δ with Maslov index 0 and a square $\square$ with Maslov index -1 which miss u, v and w. The possibilities are:

(1) $\square \in \pi_2(\mathbf{z}, \Theta_{g_1}, \Theta_{f_1}, \mathbf{y})$ and $\Delta \in \pi_2(\mathbf{x}, \Theta_{0,1}, \mathbf{z})$;
(2) $\square \in \pi_2(\mathbf{x}, \Theta_{0,1}, \Theta_{g_1}, \mathbf{z})$ and $\Delta \in \pi_2(\mathbf{z}, \Theta_{f_1}, \mathbf{y})$;
(3) $\square \in \pi_2(\mathbf{x}, \Theta_{0,1}, \Theta, \mathbf{y})$ and $\Delta \in \pi_2(\Theta_{g_1}, \Theta_{f_1}, \Theta)$;

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(4) $\square\in\pi_2(\mathbf{x},\Theta,\Theta_{f_1},\mathbf{y})$ and $\Delta\in\pi_2(\Theta_{0,1},\Theta_{g_1},\Theta)$;
(5) $\square\in\pi_2(\Theta_{0,1},\Theta_{g_1},\Theta_{f_1},\Theta)$ and $\Delta\in\pi_2(\mathbf{x},\Theta,\mathbf{y})$.

Degenerations of type 1 correspond to the coefficient of $\mathbf{y}$ in $(H^{\mathfrak{s}}_{h_1}\circ\bar{\mathfrak{f}}^{\mathfrak{s}}_{\infty})(\mathbf{x})$, where the map induced by

$$\bar{\mathfrak{f}}^{\mathfrak{s}}_{\infty}:\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0;u,v,w;\mathfrak{s})\longrightarrow\widehat{\mathrm{CFK}}(K_1;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_1;u,v,w;\mathfrak{s})$$

in homology is the homomorphism $\bar{\mathfrak{f}}^{\mathfrak{s}}_{\infty}:\mathbb{H}_0(K;\mathfrak{s})\to\mathbb{H}_1(K;\mathfrak{s})$, which appears in the splicing formula of [Eft15]. We abuse the notation and denote both the chain map and the induced map on homology by $\bar{\mathfrak{f}}^{\mathfrak{s}}_{\infty}$. Degenerations of type 2 correspond to the coefficient of $\mathbf{y}$ in $f_1^{\mathfrak{s}}\circ H^{\mathfrak{s}}(\mathbf{x})$, where

$$H^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0;u,v,w;\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v,w)$$

is defined by setting

$$H^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_m}}\ \sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\mathbf{z};u,v,w)}\#(\mathcal{M}(\square))\cdot\mathbf{z}.$$

If $\square\in\pi_2^{-1}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\mathbf{z};u,v,w)$ is a square class which contributes to $H^{\mathfrak{s}}$, from $n_u(\square)=n_v(\square)=n_w(\square)=0$ it follows that

$$n_c(\square)=n_b(\square)+1\quad\text{and}\quad n_e(\square)=n_u(\square)+1=1.$$

Furthermore, since q_m is one of the corners of the domain $\mathcal{D}(\square)$, we find that

$$\begin{aligned}n_a(\square)&=n_e(\square)+n_v(\square)-n_c(\square)+1\\&=1-n_b(\square).\end{aligned}$$

From here, we are left with two possibilities:

$$n_a(\square)+n_b(\square)=1\ \Rightarrow\ \begin{cases}\text{(i) } n_a(\square)=1\quad\text{and}\quad n_b(\square)=0,\\ \text{(ii) } n_a(\square)=0\quad\text{and}\quad n_b(\square)=1.\end{cases}$$

We may thus write $H^{\mathfrak{s}}(\mathbf{x})=H^{\mathfrak{s}}_a(\mathbf{x})+H^{\mathfrak{s}}_b(\mathbf{x})$, where $H^{\mathfrak{s}}_a(\mathbf{x})$ and $H^{\mathfrak{s}}_b(\mathbf{x})$ correspond to the contributions from holomorphic squares of types (i) and (ii), respectively.

Lemma 3.1. *For every generator* $\mathbf{x}\in\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0;u,v,w;\mathfrak{s})$, *we have*

$$H^{\mathfrak{s}}_a(\mathbf{x})\in\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v,w;\mathfrak{s})$$

and

$$H^{\mathfrak{s}}_b(\mathbf{x})\in\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v,w;\mathfrak{s}+m-1).$$

Proof. Choose $\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ so that $\mathfrak{s}(\mathbf{w})=\mathfrak{s}$. There are a particular square class

$$\square_0\in\pi_2^0(\mathbf{w}_0^{(0)},\Theta_{0,1},\Theta_{g_1}\mathbf{w}_0;u,v,w,b)$$

and a corresponding domain $\mathcal{D}_0=\mathcal{D}(\square_0)$ which intersects the winding region in a 4-gon with one obtuse angle. This 4-gon, which is shaded in Figure 3, has vertices at q_0,q_m,x_0 and x_0^0, and its obtuse angle is at q_m. Since the relative Spin^c class associated with both $\mathbf{x}$ and $\mathbf{w}_0^{(0)}$ is $\mathfrak{s}$, there is a Whitney disc $\phi\in\pi_2(\mathbf{x},\mathbf{w}_0^{(0)};u,v)$, which corresponds to a domain $\mathcal{D}_1=\mathcal{D}(\phi)$. Since

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b,w and u are not separated by $\boldsymbol{\alpha}\cup\boldsymbol{\beta}_0$, it follows that $n_b(\mathcal{D}_1)=n_w(\mathcal{D}_1)=0$. Finally, associated with every square class

$$\square\in\pi_2^{-1}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\mathbf{z};b,u,v,w)$$

which contributes to $H_a^{\mathfrak{s}}(\mathbf{x})$ we obtain a domain $\mathcal{D}_2=\mathcal{D}(\square)$. From these three domains, we obtain the domain $\mathcal{D}=\mathcal{D}_2-\mathcal{D}_1-\mathcal{D}_0$ with boundary on the union $\boldsymbol{\alpha}\cup\boldsymbol{\beta}_0\cup\boldsymbol{\beta}_1\cup\boldsymbol{\beta}_m$. After subtracting suitable multiples of periodic domains bounded between the curves in $\boldsymbol{\beta}_\bullet\backslash\{\lambda_\bullet\}$, $\bullet\in\{0,1,m\}$, we may in fact assume that

$$\partial\mathcal{D}\subset\boldsymbol{\alpha}\cup\boldsymbol{\beta}_m\cup\{\lambda_0,\lambda_1\}.$$

Since $\mathcal{D}$ connects $\mathbf{w}_0$ to $\mathbf{y}$, the boundary of $\mathcal{D}$ on each one of λ_0 and λ_1 is an integer multiple of these two closed curves. From $n_v(\mathcal{D})=n_w(\mathcal{D})=0$, it follows that $\mathcal{D}$ has no boundary on λ_0 and, from $n_w(\mathcal{D})=n_b(\mathcal{D})=0$, it follows that $\mathcal{D}$ has no boundary on λ_1. In other words, $\mathcal{D}$ is the domain of a Whitney disc $\psi\in\pi_2(\mathbf{w}_0,\mathbf{y};u,v,w,a)$. In particular, it follows that $\mathfrak{s}(\mathbf{y})=\mathfrak{s}(\mathbf{w}_0)=\mathfrak{s}$.

Let us now assume that $\square'\in\pi_2^{-1}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\mathbf{z};a,u,v,w)$ contributes to $H_b^{\mathfrak{s}}(\mathbf{x})$ and, correspondingly, we obtain a domain $\mathcal{D}_2'=\mathcal{D}(\square')$. From the above discussion, we may find a domain $\mathcal{D}_0'$ which corresponds to a class

$$\square_0'\in\pi_2(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\mathbf{w}_0;b,u,v,w),$$

with $\mathfrak{s}(\mathbf{w})=\mathfrak{s}$. From the domains $\mathcal{D}_0'$ and $\mathcal{D}_2'$, we obtain a domain $\mathcal{D}'=\mathcal{D}_2'-\mathcal{D}_0'$ which misses the markings u,v and w, and connects $\mathbf{w}_0$ to $\mathbf{z}$. We may assume that this domain has its boundary on $\boldsymbol{\alpha}\cup\boldsymbol{\beta}_m\cup\{\lambda_0,\lambda_1\}$. Moreover, from the equalities $n_v(\mathcal{D}')=n_w(\mathcal{D}')=0$, we conclude that $\mathcal{D}'$ has no boundary on λ_0. The coefficient of λ_1 is, however, equal to

$$n_b(\mathcal{D}')-n_w(\mathcal{D}')=n_b(\mathcal{D}_2')=1.$$

Next, note that there is a domain $\mathcal{D}''$ with boundary on $\boldsymbol{\alpha}\cup\boldsymbol{\beta}_m\cup\lambda_\infty$, which connects $\mathbf{w}_1$ to $\mathbf{w}_0$ while $n_\bullet(\mathcal{D}'')=0$ for $\bullet\in\{a,b,u,v,w\}$ and $n_z(\mathcal{D}'')=m-1$. There is also a periodic domain $\mathcal{P}$ with boundary on $\lambda_1\cup\lambda_m\cup\lambda_\infty$ such that

$$n_v(\mathcal{P})=n_w(\mathcal{P})=n_u(\mathcal{P})=0,\quad n_b(\mathcal{P})=-n_a(\mathcal{P})=1\quad\text{and}\quad n_z(\mathcal{P})=m-1.$$

From the three domains $\mathcal{D}',\mathcal{D}''$ and $\mathcal{P}$, we obtain the domain $\widehat{\mathcal{D}}=\mathcal{D}'+\mathcal{D}''-\mathcal{P}$ which connects $\mathbf{w}_1$ to $\mathbf{z}$ and

$$\begin{aligned}
n_\bullet(\widehat{\mathcal{D}})&=n_\bullet(\mathcal{D}')+n_\bullet(\mathcal{D}'')-n_\bullet(\mathcal{P})=0+0-0=0,\quad \bullet\in\{u,v,w\},\\
n_a(\widehat{\mathcal{D}})&=n_a(\mathcal{D}')+n_a(\mathcal{D}'')-n_a(\mathcal{P})=-1+0-(-1)=0,\\
n_a(\widehat{\mathcal{D}})&=n_b(\mathcal{D}')+n_b(\mathcal{D}'')-n_b(\mathcal{P})=1+0-1=0,\\
n_z(\widehat{\mathcal{D}})&=n_z(\mathcal{D}')+n_z(\mathcal{D}'')-n_z(\mathcal{P})=0+(m-1)-(m-1)=0.
\end{aligned}$$

We thus find that $\mathfrak{s}(\mathbf{z})=\mathfrak{s}(\mathbf{w}_1)=\mathfrak{s}+m-1$. This completes the proof of the lemma. $\square$

In a degeneration of type 3, the contributing triangle classes Δ come in cancelling pairs. The total count of such degenerations is thus trivial. Furthermore, there are no holomorphic representatives for the square classes of Maslov index -1 which appear in the boundary degenerations of type 5. Note that there are positive square classes with Maslov index 0, but no such square with Maslov index -1. It follows that there are no such degenerations.

In a degeneration of type 4, the moduli space corresponding to Δ is trivial unless Θ includes one of the two intersection points q or q'. In the former case, we need to have $\Theta=\Theta_{g_0}$ while Δ

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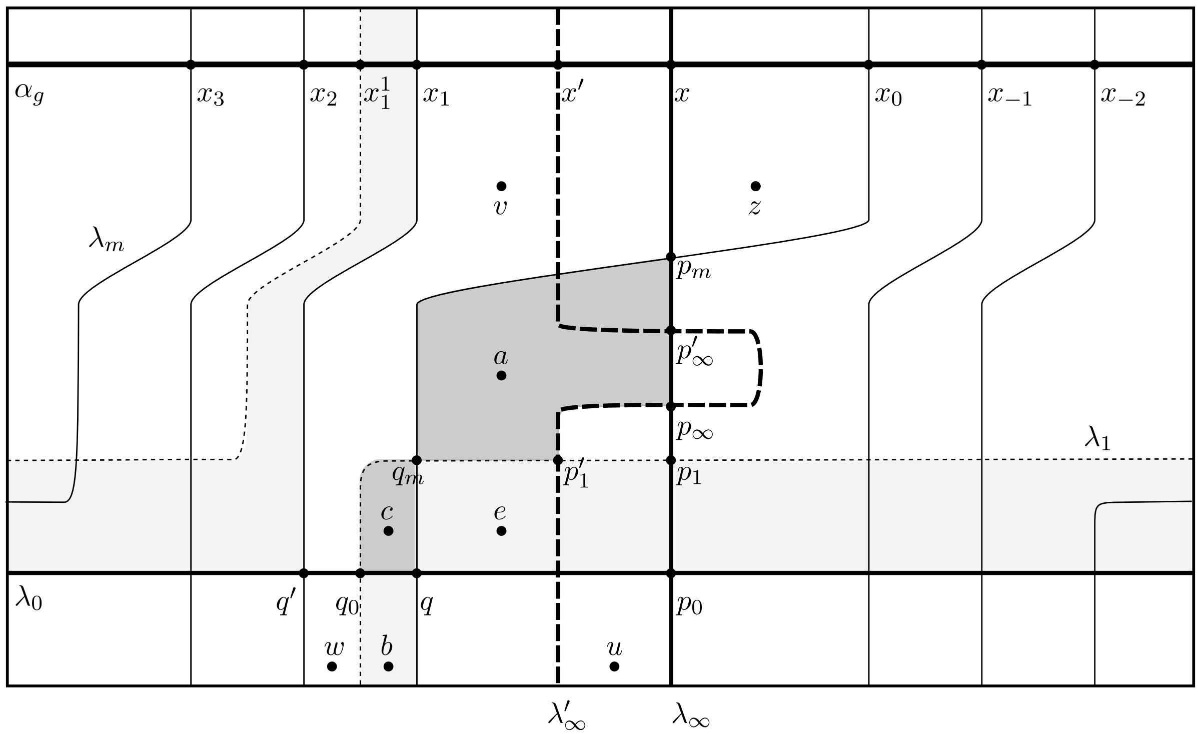

Figure 5. A domain, which is an immersion of a triangle on the surface, is lightly shaded. This domain has vertices q_0, q_m and q'. The coefficient of the shaded domain in the small triangle with vertices q_0, q_m and q (which is strongly shaded) is 2, while its coefficient in the rest of the shaded areas is 1. Moreover, a 4-gon with vertices q_m, p_m, p_∞ and p'_1 is strongly shaded.

corresponds to the union of the small triangle with vertices q_0, q_m and q in the winding region and the small triangles with vertices at top intersection points of β_i^0, β_i^1 and β_i^m. In the latter case, Θ is the generator Θ'_{g_0} (which is obtained from Θ_{g_0} by replacing q with q') and the intersection of the domain of Δ with the winding region is the region shaded in Figure 5. The total contribution of the triangle classes in both these cases is 1. Such boundary degenerations thus correspond to the coefficient of $\mathbf{y}$ in $H_q^{\mathfrak{s}}(\mathbf{x}) + H_{q'}^{\mathfrak{s}}(\mathbf{x})$, where

$$H_q^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta_{g_0},\Theta_{f_1},\mathbf{y};u,v,w)} \#(\mathcal{M}(\square))\cdot\mathbf{y}$$

and

$$H_{q'}^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta'_{g_0},\Theta_{f_1},\mathbf{y};u,v,w)} \#(\mathcal{M}(\square))\cdot\mathbf{y}.$$

Lemma 3.2. *With the above notation fixed, and under the identification of the chain complex* $\widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_\bullet; u, v)$ *with* $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_\bullet; u, v, w)$ *for* $\bullet \in \{0, \infty\}$, *we have*

$$d\circ P^{\mathfrak{s}} + P^{\mathfrak{s}}\circ d + H_{h_0}^{\mathfrak{s}} = f_1^{\mathfrak{s}}\circ H^{\mathfrak{s}} + H_{h_1}^{\mathfrak{s}}\circ \bar{\mathfrak{f}}_\infty^{\mathfrak{s}}. \tag{6}$$

Proof. As discussed in § 3.1, the holomorphic square map

$$H_{h_0}^{\mathfrak{s}} : \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_0; u, v, \mathfrak{s}) \to \widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v),$$

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which is defined from the Heegaard 4-tuple $(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_0, \boldsymbol{\beta}_{m-1}; u, v)$, may be written as

$$H^{\mathfrak{s}}_{h_0} = H^{\mathfrak{s}}_{h_0,a} + H^{\mathfrak{s}}_{h_0,b},$$

where $H^{\mathfrak{s}}_{h_0,a}$ counts the squares with $n_b(\square) = 0$ and $H^{\mathfrak{s}}_{h_0,b}$ counts the squares with $n_a(\square) = 0$. Under the identification of $\widehat{\mathrm{CF}}(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}_\bullet; u, v)$ with $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_\bullet; u, v, w)$ for $\bullet \in \{0, \infty\}$, we may also identify $H^{\mathfrak{s}}_q$ with $H^{\mathfrak{s}}_{h_0,a}$ and $H^{\mathfrak{s}}_{q'}$ with $H^{\mathfrak{s}}_{h_0,b}$. In particular, $H^{\mathfrak{s}}_q + H^{\mathfrak{s}}_{q'} = H^{\mathfrak{s}}_{h_0}$. Together with the discussion preceding the lemma, this completes the proof of (6). $\square$

Next, we analyse $H^{\mathfrak{s}}$ via degenerations of holomorphic squares. For a square class $\square \in \pi_2^0(\mathbf{x}, \Theta_{0,1}, \Theta_{g_1}, \mathbf{y}; u, v, w)$, the moduli space $\mathcal{M}(\square)$ is one dimensional, and has six types of boundary ends. Since $\Theta_{0,1}$ and Θ_{g_1} are closed, the four types of degenerations of the square class to a square and a bigon correspond to the coefficient of $\mathbf{y}$ in $(d \circ H^{\mathfrak{s}} + H^{\mathfrak{s}} \circ d)(\mathbf{x})$. The remaining boundary ends correspond to a degeneration of $\square$ to a pair of triangles. The degenerations $\square = \Delta' \star \Delta$ with $\Delta \in \pi_2(\mathbf{x}, \Theta_{0,1}, \mathbf{z})$ and $\Delta' \in \pi_2(\mathbf{z}, \Theta_{g_1}, \mathbf{y})$ correspond to the coefficient of $\mathbf{y}$ in $(g_1^{\mathfrak{s}} \circ \overline{\mathfrak{f}}^{\mathfrak{s}}_\infty)(\mathbf{x})$.

As observed in the proof of Lemma 3.2, in degenerations of the form $\square = \Delta' \star \Delta$ with $\Delta' \in \pi_2^0(\Theta_{0,1}, \Theta_{g_1}, \Theta; u, v, w)$ and $\Delta \in \pi_2^0(\mathbf{x}, \Theta, \mathbf{y}; u, v, w)$, there are precisely two generators Θ corresponding to a triangle class $\Delta' = \Delta_\Theta$ such that $\mathcal{M}(\Delta_\Theta)$ is non-empty. One of them corresponds to $\Theta = \Theta_{g_0}$ and the other one corresponds to Θ'_{g_0} (see Figure 5). Such degenerations thus correspond to the coefficient of $\mathbf{y}$ in the expression $g^{\mathfrak{s}}_{0,q}(\mathbf{x}) + g^{\mathfrak{s}}_{0,q'}(\mathbf{x})$, where

$$g^{\mathfrak{s}}_{0,q}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_m}} \sum_{\Delta \in \pi_2^0(\mathbf{x}, \Theta_{g_0}, \mathbf{y}; u, v, w)} \#(\mathcal{M}(\Delta)) \cdot \mathbf{y}$$

and

$$g^{\mathfrak{s}}_{0,q'}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_m}} \sum_{\Delta \in \pi_2^0(\mathbf{x}, \Theta'_{g_0}, \mathbf{y}; u, v, w)} \#(\mathcal{M}(\Delta)) \cdot \mathbf{y}.$$

Lemma 3.3. *With the above notation fixed, for every $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_0}$ with $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}$, we have*

$$g^{\mathfrak{s}}_{0,q}(\mathbf{x}) \in \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_m; u, v, w; \mathfrak{s})$$

and

$$g^{\mathfrak{s}}_{0,q'}(\mathbf{x}) \in \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_m; u, v, w; \mathfrak{s} + m - 1).$$

Proof. Let us assume that $\Delta \in \pi_2^0(\mathbf{x}, \Theta_{g_0}, \mathbf{y}; u, v, w)$ is a triangle class which contributes to $g^{\mathfrak{s}}_{0,q}$ and let $\mathcal{D}_1 = \mathcal{D}(\Delta)$. Choose $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}(\mathbf{w}) = \mathfrak{s}$ and let $\phi \in \pi_2(\mathbf{w}_0^{(0)}, \mathbf{x}; u, v, w)$ denote a Whitney disc which connects $\mathbf{w}_0^{(0)}$ to $\mathbf{x}$. Set $\mathcal{D}_2 = \mathcal{D}(\phi)$. There is a triangle in the Heegaard diagram with vertices x_0^0, q and x_0 which is obtained from the shaded 4-gon in Figure 3 by subtracting the small triangle with vertices q_0, q_m and q. This triangle may be paired with the small triangles bounded between β_i^0 and β_i^m, $i = 1, \ldots, g-1$, to give the domain of a distinguished triangle class

$$\Delta_0 \in \pi_2(\mathbf{w}_0^{(0)}, \Theta_{g_0}, \mathbf{w}_0; u, v, w).$$

We let $\mathcal{D}_0 = \mathcal{D}(\Delta_0)$ and $\mathcal{D} = \mathcal{D}_0 - \mathcal{D}_1 - \mathcal{D}_2$. It is then clear that $\mathcal{D}$ is a domain missing u, v and w, which connects $\mathbf{y}$ to $\mathbf{w}_0$. After subtracting appropriate multiples of the periodic domains in the Heegaard diagram $(\Sigma, \boldsymbol{\beta}_0, \boldsymbol{\beta}_m; u, v, w)$, we may also assume that $\partial \mathcal{D} \subset \boldsymbol{\alpha} \cup \boldsymbol{\beta}_m \cup \lambda_0$. The condition $n_v(\mathcal{D}) = n_w(\mathcal{D}) = 0$ implies that $\mathcal{D}$ does not have any boundary on λ_0. It is thus the domain of a Whitney disc $\psi \in \pi_2(\mathbf{y}, \mathbf{w}_0; u, v, w)$. It follows that $\mathfrak{s}(\mathbf{y}) = \mathfrak{s}(\mathbf{w}_0) = \mathfrak{s}$. The proof of the second claim is quite similar. $\square$

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As before, one may identify $\widehat{\mathrm{CF}}(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta}_{m-1};u,v;\mathfrak{s})$ with $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v;\mathfrak{s})$ and, under this identification, it is clear that $g^{\mathfrak{s}}_{0,q}=g^{\mathfrak{s}}_{0,a}$. The complex

$$\widehat{\mathrm{CF}}(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta}_{m-1};u,v;\mathfrak{s}+m-1)$$

is identified (via the inclusion map $J^{\mathfrak{s}}$) with the subcomplex of

$$\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v;\mathfrak{s}+m-1)$$

which is generated by $\mathbf{x}_i$ with $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, $i\geqslant 2$ and $\mathfrak{s}(\mathbf{x}_i)=\mathfrak{s}+m-1$. For a triangle class $\Delta\in\pi_2^0(\mathbf{x},\Theta'_{g_0},\mathbf{y};u,v,w)$ with non-trivial contribution to

$$g^{\mathfrak{s}}_{0,q'}:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0;u,v,w;\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v,w;\mathfrak{s}+m-1),$$

we have $\mathbf{y}=\mathbf{z}_i$ with $\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ and $i\geqslant 2$. Every such Δ corresponds to a triangle class $\Delta'\in\pi_2^0(\mathbf{x},\Theta_{g_0},\mathbf{z}_{i-1};u,v)$. Furthermore, $\mathcal{M}(\Delta)$ and $\mathcal{M}(\Delta')$ may be identified if the complex structures on Σ and Σ' are chosen as described in § 3.1. Such Δ' are the triangle classes which contribute to the holomorphic triangle map $g^{\mathfrak{s}}_{0,b}$. This means that $g^{\mathfrak{s}}_{0,q'}=J^{\mathfrak{s}}\circ g^{\mathfrak{s}}_{0,b}$ (again, after we identify $\widehat{\mathrm{CF}}(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta}_\bullet;u,v)$ with $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_\bullet;u,v)$ for $\bullet\in\{0,\infty\}$).

Let us define

$$G^{\mathfrak{s}}_\infty:C_{0,m-1}(\mathfrak{s})\to C_{1,m-1}(\mathfrak{s})$$

by setting

$$G^{\mathfrak{s}}_\infty(\mathbf{x}_1,\mathbf{x}_2):=(\mathbf{x}_1,J^{\mathfrak{s}}(\mathbf{x}_2)),\quad\forall\begin{cases}\mathbf{x}_1\in\widehat{\mathrm{CF}}(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta}_{m-1};u,v;\mathfrak{s}),\\ \mathbf{x}_2\in\widehat{\mathrm{CF}}(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta}_{m-1};u,v;\mathfrak{s}+m-1).\end{cases}$$

Lemma 3.4. *With the above notation fixed, we have:*

(i) $d\circ H^{\mathfrak{s}}+H^{\mathfrak{s}}\circ d=g^{\mathfrak{s}}_1\circ\bar{\mathfrak{f}}^{\mathfrak{s}}_\infty+G^{\mathfrak{s}}_\infty\circ g^{\mathfrak{s}}_0$;

(ii) $f^{\mathfrak{s}}_1\circ G^{\mathfrak{s}}_\infty=f^{\mathfrak{s}}_0,\forall m\gg 1$.

Proof. With our earlier considerations in place, the first claim already follows. We thus only need to prove the second claim. If $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_{m-1}}$ satisfies $\mathfrak{s}(\mathbf{x})=\mathfrak{s}$, then the maps $f^{\mathfrak{s}}_0$ and $f^{\mathfrak{s}}_1$ are identical on $\mathbf{x}$, by definition. Since $G^{\mathfrak{s}}_\infty(\mathbf{x})=\mathbf{x}$, the claim follows. Let us now assume that $\mathfrak{s}(\mathbf{x})=\mathfrak{s}+m-1$. It follows that $\mathbf{x}=\mathbf{z}_i$ for some $i>0$, while $\mathfrak{s}(\mathbf{z})=\mathfrak{s}+i$. Thus, $J^{\mathfrak{s}}(\mathbf{x})=\mathbf{z}_{i+1}$ and

$$f^{\mathfrak{s}}_1(G^{\mathfrak{s}}_\infty(\mathbf{x}))=f^{\mathfrak{s}}_1(\mathbf{z}_{i+1})=f^{\mathfrak{s}}_0(\mathbf{z}_i)=f^{\mathfrak{s}}_0(\mathbf{x}).$$

The third equality follows, since every contributing triangle in $\pi_2^0(\mathbf{z}_i,\Theta_{f_0},\mathbf{y};u,v)$ is in correspondence with a contributing triangle in $\pi_2^0(\mathbf{z}_{i+1},\Theta_{f_1},\mathbf{y};u,v)$. □

3.3 The bypass homomorphisms

Let us now consider the Heegaard 5-tuple $\mathcal{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_1,\boldsymbol{\beta}_m,\boldsymbol{\beta},\boldsymbol{\beta}';u,v,z)$. Since u and z are not separated by $\boldsymbol{\alpha}\cup\boldsymbol{\beta}'$, by choosing the Hamiltonian isotopy which changes $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$ close to the identity, we may assume that the chain complex associated with the punctured Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}';u,v,z)$ and $\mathfrak{s}\in\underline{\mathrm{Spin}}^c(Y,K)=\mathbb{Z}$ is identified with

$$\widehat{\mathrm{CFK}}(K;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,z;\mathfrak{s}).$$

Associated with $(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}';u,v,z)$ there is a top generator, which may be denoted by Θ'_∞. This generator uses the intersection point p'_∞. Unlike most of similar situations, Θ'_∞ is not closed and $d(\Theta'_\infty)=\Theta_\infty$ is the generator which is obtained from Θ'_∞ by changing the choice of intersection point in $\lambda_\infty\cap\lambda'_\infty$ from p'_∞ to p_∞. By construction, Θ_∞ is closed. The triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_1,\boldsymbol{\beta}';u,v,z)$

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may be used to define a holomorphic triangle map

$$\bar{\mathfrak{f}}_0^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_1;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_1;u,v,z;\mathfrak{s})\longrightarrow\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}';u,v,z;\mathfrak{s})=\widehat{\mathrm{CFK}}(K;\mathfrak{s}).$$

The map on homology induced by $\bar{\mathfrak{f}}_0^{\mathfrak{s}}$ coincides with the map used in the splicing formula of [Eft15]. We may also define a map

$$I^{\mathfrak{s}}:C_{1,m-1}(\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}';u,v,z;\mathfrak{s})=\widehat{\mathrm{CFK}}(K,\mathfrak{s}),$$

which is defined by setting

$$I^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\mathfrak{s}(\mathbf{y})=\mathfrak{s}}\ \sum_{\square\in\pi_2^{-1}(\mathbf{z},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)}\#\mathcal{M}(\square)\cdot\mathbf{y},\quad\forall\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_m}.$$

Lemma 3.5. *The map $I^{\mathfrak{s}}$ is a chain map.*

Proof. For $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_m}$ and $\square\in\pi_2^0(\mathbf{z},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)$, the ends of the moduli space $\mathcal{M}(\square)$ are in correspondence with degenerations of $\square$. Since Θ_{f_1} and Θ_∞ are closed, degenerations into a bigon and a square correspond to the coefficient of $\mathbf{y}$ in $(d\circ I^{\mathfrak{s}}+I^{\mathfrak{s}}\circ d)(\mathbf{x})$. There are no holomorphic triangles $\Delta\in\pi_2^0(\Theta_{f_1},\Theta_\infty,\Theta;u,v,z)$, implying that there are no degenerations of the form $\square=\Delta\star\Delta'$ with

$$\Delta\in\pi_2^0(\Theta_{f_1},\Theta_\infty,\Theta;u,v,z)\quad\text{and}\quad\Delta'\in\pi_2^0(\mathbf{x},\Theta,\mathbf{y};u,v,z).$$

Finally, by looking at local multiplicities around x, we may conclude that there are no positive triangle classes $\Delta\in\pi_2^0(\mathbf{x},\Theta_{f_1},\mathbf{z};u,v,z)$. The contribution of degenerations of the form $\square=\Delta\star\Delta'$ with

$$\Delta\in\pi_2^0(\mathbf{x},\Theta_{f_1},\mathbf{z};u,v,z)\quad\text{and}\quad\Delta'\in\pi_2^0(\mathbf{z},\Theta_\infty,\mathbf{y};u,v,z)$$

is thus trivial. From these observations, we conclude that $(d\circ I^{\mathfrak{s}}+I^{\mathfrak{s}}\circ d)(\mathbf{x})=0$ for all $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_m}$, completing the proof of the lemma. $\square$

Lemma 3.6. *With the above notation fixed, there is a map*

$$Q^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_1;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_1;u,v,z;\mathfrak{s})\longrightarrow\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,v,z;\mathfrak{s})=\widehat{\mathrm{CFK}}(K;\mathfrak{s})$$

which satisfies

$$d\circ Q^{\mathfrak{s}}+Q^{\mathfrak{s}}\circ d=I^{\mathfrak{s}}\circ g_1^{\mathfrak{s}}+\bar{\mathfrak{f}}_0^{\mathfrak{s}}.$$

Proof. The diagram $\mathcal{H}$ defines a pentagon map

$$Q^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_1;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_1;u,v,z;\mathfrak{s})\longrightarrow\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,v,z;\mathfrak{s})=\widehat{\mathrm{CFK}}(K;\mathfrak{s})$$

by setting

$$Q^{\mathfrak{s}}(\mathbf{x})=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\mathfrak{s}(\mathbf{y})=\mathfrak{s}}\ \sum_{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_{g_1},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)}\#(\mathcal{M}(\pentagon))\cdot\mathbf{y}.$$

For $\pentagon\in\pi_2^{-1}(\mathbf{x},\Theta_{g_1},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)$, the ends of the moduli space $\mathcal{M}(\pentagon)$ which correspond to the degenerations of the pentagon to a bigon and a pentagon contribute to the coefficient of $\mathbf{y}$ in $(d\circ Q^{\mathfrak{s}}+Q^{\mathfrak{s}}\circ d)(\mathbf{x})$. Other ends correspond to the degenerations of the form $\pentagon=\square\star\Delta$ of one of the following five types:

(1) $\square\in\pi_2(\mathbf{z},\Theta_{f_1},\Theta_\infty,\mathbf{y})$ and $\Delta\in\pi_2(\mathbf{x},\Theta_{g_1},\mathbf{z})$;
(2) $\square\in\pi_2(\mathbf{x},\Theta_{g_1},\Theta_{f_1},\mathbf{z})$ and $\Delta\in\pi_2(\mathbf{z},\Theta_\infty,\mathbf{y})$;
(3) $\square\in\pi_2(\mathbf{x},\Theta_{g_1},\Theta,\mathbf{y})$ and $\Delta\in\pi_2(\Theta_{f_1},\Theta_\infty,\Theta)$;

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(4) $\square \in \pi_2(\mathbf{x}, \Theta, \Theta_\infty, \mathbf{y})$ and $\Delta \in \pi_2(\Theta_{g_1}, \Theta_{f_1}, \Theta)$;
(5) $\square \in \pi_2(\Theta_{g_1}, \Theta_{f_1}, \Theta_\infty, \Theta)$ and $\Delta \in \pi_2(\mathbf{x}, \Theta, \mathbf{y})$.

Degenerations of types 1 and 2 correspond to the coefficient of $\mathbf{y}$ in $(I^{\mathfrak{s}} \circ g_1^{\mathfrak{s}})(\mathbf{x})$ and $(X^{\mathfrak{s}} \circ H_{h_1}^{\mathfrak{s}})(\mathbf{x})$, respectively, where

$$X^{\mathfrak{s}}(\mathbf{z}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}, \mathfrak{s}(\mathbf{y}) = \mathfrak{s}} \; \sum_{\Delta \in \pi_2^0(\mathbf{z}, \Theta_\infty, \mathbf{y}; u, v, z)} \#(\mathcal{M}(\Delta)) \cdot \mathbf{y}.$$

Considering the local multiplicities around $\lambda_\infty \cap \lambda'_\infty$, one concludes that there are no triangle classes $\Delta \in \pi_2^0(\mathbf{z}, \Theta_\infty, \mathbf{y}; u, v, z)$ with positive domain. In particular, $X^{\mathfrak{s}}$ is trivial. There are no triangle classes which contribute in the degenerations of type 3. The contributing triangles in degenerations of type 4 come in cancelling pairs. Thus, the total number of boundary ends corresponding to degenerations of types 3 and 4 is zero. There is a unique square class in $\pi_2^{-1}(\Theta_{g_1}, \Theta_{f_1}, \Theta_\infty, \Theta; u, v, z)$ with non-trivial contribution to degenerations of type 5. The intersection of the domain of this square class with the winding region is the rectangle with vertices q_m, p_m, p_∞ and p'_1, which is shaded yellow in Figure 5. For this square class, $\Theta \in \mathbb{T}_{\beta_1} \cap \mathbb{T}_{\beta'}$ is the top generator and $\square$ has a unique holomorphic representative. Using the generator Θ, we define the map

$$\overline{\mathfrak{f}}_0^{\mathfrak{s}} : \widehat{\mathrm{CFK}}(K_1; \mathfrak{s}) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_1; u, v, z; \mathfrak{s}) \longrightarrow \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'; u, v, z; \mathfrak{s}) = \widehat{\mathrm{CFK}}(K; \mathfrak{s}),$$

which is again one of the maps which appeared in the splicing formula of [Eft15]. The contribution of the degenerations of type 5 thus corresponds to the coefficient of $\mathbf{y}$ in $\overline{\mathfrak{f}}_0^{\mathfrak{s}}(\mathbf{x})$. These observations complete the proof of the lemma. $\square$

Let $\widehat{\mathrm{CF}}(Y)$ denote $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$, as before. Define the maps

$$\overline{F}_0^{\mathfrak{s}} : M(f_1^{\mathfrak{s}}) \to \widehat{\mathrm{CFK}}(K; \mathfrak{s}) = \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'; u, v, z; \mathfrak{s}) \quad \text{and} \quad \overline{F}_\infty^{\mathfrak{s}} : M(f_0^{\mathfrak{s}}) \to M(f_1^{\mathfrak{s}})$$

by setting

$$\overline{F}_0^{\mathfrak{s}}(\mathbf{x}_1, \mathbf{x}_2) := I^{\mathfrak{s}}(\mathbf{x}_1), \quad \forall \begin{cases} \mathbf{x}_1 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_m}, & \mathfrak{s}(\mathbf{x}_1) \in \{\mathfrak{s}, \mathfrak{s} + m - 1\}, \\ \mathbf{x}_2 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, \end{cases}$$
$$\overline{F}_\infty^{\mathfrak{s}}(\mathbf{x}_1, \mathbf{x}_2) := (G_\infty^{\mathfrak{s}}(\mathbf{x}_1), \mathbf{x}_2), \quad \forall \begin{cases} \mathbf{x}_1 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_{m-1}}, & \mathfrak{s}(\mathbf{x}_1) \in \{\mathfrak{s}, \mathfrak{s} + m - 1\}, \\ \mathbf{x}_2 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta. \end{cases}$$

The outcome of the above observations, together with Lemmas 3.2, 3.4 and 3.6, is the following theorem.

Theorem 3.7. *With the above notation fixed and up to chain homotopy, the following diagram is commutative:*

$$\begin{tikzcd} \widehat{\mathrm{CFK}}(K_0; \mathfrak{s}) \arrow[r, "\overline{\mathfrak{f}}_\infty^{\mathfrak{s}}"] \arrow[d, "\jmath_0^{\mathfrak{s}}"'] & \widehat{\mathrm{CFK}}(K_1; \mathfrak{s}) \arrow[r, "\overline{\mathfrak{f}}_0^{\mathfrak{s}}"] \arrow[d, "\jmath_1^{\mathfrak{s}}"'] & \widehat{\mathrm{CFK}}(K; \mathfrak{s}) \arrow[d, "\mathrm{Id}"'] \\ M(f_0^{\mathfrak{s}}) \arrow[r, "\overline{F}_\infty^{\mathfrak{s}}"] & M(f_1^{\mathfrak{s}}) \arrow[r, "\overline{F}_0^{\mathfrak{s}}"] & \widehat{\mathrm{CFK}}(K; \mathfrak{s}) \end{tikzcd} \tag{7}$$

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Proof. We first need to verify that $\overline{F}^{\mathfrak{s}}_{\infty}$ and $\overline{F}^{\mathfrak{s}}_{0}$ are chain maps. Note that

$$\begin{aligned}(\overline{F}^{\mathfrak{s}}_{\infty}\circ d+d\circ\overline{F}^{\mathfrak{s}}_{\infty})(\mathbf{x}_1,\mathbf{x}_2)&=((G^{\mathfrak{s}}_{\infty}\circ d+d\circ G^{\mathfrak{s}}_{\infty})(\mathbf{x}_1),(f^{\mathfrak{s}}_1\circ G^{\mathfrak{s}}_{\infty}+f^{\mathfrak{s}}_0)(\mathbf{x}_1))\\&=(0,(f^{\mathfrak{s}}_1\circ G^{\mathfrak{s}}_{\infty}+f^{\mathfrak{s}}_0)(\mathbf{x}_1))\\&=0,\end{aligned}$$

where the last equality follows from the second part of Lemma 3.4. Since $I^{\mathfrak{s}}$ is a chain map by Lemma 3.5, it follows that $\overline{F}^{\mathfrak{s}}_{0}$ is also a chain map. Lemma 3.6 implies that

$$\bar{\mathfrak{f}}^{\mathfrak{s}}_0+\overline{F}^{\mathfrak{s}}_0\circ\jmath^{\mathfrak{s}}_1=d\circ Q^{\mathfrak{s}}+Q^{\mathfrak{s}}\circ d.$$

This proves the commutativity of the right-hand-side square up to chain homotopy. To prove the commutativity of the left-hand-side square, define

$$\begin{gathered}R^{\mathfrak{s}}:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0;u,v,w;\mathfrak{s})=\widehat{\mathrm{CF}}(\Sigma',\boldsymbol{\alpha},\boldsymbol{\beta}_0;u,v;\mathfrak{s})\to M(f^{\mathfrak{s}}_1),\\ R^{\mathfrak{s}}(\mathbf{x}):=(H^{\mathfrak{s}}(\mathbf{x}),P^{\mathfrak{s}}(\mathbf{x})),\quad\forall\mathbf{x}\in\mathbb{T}_{\alpha}\cap\mathbb{T}_{\beta_0}\text{ with }\mathfrak{s}(\mathbf{x})=\mathfrak{s}.\end{gathered}$$

We thus find that

$$\begin{aligned}(d\circ R^{\mathfrak{s}}+R^{\mathfrak{s}}\circ d)(\mathbf{x})&=d(H^{\mathfrak{s}}(\mathbf{x}),P^{\mathfrak{s}}(\mathbf{x}))+(H^{\mathfrak{s}}(d(\mathbf{x})),P^{\mathfrak{s}}(d(\mathbf{x})))\\&=((G^{\mathfrak{s}}_{\infty}\circ g^{\mathfrak{s}}_0+g^{\mathfrak{s}}_1\circ\bar{\mathfrak{f}}^{\mathfrak{s}}_{\infty})(\mathbf{x}),(d\circ P^{\mathfrak{s}}+P^{\mathfrak{s}}\circ d+f^{\mathfrak{s}}_1\circ H^{\mathfrak{s}})(\mathbf{x}))\\&=((G^{\mathfrak{s}}_{\infty}\circ g^{\mathfrak{s}}_0+g^{\mathfrak{s}}_1\circ\bar{\mathfrak{f}}^{\mathfrak{s}}_{\infty})(\mathbf{x}),(H^{\mathfrak{s}}_{h_1}\circ\bar{\mathfrak{f}}^{\mathfrak{s}}_{\infty}+H^{\mathfrak{s}}_{h_0})(\mathbf{x}))\\&=(\overline{F}^{\mathfrak{s}}_{\infty}\circ\jmath^{\mathfrak{s}}_0+\jmath^{\mathfrak{s}}_1\circ\bar{\mathfrak{f}}^{\mathfrak{s}}_{\infty})(\mathbf{x}).\end{aligned}$$

The second equality follows from the first part of Lemma 3.4, while the third equality follows from Lemma 3.2. This observation completes the proof of Theorem 3.7. □

3.4 The proof of the splicing formula

We now turn to understanding the maps $\overline{F}^{\mathfrak{s}}_0$ and $\overline{F}^{\mathfrak{s}}_{\infty}$ (which will be called the *bypass homomorphisms*) under the identifications of Theorem 2.5. To understand $\overline{F}^{\mathfrak{s}}_0$, one should identify $I^{\mathfrak{s}}$ on

$$\widehat{\mathrm{CFK}}(K_m;\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_m;\mathfrak{s}+m-1)=C\{i\leqslant\mathfrak{s},j=0\}\oplus C\{i=0,j\leqslant-\mathfrak{s}\}.$$

Let $\mathbf{x}\in\mathbb{T}_{\alpha}\cap\mathbb{T}_{\beta}$, $\mathbf{x}_i$ be the corresponding generator in $\widehat{\mathrm{CFK}}(K_m)$ and suppose that $\square\in\pi_2^{-1}(\mathbf{x}_i,\Theta_{f_1},\Theta_{\infty},\mathbf{y};u,v,z)$ contributes to $I^{\mathfrak{s}}$. Looking at local coefficients in the regions pictured in Figure 3 implies that $i=1$ and that the intersection of the domain of $\square$ with the winding region is the rectangle with vertices x_1,p_m,p_{∞} and x, which contains the markings a,e and s. In particular, $\mathfrak{s}(\mathbf{x})=\mathfrak{s}(\mathbf{y})=\mathfrak{s}$ and $\mathbf{x}_1$ corresponds to the generator $[\mathbf{x},0,-\mathfrak{s}]\in C\{i=0,j\leqslant-\mathfrak{s}\}$. There is a particular class $\square\in\pi_2(\mathbf{x}_1,\Theta_{f_1},\Theta_{\infty},\mathbf{x})$ with small domain and non-trivial contribution to $I^{\mathfrak{s}}$. Modifying $\widehat{\mathrm{CFK}}(K;\mathfrak{s})=C\{i=0,j=-\mathfrak{s}\}$ by the chain map $I^{\mathfrak{s}}|_{C\{i=0,j=-\mathfrak{s}\}}$, which is a change of basis using the energy filtration, we may thus assume that $\overline{F}^{\mathfrak{s}}_0$ is induced by projecting the factor $C\{i=0,j\leqslant-\mathfrak{s}\}$ in the mapping cone of $\jmath^{\mathfrak{s}}_1$ over the quotient complex $C\{i=0,j=-\mathfrak{s}\}=\widehat{\mathrm{CFK}}(K;\mathfrak{s})$. On the other hand, the image of

$$g^{\mathfrak{s}}_{0,q'}=J^{\mathfrak{s}}\circ g^{\mathfrak{s}}_{0,b}:\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})\to\widehat{\mathrm{CFK}}(K_m;\mathfrak{s}+m-1)$$

is in the subcomplex

$$C\{i=0,j\leqslant-\mathfrak{s}-1\}\subset\widehat{\mathrm{CFK}}(K_m;\mathfrak{s}+m-1)=C\{i=0,j\leqslant-\mathfrak{s}\}.$$

The above observations imply the following theorem.

Theorem 3.8. *Under the identification of $\widehat{\mathrm{CFK}}(K_\bullet;\mathfrak{s})$ with $M(i_\bullet^{\mathfrak{s}})$ for $\bullet = 0,1$, $\overline{F}_\infty^{\mathfrak{s}}$ is given by the inclusion of $M(i_0^{\mathfrak{s}})$ in $M(i_1^{\mathfrak{s}})$ as a subcomplex, while $\overline{F}_0^{\mathfrak{s}}$ is given by the quotient map corresponding to this inclusion map. In particular, we have a short exact sequence*

$$0 \longrightarrow M(i_0^{\mathfrak{s}}) \xrightarrow{\overline{F}_\infty^{\mathfrak{s}} = \hookrightarrow} M(i_1^{\mathfrak{s}}) \xrightarrow{\overline{F}_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K;\mathfrak{s}) = \frac{M(i_1^{\mathfrak{s}})}{M(i_0^{\mathfrak{s}})} \longrightarrow 0.$$

Theorem 3.8 implies that the second row in (7) is part of a short exact sequence. The discussion preceding [Eft15, Theorem 4.6] implies that the initial Heegaard diagram may be chosen so that the first row is also completed to a short exact sequence. We thus have the following commutative diagram (up to chain homotopy):

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathrm{CFK}}(K_0;\mathfrak{s}) \arrow[r, "\bar{\mathfrak{f}}_\infty^{\mathfrak{s}}"] \arrow[d, "\jmath_0^{\mathfrak{s}}"'] & \widehat{\mathrm{CFK}}(K_1;\mathfrak{s}) \arrow[r, "\bar{\mathfrak{f}}_0^{\mathfrak{s}}"] \arrow[d, "\jmath_1^{\mathfrak{s}}"'] & \widehat{\mathrm{CFK}}(K;\mathfrak{s}) \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
0 \arrow[r] & M(i_0^{\mathfrak{s}}) \arrow[r, "\overline{F}_\infty^{\mathfrak{s}}"] & M(i_1^{\mathfrak{s}}) \arrow[r, "\overline{F}_0^{\mathfrak{s}}"] & \widehat{\mathrm{CFK}}(K;\mathfrak{s}) \arrow[r] & 0
\end{tikzcd} \tag{8}$$

In particular, in the level of homology, the connecting homomorphism of the short exact sequence in the second row of (8) is identified with the connecting homomorphism $\bar{\mathfrak{f}}_1^{\mathfrak{s}}$ of the first row, which is used in the splicing formula of [Eft15]. A completely similar argument identifies $\mathfrak{f}_\infty^{\mathfrak{s}}$ with the inclusion map $F_\infty^{\mathfrak{s}}$ from $M(i_0^{\mathfrak{s}-1})$ to $M(i_1^{\mathfrak{s}})$ and $\mathfrak{f}_0^{\mathfrak{s}}$ with the quotient map $F_0^{\mathfrak{s}}$ from $M(i_1^{\mathfrak{s}})$ to $\widehat{\mathrm{CFK}}(K;\mathfrak{s})$, while $\mathfrak{f}_1^{\mathfrak{s}}$ is identified with the connecting homomorphism of the short exact sequence

$$0 \longrightarrow M(i_0^{\mathfrak{s}-1}) \xrightarrow{F_\infty^{\mathfrak{s}}} M(i_1^{\mathfrak{s}}) \xrightarrow{F_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K;\mathfrak{s}) \longrightarrow 0. \tag{9}$$

Proof of Theorem 1.6. Let $C_\bullet(K) = \bigoplus_{\mathfrak{s}\in\mathbb{Z}} C_\bullet(K;\mathfrak{s})$, where $C_\bullet(K;\mathfrak{s}) = M(i_\bullet^{\mathfrak{s}})$ for $\bullet = 0,1$ and $C_\infty(K;\mathfrak{s}) = C\{i=\mathfrak{s}, j=0\}$. The maps $F_\infty, \overline{F}_\infty\colon C_0(K) \to C_1(K)$ and $F_0, \overline{F}_0\colon C_1(K) \to C_\infty(K)$ sit in the short exact sequences

$$0 \longrightarrow C_0(K) \xrightarrow{F_\infty} C_1(K) \xrightarrow{F_0} C_\infty(K) \longrightarrow 0$$

and

$$0 \longrightarrow C_0(K) \xrightarrow{\overline{F}_\infty} C_1(K) \xrightarrow{\overline{F}_0} C_\infty(K) \longrightarrow 0.$$

The maps induced by $F_\bullet$ and $\overline{F}_\bullet$ in homology are $\mathfrak{f}_\bullet$ and $\bar{\mathfrak{f}}_\bullet$, respectively. Thus, [Eft15, Proposition 7.2] may be applied here to complete the proof of Theorem 1.6. $\square$

4. The linear algebra of bypass homomorphisms

4.1 Alternative compositions

Let K be a knot inside the homology sphere Y. Let $(C,d) = (C_K.d_K)$ denote the chain complex associated with K which was discussed in the previous two sections. Correspondingly, one may define the maps $F_\bullet^{\mathfrak{s}}$ and $\overline{F}_\bullet^{\mathfrak{s}}$ for $\bullet \in \{0,\infty\}$. We will denote the maps induced by $F_\bullet^{\mathfrak{s}}$ and $\overline{F}_\bullet^{\mathfrak{s}}$ in the level of homology by $\mathfrak{f}_\bullet^{\mathfrak{s}}$ and $\bar{\mathfrak{f}}_\bullet^{\mathfrak{s}}$, respectively, for $\bullet \in \{0,\infty\}$. The connecting homomorphisms corresponding to the short exact sequences

$$0 \longrightarrow M(i_0^{\mathfrak{s}-1}) \xrightarrow{F_\infty^{\mathfrak{s}}} M(i_1^{\mathfrak{s}}) \xrightarrow{F_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \longrightarrow 0$$

and

$$0 \longrightarrow M(i_0^{\mathfrak{s}}) \xrightarrow{\overline{F}_\infty^{\mathfrak{s}}} M(i_1^{\mathfrak{s}}) \xrightarrow{\overline{F}_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \longrightarrow 0$$

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will be denoted by $\mathfrak{f}_1^{\mathfrak{s}}$ and $\overline{\mathfrak{f}}_1^{\mathfrak{s}}$, respectively. By Theorem 3.8, this notation is compatible with the notation of [Eft15].

Lemma 4.1. *Let* $\mathbf{x}\in\widehat{\mathrm{CFK}}(K,\mathfrak{s})$ *be a closed element and* $[\mathbf{x}]$ *denote the class represented by* $\mathbf{x}$ *in* $\widehat{\mathrm{HFK}}(K,\mathfrak{s})$. *Then*

$$(\mathfrak{f}_0^{\mathfrak{s}-1}\circ\overline{\mathfrak{f}}_\infty^{\mathfrak{s}-1}\circ\mathfrak{f}_1^{\mathfrak{s}})[\mathbf{x}]=[d^{1,0}(\mathbf{x})]\quad\textit{and}\quad(\overline{\mathfrak{f}}_0^{\mathfrak{s}+1}\circ\mathfrak{f}_\infty^{\mathfrak{s}+1}\circ\overline{\mathfrak{f}}_1^{\mathfrak{s}})[\mathbf{x}]=[d^{0,1}(\mathbf{x})].$$

Proof. Since $\mathfrak{f}_1^{\mathfrak{s}}$ is the connecting homomorphism associated with the short exact sequence (9), to compute $\mathfrak{f}_1^{\mathfrak{s}}[\mathbf{x}]$ note that $\mathbf{x}$ is the image of $([\mathbf{x},\mathfrak{s},0],0,0)\in M(i_1^{\mathfrak{s}})$ under the quotient map. The differential of $M(i_1^{\mathfrak{s}})$ takes this element to

$$\left(\sum_{i=0}^{\infty}[d^{i,0}(\mathbf{x}),\mathfrak{s}-i,0],0,[\mathbf{x},\mathfrak{s},0]\right)\in M(i_1^{\mathfrak{s}}).$$

Since $d^{0,0}(\mathbf{x})=0$, this latter element is in $M(i_0^{\mathfrak{s}-1})$. The map $\overline{F}_\infty^{\mathfrak{s}-1}$ is the inclusion; thus,

$$(\overline{F}_\infty^{\mathfrak{s}-1}\circ F_1^{\mathfrak{s}})(\mathbf{x})=\left(\sum_{i=1}^{\infty}[d^{i,0}(\mathbf{x}),\mathfrak{s}-i,0],0,[\mathbf{x},\mathfrak{s},0]\right)\in M(i_1^{\mathfrak{s}-1}).$$

The projection map $F_0^{\mathfrak{s}-1}$ takes this latter element to the closed element $d^{1,0}(\mathbf{x})$ in $\widehat{\mathrm{CFK}}(K;\mathfrak{s}-1)$. This completes the proof of the first claim. The second claim is proved similarly. □

Consider the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complexes $C_1=C_{K_1}$ and $C_0=C_{K_0}=\bigoplus_{\mathfrak{s}}C_0^{\mathfrak{s}}$ associated with the knots K_1 and K_0, respectively. Note that $\mathbb{H}_1(K,\mathfrak{s})$ and $\mathbb{H}_0(K,\mathfrak{s})$ may be identified with the homology of the complexes $C_1\{i=\mathfrak{s},j=0\}$ and $C_0^{\mathfrak{s}}\{i=0,j=0\}$, respectively. It thus make sense to talk about $d_\bullet^{1,0}(\mathbf{x}_\bullet)$ and $d_\bullet^{0,1}(\mathbf{x}_\bullet)$ for $\mathbf{x}\in\mathbb{H}_\bullet(K,\mathfrak{s})$. It is important to note that the chain complexes $C_0^{\mathfrak{s}}\{i=a,j=b\}$ for different integer values a and b are all isomorphic. In fact, the longitude λ_0 is homologically trivial and the condition

$$\mathfrak{s}(\mathbf{x})+(j-i)\mathrm{PD}[\lambda_0]=\mathfrak{s}$$

only means that $\mathfrak{s}(\mathbf{x})=\mathfrak{s}$, and does not put any restrictions on the pair (i,j). We can then prove the following analogue of Lemma 4.1 .

Lemma 4.2. *Let* $\mathbf{x}_\bullet\in\widehat{\mathrm{CFK}}(K_\bullet,\mathfrak{s})$ *be a closed element and* $[\mathbf{x}_\bullet]$ *denote the class represented by* $\mathbf{x}$ *in* $\widehat{\mathrm{HFK}}(K,\mathfrak{s})$. *Then*

$$\begin{aligned}&(\mathfrak{f}_1^{\mathfrak{s}+1}\circ\overline{\mathfrak{f}}_0^{\mathfrak{s}+1}\circ\mathfrak{f}_\infty^{\mathfrak{s}+1})[\mathbf{x}_0]=[d_0^{1,0}(\mathbf{x}_0)],\quad(\overline{\mathfrak{f}}_1^{\mathfrak{s}}\circ\mathfrak{f}_0^{\mathfrak{s}}\circ\overline{\mathfrak{f}}_\infty^{\mathfrak{s}})[\mathbf{x}_0]=[d_0^{0,1}(\mathbf{x}_0)],\\&(\mathfrak{f}_\infty^{\mathfrak{s}}\circ\overline{\mathfrak{f}}_1^{\mathfrak{s}}\circ\mathfrak{f}_0^{\mathfrak{s}})[\mathbf{x}_1]=[d_1^{1,0}(\mathbf{x}_1)]\quad\textit{and}\quad(\overline{\mathfrak{f}}_\infty^{\mathfrak{s}-1}\circ\mathfrak{f}_1^{\mathfrak{s}}\circ\overline{\mathfrak{f}}_0^{\mathfrak{s}})[\mathbf{x}_1]=[d_1^{0,1}(\mathbf{x}_1)].\end{aligned}$$

Proof. We sketch the proof of the first statement, which is a combination of degeneration arguments for holomorphic polygons, similar to the arguments used in §§ 2 and 3. The proof of the other statements is similar.

Theorem 3.8 reduces the proof to showing the commutativity of the following diagram in the level of homology groups:

$$\begin{tikzcd}\widehat{\mathrm{CFK}}(K_0;\mathfrak{s}) \arrow[r, "d^{1,0}"] \arrow[d, "\jmath_0^{\mathfrak{s}}"'] & \widehat{\mathrm{CFK}}(K_0;\mathfrak{s}) \arrow[d, "\jmath_0^{\mathfrak{s}}"] \\ M(i_0^{\mathfrak{s}}) \arrow[r, "\mathfrak{f}_1^{\mathfrak{s}+1}\circ\overline{\mathfrak{f}}_0^{\mathfrak{s}+1}\circ\mathfrak{f}_\infty^{\mathfrak{s}+1}"] & M(i_0^{\mathfrak{s}})\end{tikzcd}\qquad(10)$$

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Let $\jmath_0^{\mathfrak{s}}(\mathbf{x}_0)=(x,y,z)$ with $x\in C\{i\leqslant \mathfrak{s}, j=0\}$, $y\in C\{i=0, j\leqslant -\mathfrak{s}-1\}$ and $z\in C\{j=0\}$. One may then check that $(\bar{\mathfrak{f}}_0^{\mathfrak{s}+1}\circ \mathfrak{f}_\infty^{\mathfrak{s}+1})(\mathbf{x}_0)=[\mathbf{y}_1]\in \mathbb{H}_\infty(K;\mathfrak{s}+1)$, where $y=\sum_{i\geqslant 1}[\mathbf{y}_i, 0, -\mathfrak{s}-i]$. Correspondingly, we may compute

$$(\mathfrak{f}_1^{\mathfrak{s}+1}\circ \bar{\mathfrak{f}}_0^{\mathfrak{s}+1}\circ \mathfrak{f}_\infty^{\mathfrak{s}+1})(\mathbf{x}_0)=(d^{*,0}[\mathbf{y}_1,\mathfrak{s}+1,0],0,[\mathbf{y}_1,\mathfrak{s}+1,0]).$$

Under the identification of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha}_m,\boldsymbol{\beta};u,v;\mathfrak{s})$ with $C\{i\leqslant \mathfrak{s}, j=0\}$ and the identification of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v;\mathfrak{s}+m)$ with $C\{i=0,j\leqslant -\mathfrak{s}-1\}$, we have $y=g_{0,b}^{\mathfrak{s}+1}(\mathbf{x}_0)$. We denote $[\mathbf{y}_i,\mathfrak{s}+i,0]$ by y_i. To show the commutativity of the diagram in (10), we need to show that the map

$$\Phi:\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})\to M(i_0^{\mathfrak{s}}),$$
$$\Phi(\mathbf{x}_0):=(g_{0,a}^{\mathfrak{s}}(d_0^{1,0}(\mathbf{x}_0))+d^{*,0}(y_1), g_{0,b}^{\mathfrak{s}}(d_0^{1,0}(\mathbf{x}_0)), y_1+H_{f_0}^{\mathfrak{s}}(d_0^{1,0}(\mathbf{x}_0)))$$

takes closed elements of $\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})$ to exact elements. For this purpose, using the notation of § 3.1 and the labelling of Figure 3, define $P_a^{\mathfrak{s}}$ and $P_b^{\mathfrak{s}}$ by

$$P_a^{\mathfrak{s}},P_b^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})\to \widehat{\mathrm{CF}}(Y)$$
$$P_a^{\mathfrak{s}}(\mathbf{x}_0)=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\Box\in\pi_2^{-1}(\mathbf{x}_0,\Theta_q,\Theta_{pm},\mathbf{y};v)\\ n_u(\Box)=n_a(\Box)=n_b(\Box)=1}}\#\mathcal{M}(\Box)\mathbf{y},$$
$$P_b^{\mathfrak{s}}(\mathbf{x}_0)=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\Box\in\pi_2^{-1}(\mathbf{x}_0,\Theta_q,\Theta_{pm},\mathbf{y};v,a)\\ n_u(\Box)=1,n_b(\Box)=2}}\#\mathcal{M}(\Box)\mathbf{y}.$$

Here, Θ_q and Θ_{p_m} are the top intersection points in $\mathbb{T}_{\beta_0}\cap\mathbb{T}_{\beta_m}$ and $\mathbb{T}_{\beta_m}\cap\mathbb{T}_\beta$ which use the intersection points q and p_m, respectively. Considering different possible degenerations of a square with Maslov index 0, for a closed generator $\mathbf{x}_0\in \mathrm{CFK}(K_0;\mathfrak{s})$ as above, we obtain

$$(H_{h_0}^{\mathfrak{s}}\circ d_0^{1,0})(\mathbf{x}_0)+y_1=(d^{*,0}\circ P_a^{\mathfrak{s}})(\mathbf{x}_0)+(d^{*,0}\circ P_b^{\mathfrak{s}})(\mathbf{x}_0)+(f_0^{\mathfrak{s}}\circ Q^{\mathfrak{s}})(\mathbf{x}_0),$$

where $Q^{\mathfrak{s}}=Q_a^{\mathfrak{s}}+Q_b^{\mathfrak{s}}$ is defined by

$$Q_a^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v;\mathfrak{s})$$
$$Q_a^{\mathfrak{s}}(\mathbf{x}_0)=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\Delta\in\pi_2^0(\mathbf{x}_0,\Theta_{pm},\mathbf{y};v)\\ n_u(\Delta)=n_a(\Delta)=n_b(\Delta)=1}}\#\mathcal{M}(\Delta)\mathbf{y},$$
$$Q_a^{\mathfrak{s}},Q_b^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_m;u,v;\mathfrak{s}+m)$$
$$Q_b^{\mathfrak{s}}(\mathbf{x}_0)=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\Delta\in\pi_2^0(\mathbf{x}_0,\Theta_{pm},\mathbf{y};v,a)\\ n_u(\Delta)=1,n_b(\Delta)=2}}\#\mathcal{M}(\Delta)\mathbf{y}$$

and $f_0^{\mathfrak{s}}\simeq \Xi\overline{f_0}^{\mathfrak{s}}$ give the identification of $C_{0,m}(\mathfrak{s})$ with $C\{i\leqslant\mathfrak{s},j=0\}\oplus C\{i=0,j<-\mathfrak{s}\}$ and are also discussed in Lemma 2.4. Note that $(f_0^{\mathfrak{s}}\circ Q^{\mathfrak{s}})(\mathbf{x}_0)$ is equal to

$$(f_0^{\mathfrak{s}}\circ Q_a^{\mathfrak{s}})(\mathbf{x}_0)+(\Xi\circ\overline{f}_0^{\mathfrak{s}}\circ Q^{\mathfrak{s}})(\mathbf{x}_0)$$

up to an exact element denoted by $d^{*,0}(P_c(\mathbf{x}_0))$. Set $P^{\mathfrak{s}}=P_a^{\mathfrak{s}}+P_b^{\mathfrak{s}}+P_c^{\mathfrak{s}}$. Considering different possible degenerations of a triangle of Maslov index 1, for a closed generator $\mathbf{x}_0$ as above, we obtain

$$(g_{0,a}^{\mathfrak{s}}\circ d_0^{1,0})(\mathbf{x}_0)+d^{*,0}(y_1)=(d\circ Q_a^{\mathfrak{s}})(\mathbf{x}_0),$$
$$(g_{0,b}^{\mathfrak{s}}\circ d_0^{1,0})(\mathbf{x}_0)=(d\circ Q_b^{\mathfrak{s}})(\mathbf{x}_0).$$

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Define the chain homotopy

$$\Psi_0^{\mathfrak{s}}:\mathrm{Ker}(d)\subset\widehat{\mathrm{CFK}}(K_0;\mathfrak{s})\to M(i_0^{\mathfrak{s}}),$$
$$\Psi_0^{\mathfrak{s}}(\mathbf{x}_0):=(Q_a^{\mathfrak{s}}(\mathbf{x}_0),Q_b^{\mathfrak{s}}(\mathbf{x}_0),P^{\mathfrak{s}}(\mathbf{x}_0)).$$

It is implied from the above considerations that

$$\Phi(\mathbf{x}_0)=d\Psi_0^{\mathfrak{s}}(\mathbf{x}_0)=((d\circ Q_a^{\mathfrak{s}})(\mathbf{x}_0),(d\circ Q_b^{\mathfrak{s}})(\mathbf{x}_0),(d^{*,0}\circ P^{\mathfrak{s}}+f_0^{\mathfrak{s}}\circ Q_a^{\mathfrak{s}}+\Xi\circ\overline{f}_0^{\mathfrak{s}}\circ Q_b^{\mathfrak{s}})(\mathbf{x}_0)).$$

This completes the proof. □

Corollary 4.3. *For every relative* Spin^c *class* $\mathfrak{s}\in\underline{\mathrm{Spin}}^c(Y,K)$, *the maps*

$$F_0^{\mathfrak{s}}=\mathfrak{f}_1\circ\bar{\mathfrak{f}}_0\circ\mathfrak{f}_\infty\circ\bar{\mathfrak{f}}_1\circ\mathfrak{f}_0\circ\bar{\mathfrak{f}}_\infty|_{\widehat{\mathrm{HFK}}(K_0;\mathfrak{s})}:\widehat{\mathrm{HFK}}(K_0;\mathfrak{s})\longrightarrow\widehat{\mathrm{HFK}}(K_0;\mathfrak{s}),$$
$$F_1^{\mathfrak{s}}=\mathfrak{f}_\infty\circ\bar{\mathfrak{f}}_1\circ\mathfrak{f}_0\circ\bar{\mathfrak{f}}_\infty\circ\mathfrak{f}_1\circ\bar{\mathfrak{f}}_0|_{\widehat{\mathrm{HFK}}(K_1;\mathfrak{s})}:\widehat{\mathrm{HFK}}(K_1;\mathfrak{s})\longrightarrow\widehat{\mathrm{HFK}}(K_1;\mathfrak{s}),$$
$$F_\infty^{\mathfrak{s}}=\mathfrak{f}_0\circ\bar{\mathfrak{f}}_\infty\circ\mathfrak{f}_1\circ\bar{\mathfrak{f}}_0\circ\mathfrak{f}_\infty\circ\bar{\mathfrak{f}}_1|_{\widehat{\mathrm{HFK}}(K;\mathfrak{s})}:\widehat{\mathrm{HFK}}(K;\mathfrak{s})\longrightarrow\widehat{\mathrm{HFK}}(K;\mathfrak{s})$$

are nilpotent.

Proof. By Lemma 4.1, for every closed $\mathbf{x}\in\widehat{\mathrm{CFK}}(K;\mathfrak{s})$, we have

$$F_\infty^{\mathfrak{s}}[\mathbf{x}]=[d^{1,0}(d^{0,1}(\mathbf{x}))]\;\Rightarrow\;(F_\infty^{\mathfrak{s}}\circ F_\infty^{\mathfrak{s}})[\mathbf{x}]=[(d^{1,0}\circ d^{0,1})^2(\mathbf{x})]=[((d^{1,0})^2\circ(d^{0,1})^2)(\mathbf{x})]=0,$$

where the last two equalities follow from (5). The other claims are proved similarly. □

4.2 Block decomposition for bypass homomorphisms

Let us assume that the chain complex C is defined from the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,z)$. Changing the role of punctures gives the duality maps

$$\tau_\bullet=\tau_\bullet(K):\mathbb{H}_\bullet(K)\to\mathbb{H}_\bullet(K)\quad\text{for }\bullet\in\{0,1,\infty\},$$

where $\tau_\bullet$ takes $\mathbb{H}_\bullet(K;\mathfrak{s})$ to $\mathbb{H}_\bullet(K,-\mathfrak{s})$ if $\bullet=1,\infty$ and to $\mathbb{H}_0(K,-\mathfrak{s}-1)$ when $\bullet=0$. Following the notation of [Eft15], in a basis for $\mathbb{H}_\bullet(K)$ where $\mathfrak{f}_\bullet$ takes the block form $\left(\begin{smallmatrix}0&0\\I&0\end{smallmatrix}\right)$, we assume that

$$\tau_\bullet=\begin{pmatrix}A_\bullet&B_\bullet\\C_\bullet&D_\bullet\end{pmatrix}\quad\text{and}\quad\tau_\bullet^{-1}=\begin{pmatrix}\overline{A}_\bullet&\overline{B}_\bullet\\\overline{C}_\bullet&\overline{D}_\bullet\end{pmatrix},\qquad\bullet\in\{0,1,\infty\}.\tag{11}$$

It was observed in [Eft15] that

$$\bar{\mathfrak{f}}_0=\tau_\infty\circ\mathfrak{f}_0\circ\tau_1^{-1},\quad\bar{\mathfrak{f}}_1=\tau_0\circ\mathfrak{f}_1\circ\tau_\infty^{-1}\quad\text{and}\quad\bar{\mathfrak{f}}_\infty=\tau_1\circ\mathfrak{f}_\infty\circ\tau_0^{-1}.$$

The maps B_0,B_1 and B_∞ correspond to the induced maps

$$\tau_0\colon\mathrm{Ker}(\mathfrak{f}_\infty)\to\frac{\mathbb{H}_0(K)}{\mathrm{Ker}(\mathfrak{f}_\infty)}=\mathrm{Coker}(\mathfrak{f}_1),$$
$$\tau_1\colon\mathrm{Ker}(\mathfrak{f}_0)\to\frac{\mathbb{H}_1(K)}{\mathrm{Ker}(\mathfrak{f}_0)}=\mathrm{Coker}(\mathfrak{f}_\infty),$$
$$\tau_\infty\colon\mathrm{Ker}(\mathfrak{f}_1)\to\frac{\mathbb{H}_\infty(K)}{\mathrm{Ker}(\mathfrak{f}_1)}=\mathrm{Coker}(\mathfrak{f}_0).$$

It follows that, up to a change of basis for the vector spaces $\mathrm{Ker}(\mathfrak{f}_\bullet)$ and $\mathrm{Coker}(\mathfrak{f}_\bullet)$, the matrices $B_\bullet$ are well defined and are invariants of K. In particular, their sizes, ranks, injectivity and surjectivity do not depend on the particular choice of the above presentation for $\tau_\bullet$. Denote the rank of $\mathfrak{f}_\bullet$ by $a_\bullet = a_\bullet(K)$. Thus, a_1, a_∞ and a_0+1 have the same parity. Note that B_0, B_1 and B_∞ are matrices of sizes $a_\infty \times a_1$, $a_0 \times a_\infty$ and $a_1 \times a_0$, respectively. Define $X_\bullet = X_\bullet(K)$ by $X_0 = B_1\overline{B}_0B_\infty, X_1 = B_\infty\overline{B}_1B_0$ and $X_\infty = B_0\overline{B}_\infty B_1$. Similarly, define $\overline{X}_0 = \overline{B}_1B_0\overline{B}_\infty, \overline{X}_1 = \overline{B}_\infty B_1\overline{B}_0$ and $\overline{X}_\infty = \overline{B}_0B_\infty\overline{B}_1$.

LEMMA 4.4. *With the above notation fixed, we have*

$$\overline{A}_\bullet(K) = A_\bullet(K), \quad \overline{B}_\bullet(K) = B_\bullet(K), \quad \overline{D}_\bullet(K) = D_\bullet(K) \quad \textit{and} \quad \overline{X}_\bullet(K) = X_\bullet(K)$$

for $\bullet \in \{0,1,\infty\}$. *Furthermore,* $X_\bullet(K)^2 = 0$ *for* $\bullet \in \{0,1,\infty\}$.

Proof. Applying the homomorphism $\tau_\bullet$ twice gives an involution on $\mathbb{H}_\bullet(K) = \bigoplus_{\mathfrak{s}} \mathbb{H}_\bullet(K,\mathfrak{s})$ which respects the decomposition by Spin^c structures. The resulting isomorphism $\varsigma_\bullet = \tau_\bullet \circ \tau_\bullet$ is an involution (i.e. $\varsigma_\bullet \circ \varsigma_\bullet$ is the identity map on $\mathbb{H}_\bullet(K)$), which was studied by Sarkar [Sar15] and by Hendricks and Manolescu [HM15]. In particular, [Sar15, Theorem 1.1] implies that

$$\varsigma_\bullet = \mathrm{Id} + d_\bullet^{1,0} \circ d_\bullet^{0,1}$$

for $\bullet \in \{0,1,\infty\}$; cf. [HM15, Proposition 6.6]. Lemmas 4.1 and 4.2 imply that

$$\begin{aligned}\tau_\infty^2 &= \mathrm{Id} + d^{1,0} \circ d^{0,1} = \mathrm{Id} + \mathfrak{f}_0 \circ \bar{\mathfrak{f}}_\infty \circ \mathfrak{f}_1 \circ \bar{\mathfrak{f}}_0 \circ \mathfrak{f}_\infty \circ \bar{\mathfrak{f}}_1 = \begin{pmatrix} I & 0 \\ X_0\overline{B}_1B_0\overline{A}_\infty & I + X_0\overline{X}_0 \end{pmatrix} \\ &= \mathrm{Id} + d^{0,1} \circ d^{1,0} = \mathrm{Id} + \bar{\mathfrak{f}}_0 \circ \mathfrak{f}_\infty \circ \bar{\mathfrak{f}}_1 \circ \mathfrak{f}_0 \circ \bar{\mathfrak{f}}_\infty \circ \mathfrak{f}_1 = \begin{pmatrix} I + X_1\overline{X}_1 & 0 \\ D_\infty\overline{B_1}B_0\overline{X}_1 & I \end{pmatrix}.\end{aligned}$$

It follows that $X_0\overline{X}_0 = X_1\overline{X}_1 = 0$. Similarly, we can show that $X_\infty\overline{X}_\infty = 0$ and we conclude that $\varsigma_\bullet = \left(\begin{smallmatrix} I & 0 \\ Z_\bullet & I \end{smallmatrix}\right)$. This means, in particular, that

$$\begin{aligned}\tau_\bullet^{-1} = \varsigma_\bullet\tau_\bullet &= \begin{pmatrix} A_\bullet & B_\bullet \\ C_\bullet + Z_\bullet A_\bullet & D_\bullet + Z_\bullet B_\bullet \end{pmatrix} \\ = \tau_\bullet\varsigma_\bullet &= \begin{pmatrix} A_\bullet + B_\bullet Z_\bullet & B_\bullet \\ C_\bullet + D_\bullet Z_\bullet & D_\bullet \end{pmatrix} \\ &= \begin{pmatrix} \overline{A}_\bullet & \overline{B}_\bullet \\ \overline{C}_\bullet & \overline{D}_\bullet \end{pmatrix}\end{aligned}$$

and that $\overline{X}_\bullet = X_\bullet$. □

LEMMA 4.5. *If K is a knot of genus $g>0$, then $B_1 \neq 0$ and $B_\infty \neq 0$. In particular, $a_\bullet > 0$ for $\bullet \in \{0,1,\infty\}$.*

Proof. Since $H_*(M(i_0^g)) = 0$ by Theorem 2.5, the map $\bar{\mathfrak{f}}_0^g : \mathbb{H}_1(K,g) \to \mathbb{H}_\infty(K,g)$ is an isomorphism. From here and by duality, $\mathfrak{f}_0^{-g}$ is also an isomorphism and

$$H_*(M(i_1^{-g})) \simeq \widehat{\mathrm{HFK}}(K,-g).$$

It follows from the proof of Proposition 5.3 from [Vaf15] (which may be extended to knots in arbitrary homology spheres) that

$$H_*(M(i_0^{g-1})) = \mathbb{H}_0(K,g-1) \simeq \widehat{\mathrm{HFK}}(K,g) \oplus \widehat{\mathrm{HFK}}(K,g).$$

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This was also proved for fibered knots in [Eft05]. Thus, $\mathfrak{f}_0^g$ is trivial, $\mathfrak{f}_1^g$ is injective and $\mathfrak{f}_\infty^g$ is surjective. The triviality of $\mathfrak{f}_0^g$ implies that $\mathrm{Ker}(\mathfrak{f}_0)\backslash\mathrm{Ker}(\bar{\mathfrak{f}}_0)$ and $\mathrm{Im}(\bar{\mathfrak{f}}_0)\backslash\mathrm{Im}(\mathfrak{f}_0)$ are both non-empty. Since $\mathrm{Ker}(\mathfrak{f}_0)\backslash\mathrm{Ker}(\bar{\mathfrak{f}}_0)$ is non-empty, it follows that

$$\exists \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{H}_1(K) \quad \text{s.t.} \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} A_\infty & B_\infty \\ C_\infty & D_\infty \end{pmatrix}\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \neq 0.$$

Thus, $a = 0$ and $\left(\begin{smallmatrix} B_\infty B_1 b \\ D_\infty B_1 b \end{smallmatrix}\right) \neq 0$. In particular, $B_1 \neq 0$.

Similarly, from the condition $\mathrm{Im}(\bar{\mathfrak{f}}_0)\backslash\mathrm{Im}(\mathfrak{f}_0) \neq \emptyset$, it follows that $\mathrm{Ker}(\bar{\mathfrak{f}}_1)\backslash\mathrm{Ker}(\mathfrak{f}_1)$ is non-empty and thus $B_\infty \neq 0$. □

LEMMA 4.6. *For every knot K, $X_\bullet^2 = 0$ for $\bullet \in \{0, 1, \infty\}$. In particular, if K is non-trivial, the kernel and the cokernel of $X_\bullet$ are non-trivial.*

Proof. The first claim is already proved in the discussion preceding Lemma 4.5. The second claim is a consequence of the first, since $a_\bullet > 0$ by Lemma 4.5. □

If $P_\bullet$ is an invertible $a_\bullet \times a_\bullet$ matrix and the matrices $Y_\bullet$ are arbitrary matrices of correct size, we may choose a change of basis for either of $\mathbb{H}_0(K), \mathbb{H}_1(K)$ and $\mathbb{H}_\infty(K)$, which is given by the invertible matrices

$$\mathbb{P}_0 = \begin{pmatrix} P_\infty & 0 \\ Y_0 & P_1 \end{pmatrix}, \quad \mathbb{P}_1 = \begin{pmatrix} P_0 & 0 \\ Y_1 & P_\infty \end{pmatrix} \quad \text{and} \quad \mathbb{P}_\infty = \begin{pmatrix} P_1 & 0 \\ Y_\infty & P_0 \end{pmatrix}, \tag{12}$$

respectively. The block forms $\mathfrak{f}_\bullet = \left(\begin{smallmatrix} 0 & 0 \\ I & 0 \end{smallmatrix}\right)$ remain unchanged under such a change of basis. A simultaneous change of basis of the form illustrated in (12) is called an *admissible* change of basis. The following lemma will be useful through our forthcoming discussions.

LEMMA 4.7. *Suppose that K is a knot in a homology sphere and, for $\bullet \in \{0, 1, \infty\}$, let $\tau_\bullet$ denote $\tau_\bullet(K)$ and $X_\bullet$ denote the matrix $X_\bullet(K)$. Choose*

$$(\circ, \bullet, *) \in \{(0, 1, \infty), (1, \infty, 0), (\infty, 0, 1)\}.$$

(1) *If $B_\circ(K), B_\bullet(K)$ are injective and $B_*(K)$ is surjective, after an admissible change of basis we may assume that*

$$\tau_\circ = \left(\begin{array}{cc|c} 0 & 0 & I \\ 0 & \star & 0 \\ \hline I & 0 & 0 \end{array}\right), \quad \tau_\bullet = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & \star & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{array}\right) \quad \textit{and} \quad \tau_* = \left(\begin{array}{c|ccc} 0 & X_\bullet & \star & \star \\ \hline \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{array}\right). \tag{13}$$

(2) *If $B_\circ(K), B_\bullet(K)$ are surjective and $B_*(K)$ is injective, after an admissible change of basis we may assume that*

$$\tau_\circ = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline 0 & 0 & \star & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{array}\right), \quad \tau_\bullet = \left(\begin{array}{c|cc} 0 & 0 & I \\ 0 & \star & 0 \\ I & 0 & 0 \end{array}\right) \quad \textit{and} \quad \tau_* = \left(\begin{array}{ccc|c} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & X_\circ \\ \hline \star & \star & \star & 0 \end{array}\right). \tag{14}$$

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Proof. It is important to note that when $B_\bullet$ is injective, we can assume that $D_\bullet = 0$ and $Z_\bullet$ is thus zero. This implies that $\tau_\bullet^{-1} = \tau_\bullet$. Similarly, when $B_\bullet$ is surjective, we can assume that $\overline{A}_\bullet = A_\bullet = 0$ and $Z_\bullet$ is thus zero. Again, this implies that $\tau_\bullet^{-1} = \tau_\bullet$. The rest of the proof consists of straightforward linear algebra. $\square$

Definition 4.8. The knot K inside the homology sphere Y is called *full-rank* if all three matrices $B_0(K), B_1(K)$ and $B_\infty(K)$ are full-rank.

If K is full-rank, it is implied that $\varsigma_\bullet(K) = \tau_\bullet(K)^2$ is the identity. We may thus assume that $\overline{\square}_\bullet = \square_\bullet$ for $\square \in \{A, B, C, D, X\}$ and $\bullet \in \{0, 1, \infty\}$.

5. Splicing and homology sphere L-spaces

5.1 Special pairs

Given an arbitrary matrix M, denote the rank of $\mathrm{Ker}(M)$ by $k(M)$, denote the rank of $\mathrm{Coker}(M)$ by $c(M)$ and set $h(M) = k(M) + c(M)$. The matrices M_1 and M_2 are called *equivalent* if $k(M_1) = k(M_2)$ and $c(M_1) = c(M_2)$. If $M^\star \in M_{n_\star \times m_\star}(\mathbb{F})$ for $\star = 1, 2$ are a pair of matrices, $M^1 \otimes M^2 \in M_{n_1 n_2 \times m_1 m_2}(\mathbb{F})$ denotes the associated map from $\mathbb{F}^{m_1 m_2} = \mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2}$ to $\mathbb{F}^{n_1 n_2} = \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$.

Let $Y = Y(K_1, K_2)$ denote the three-manifold obtained by splicing the complements of $K_1 \subset Y_1$ and $K_2 \subset Y_2$, where Y_1 and Y_2 are homology spheres. For $\square \in \{A, B, C, D, X, \tau\}$, $\bullet \in \{0, 1, \infty\}$ and $\star \in \{1, 2\}$, let $\square_\bullet^\star = \square_\bullet(K_\star)$. Proposition 5.4 from [Eft15] and the discussion following it give the following.

Proposition 5.1. *If K_i is a knot inside the homology sphere Y_i for $i = 1, 2$,*

$$\mathrm{rnk}\ \widehat{\mathrm{HF}}(Y(K_1, K_2); \mathbb{F}) = h(\mathfrak{D}(K_1, K_2)),$$

where the matrix $\mathfrak{D} = \mathfrak{D}(K_1, K_2)$ is given by

$$\mathfrak{D} = \begin{pmatrix} B_1^1 \otimes B_1^2 & B_1^1 \otimes A_1^2 & 0 & A_1^1 \otimes B_1^2 & 0 & 0 \\ 0 & A_0^1 \otimes B_\infty^2 & B_0^1 \otimes B_\infty^2 & 0 & 0 & B_0^1 \otimes A_\infty^2 \\ D_1^1 \otimes B_1^2 & \begin{matrix} D_1^1 \otimes A_1^2 \\ +A_0^1 \otimes D_\infty^2 \end{matrix} & B_0^1 \otimes D_\infty^2 & C_1^1 \otimes B_1^2 & 0 & B_0^1 \otimes C_\infty^2 \\ 0 & 0 & 0 & B_\infty^1 \otimes A_0^2 & B_\infty^1 \otimes B_0^2 & A_\infty^1 \otimes B_0^2 \\ B_1^1 \otimes D_1^2 & B_1^1 \otimes C_1^2 & 0 & \begin{matrix} D_\infty^1 \otimes A_0^2 \\ +A_1^1 \otimes D_1^2 \end{matrix} & D_\infty^1 \otimes B_0^2 & C_\infty^1 \otimes B_0^2 \\ 0 & C_0^1 \otimes B_\infty^2 & D_0^1 \otimes B_\infty^2 & B_\infty^1 \otimes C_0^2 & B_\infty^1 \otimes D_0^2 & \begin{matrix} A_\infty^1 \otimes D_0^2 \\ +D_0^1 \otimes A_\infty^2 \\ +X_1^1 \otimes X_1^2 \end{matrix} \end{pmatrix}.$$

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For the purposes of this paper, it is usually easier to work with a matrix $\mathfrak{D}'(K_1,K_2)$, which is equivalent to $\mathfrak{D}(K_1,K_2)$ and is given by the block form

$$\begin{pmatrix}
D^1_\infty B^1_1\otimes B^2_1A^2_0 & B^1_1A^1_0\otimes I & B^1_1B^1_0\otimes I & D^1_\infty A^1_1\otimes B^2_1A^2_0 & I\otimes B^2_1B^2_0 & 0\\
I\otimes B^2_\infty B^2_1 & D^1_1A^1_0\otimes B^2_\infty A^2_1 & D^1_1B^1_0\otimes B^2_\infty A^2_1 & 0 & B^1_0B^1_\infty\otimes I & B^1_0A^1_\infty\otimes I\\
I\otimes D^2_\infty B^2_1 & \begin{matrix} I\otimes I+\\ D^1_1A^1_0\otimes D^2_\infty A^2_1\end{matrix} & D^1_1B^1_0\otimes D^2_\infty A^2_1 & 0 & 0 & 0\\
B^1_\infty B^1_1\otimes I & 0 & I\otimes B^2_0B^2_\infty & B^1_\infty A^1_1\otimes I & \begin{matrix} D^1_0B^1_\infty\otimes B^2_0A^2_\infty\\ +X^1_1B^1_\infty\otimes B^2_0X^2_1\end{matrix} & \begin{matrix} D^1_0A^1_\infty\otimes B^2_0A^2_\infty\\ +X^1_1A^1_\infty\otimes B^2_0X^2_1\end{matrix}\\
D^1_\infty B^1_1\otimes D^2_1A^2_0 & 0 & 0 & \begin{matrix} I\otimes I+\\ D^1_\infty A^1_1\otimes D^2_1A^2_0\end{matrix} & I\otimes D^2_1B^2_0 & 0\\
0 & 0 & I\otimes D^2_0B^2_\infty & 0 & \begin{matrix} D^1_0B^1_\infty\otimes D^2_0A^2_\infty\\ +X^1_1B^1_\infty\otimes D^2_0X^2_1\end{matrix} & \begin{matrix} I\otimes I+\\ D^1_0A^1_\infty\otimes D^2_0A^2_\infty\\ +X^1_1A^1_\infty\otimes D^2_0X^2_1\end{matrix}
\end{pmatrix}.$$

A change of basis changes $\mathfrak{D}(K_1,K_2)$ to $\mathfrak{D}'(K_1,K_2)$, as discussed in [Eft15]. Here, it is important to note that $\overline{\square}^i_\bullet=\square^i_\bullet$ for $\bullet\in\{0,1,\infty\}$, $i=1,2$ and $\square\in\{A,B,D,X\}$.

This is the original formulation of [Eft15, Proposition 5.4], which was proved based on the incorrect assumption that $\tau_\bullet(K_i)$ is an involution for $i=1,2$ and $\bullet\in\{0,1,\infty\}$. The above two splicing formulas are corrected in [Eft17]. The first matrix presentation remains unchanged, while in the second matrix presentation some of the matrices $\square^i_\bullet$ are changed to $\overline{\square}^i_\bullet$ for $\square\in\{A,B,C,D,X\}$, $\bullet\in\{0,1,\infty\}$ and $i\in\{1,2\}$. It is then important to note that the matrices $\overline{C}^i_\bullet$ are not used in the original and revised splicing formula. Lemma 4.4 thus implies that the above splicing formulas are in fact valid.

There is a *symmetry* between K_1 and K_2 in the block presentation of $\mathfrak{D}(K_1,K_2)$, which may be made precise as follows. If we re-order the row blocks and the column blocks of $\mathfrak{D}(K_1,K_2)$ using the permutation

$$(1,2,3,4,5,6)\longrightarrow(1,4,5,2,3,6),$$

we obtain a new matrix with entries of the form $X^1\otimes X^2$ (or a sum of such elements). We may further change every such entry to $X^2\otimes X^1$. The resulting matrix is $\mathfrak{D}(K_2,K_1)$. This symmetry shows that $\mathfrak{D}(K_1,K_2)$ is equivalent to $\mathfrak{D}(K_2,K_1)$. The disadvantage of using $\mathfrak{D}'(K_1,K_2)$ is that the symmetry between K_1 and K_2 is not seen in the splicing formula when we use $\mathfrak{D}'(K_1,K_2)$. Yet, the equivalence of $\mathfrak{D}'(K_1,K_2)$ with $\mathfrak{D}(K_1,K_2)$ implies that $\mathfrak{D}'(K_1,K_2)$ is equivalent to $\mathfrak{D}'(K_2,K_1)$.

DEFINITION 5.2. The pair (K_1,K_2) is called a *special pair* if $\widehat{\mathrm{HF}}(Y(K_1,K_2);\mathbb{F})=\mathbb{F}$.

Suppose, throughout this section, that (K_1,K_2) is a special pair. Let $k^\bullet_\star=k(B^\bullet_\star)$ and $c^\bullet_\star=c(B^\bullet_\star)$ for $\star\in\{0,1,\infty\}$ and $\bullet=1,2$. Define $\imath:\{0,1,\infty\}\to\{0,1,\infty\}$ by $\imath(0)=\infty,\imath(1)=1$ and $\imath(\infty)=0$. For $\mathfrak{D}=\mathfrak{D}(K_1,K_2)$, the cokernel and kernel of $\mathfrak{D}$ include subspaces $\mathrm{C}(\mathfrak{D})$ and $\mathrm{K}(\mathfrak{D})$ (respectively), which are isomorphic to

$$\bigoplus_{\bullet\in\{0,1,\infty\}}\mathrm{Coker}(B^1_\bullet)\otimes\mathrm{Coker}(B^2_{\imath(\bullet)})\quad\text{and}\quad\bigoplus_{\bullet\in\{0,1,\infty\}}\mathrm{Ker}(B^1_\bullet)\otimes\mathrm{Ker}(B^2_{\imath(\bullet)}),$$

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respectively, and correspond to the first, second and fourth rows, and to the first, third and fifth columns, respectively. Moreover, if $A^1_\infty \otimes D^2_0 + D^1_0 \otimes A^2_\infty = 0$ (which may be assumed after an admissible change of basis if $c^1_\infty k^2_0 = k^1_0 c^2_\infty = 0$), the cokernel also includes a subspace isomorphic to $\mathrm{Coker}(B^1_\infty) \otimes \mathrm{Coker}(B^2_\infty)$ and the kernel includes a subspace isomorphic to $\mathrm{Ker}(B^1_0) \otimes \mathrm{Ker}(B^2_0)$. These two subspaces correspond to the last row and the last column of the matrix $\mathfrak{D}(K_1, K_2)$, since $\mathrm{Ker}(B^\star_0) \subset \mathrm{Ker}(X^\star_1)$ for $\star = 1, 2$, while $\mathrm{Coker}(B^\star_\infty)$ is a quotient of $\mathrm{Coker}(X^\star_1)$.

Denote the ranks of $\mathrm{K}(\mathfrak{D})$ and $\mathrm{C}(\mathfrak{D})$ by $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$, respectively. Thus, $k(\mathfrak{D}) + c(\mathfrak{D}) \leqslant 1$ and

$$\widehat{k}(\mathfrak{D}) = \sum_{\bullet \in \{0,1,\infty\}} k^1_\bullet k^2_{\imath(\bullet)} \leqslant k(\mathfrak{D}) \quad \text{and} \quad \widehat{c}(\mathfrak{D}) = \sum_{\bullet \in \{0,1,\infty\}} c^1_\bullet c^2_{\imath(\bullet)} \leqslant c(\mathfrak{D}).$$

Proposition 5.3. *If (K_1, K_2) is a special pair, then possibly after interchanging K_1 and K_2, one of the following is the case.*

- (G) *K_1 is full-rank.*
- (S-1) *The matrix B^2_0 is invertible, B^1_0 is surjective and B^1_1 and B^2_∞ are injective.*
- (S-2) *The matrix B^2_0 is invertible, B^1_0 is injective and B^1_1 and B^2_∞ are surjective.*

Proof. We assume that (K_1, K_2) is a special pair, while none of K_1 and K_2 is full-rank. Let us first assume that both $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$ are zero. From the above assumption, we find that $k^1_\bullet k^2_{\imath(\bullet)} = c^1_\bullet c^2_{\imath(\bullet)} = 0$ for $\bullet = 0, 1, \infty$. If $B^1_\bullet$ is not a full-rank matrix, then both $c^1_\bullet$ and $k^1_\bullet$ are non-zero. From here, $k^2_{\imath(\bullet)} = c^2_{\imath(\bullet)} = 0$, i.e. $B^2_{\imath(\bullet)}$ is invertible. Since the parity of a^2_0 is different from the parity of a^2_1 and a^2_∞, the matrices B^2_1 and B^2_∞ cannot be square matrices. Thus, $\imath(\bullet) = 0$ and $\bullet = \infty$. In other words, we conclude that B^1_0 and B^1_1 are full-rank and B^2_0 is invertible, while B^2_∞ is not full-rank. Similarly, we may conclude that B^2_1 is full-rank and B^1_0 is invertible, while B^1_∞ is not full-rank. Moreover, since $c^1_1 c^2_1 = k^1_1 k^2_1 = 0$, precisely one of B^1_1 and B^2_1 is injective and the other one is surjective. Without loss of generality, we may thus assume that:

- B^1_0 and B^2_0 are invertible, B^1_1 is injective and B^2_1 is surjective;
- none of B^1_∞ and B^2_∞ is full-rank.

In particular, $k^1_\infty > c^1_\infty > 0$ and $c^2_\infty > k^2_\infty > 0$. Since B^1_0 and B^2_0 are both invertible, we may assume that $D^1_0 = 0$ and $D^2_0 = 0$. From here, the cokernel of $\mathfrak{D}$ includes a subspace isomorphic to $\mathrm{Coker}(B^1_\infty) \otimes \mathrm{Coker}(B^2_\infty)$, which is of size $c^1_\infty c^2_\infty \geqslant 2$. This implies that (K_1, K_2) is not special.

From this contradiction, we conclude that one of $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$ is non-zero. Suppose that $\widehat{c}(\mathfrak{D}) = 1$ and $\widehat{k}(\mathfrak{D}) = 0$. For some $\bullet \in \{0, 1, \infty\}$, we thus have $c^1_\bullet = c^2_{\imath(\bullet)} = 1$, while $k^1_\bullet k^2_{\imath(\bullet)} = 0$ and, for $\star \neq \bullet$, we have $c^1_\star c^2_{\imath(\star)} = k^1_\star k^2_{\imath(\star)} = 0$. Without loss of generality, we may assume that $k^1_\bullet = 0$. Thus, $B^1_\bullet$ is injective with a one-dimensional cokernel. In particular, the parities of the number of rows and the number of columns for $B^1_\bullet$ are different, i.e. $\bullet \neq 0$. Thus, $c^1_0 c^2_\infty = k^1_0 k^2_\infty = 0$. Since B^2_∞ is not a square matrix, at least one of c^2_∞ and k^2_∞ is non-zero, implying that at least one of c^1_0 and k^1_0 is zero, i.e. B^1_0 is full-rank. The assumption that K_1 is not full-rank implies that $B^1_\star$ is not full-rank, where $\{\star\} = \{1, \infty\} \backslash \{\bullet\}$. From here, $c^1_\star, k^1_\star > 0$. Together with $c^1_\star c^2_{\imath(\star)} = k^1_\star k^2_{\imath(\star)} = 0$, this implies that $c^2_{\imath(\star)} = k^2_{\imath(\star)} = 0$, i.e. $B^2_{\imath(\star)}$ is invertible. Thus, $\imath(\star) = 0, \star = \infty$ and $\bullet = 1$. We thus conclude that:

- B^2_0 is invertible, B^1_0 is full-rank, B^1_1 is injective and B^1_∞ is not full-rank;
- $c^1_1 = c^2_1 = 1$.

Since B_0^2 is invertible, we may assume that $A_0^2 = D_0^2 = 0$. If B_0^1 is injective, we may also assume that $D_0^1 = 0$ and that $\mathrm{Coker}(\mathfrak{D})$ includes a subspace isomorphic to $\mathrm{Coker}(B_\infty^1) \otimes \mathrm{Coker}(B_\infty^2)$ and of size $c_\infty^1 c_\infty^2$. Since $c_\infty^1 \neq 0$, we conclude that B_∞^2 is surjective. From here, $a_\infty^2 = a_1^2 \leqslant a_0^2 - 1$ and $1 - k_1^2 = c_1^2 - k_1^2 = a_0^2 - a_\infty^2 \geqslant 1$. We thus find that $k_1^2 = 0$ and K_2 is full-rank, which is a contradiction. Thus, $k_0^1 > 0$ and $c_0^1 = 0$. From $k_0^1 k_\infty^2 = 0$, we find that $k_\infty^2 = 0$, i.e. B_∞^2 is injective and the conditions of (S-1) are satisfied. A similar argument reduces the case $\widehat{k}(\mathfrak{D}) = 1$ and $\widehat{c}(\mathfrak{D}) = 0$ to (S-2). □

Proposition 5.4. *Given the pair of knots (K_1, K_2), where K_1 is a full-rank knot and for any $(\circ, \bullet, *) \in \{(0, 1, \infty), (1, \infty, 0), (\infty, 0, 1)\}$:*

(K) *if $B_\circ^1, B_\bullet^1$ are injective and B_*^1 is surjective, then*

$$c(\mathfrak{D}) \geqslant c_\bullet^1 c_{\imath(\bullet)}^2 + c_\circ^1 c_{\imath(\circ)}^2 \quad \text{and} \quad k(\mathfrak{D}) \geqslant k(X_\bullet^1) k(B_{\imath(*)}^2 X_{\imath(\bullet)}^2);$$

(C) *if $B_\circ^1, B_\bullet^1$ are surjective and B_*^1 is injective, then*

$$k(\mathfrak{D}) \geqslant k_\bullet^1 k_{\imath(\bullet)}^2 + k_\circ^1 k_{\imath(\circ)}^2 \quad \text{and} \quad c(\mathfrak{D}) \geqslant c(X_\circ^1) c(X_{\imath(\circ)}^2 B_{\imath(*)}^2).$$

Proof. The first claim in either of cases (K) and (C) is already observed in our earlier discussions. We thus need to prove the second claim in each case. The proofs are very similar. In fact, the proof of claim (C) for $(\circ, \bullet, *)$ is almost identical to the proof of claim (K) for $(\imath(\bullet), \imath(\circ), \imath(*))$ because of the symmetry in the block presentation of $\mathfrak{D}'$. We will only go through the proof for $(\circ, \bullet, *) = (0, 1, \infty)$.

In case (K), after an admissible change of basis, we may assume that $\tau_0(K_1), \tau_1(K_1)$ and $\tau_\infty(K_1)$ take the standard form of (13). Since $D_0^1 = D_1^1 = A_\infty^1 = 0$, the $(3, 2)$ entry and the $(6, 6)$ entry of the matrix $\mathfrak{D}' = \mathfrak{D}'(K_1, K_2)$ are both the identity matrix. The matrix $\mathfrak{D}'$ is thus equivalent to the matrix

$$\begin{pmatrix}
\begin{matrix} D_\infty^1 B_1^1 \otimes B_1^2 A_0^2 + \\ B_1^1 A_0^1 \otimes D_\infty^2 B_1^2 \end{matrix} & B_1^1 B_0^1 \otimes I & D_\infty^1 A_1^1 \otimes B_1^2 A_0^2 & I \otimes B_1^2 B_0^2 \\
I \otimes B_\infty^2 B_1^2 & 0 & 0 & B_0^1 B_\infty^1 \otimes I \\
B_\infty^1 B_1^1 \otimes I & I \otimes B_0^2 B_\infty^2 & B_\infty^1 A_1^1 \otimes I & X_1^1 B_\infty^1 \otimes B_0^2 X_1^2 \\
D_\infty^1 B_1^1 \otimes D_1^2 A_0^2 & 0 & \begin{matrix} I \otimes I + \\ D_\infty^1 A_1^1 \otimes D_1^2 A_0^2 \end{matrix} & I \otimes D_1^2 B_0^2
\end{pmatrix}. \tag{15}$$

Replacing the block forms for $\tau_\star(K_1)$ gives the following presentation of the above matrix:

$$\left(\begin{array}{cc|c|ccc|ccc}
\star & \star & I \otimes I & 0 & 0 & \star & I \otimes B_1^2 B_0^2 & 0 & 0 \\
\star & \star & 0 & 0 & 0 & \star & 0 & \star & 0 \\
\star & \star & 0 & 0 & 0 & \star & 0 & 0 & \star \\
\hline
\star & 0 & 0 & 0 & 0 & 0 & X_1^1 \otimes I & \star & \star \\
0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\star & \star & I \otimes B_0^2 B_\infty^2 & 0 & 0 & \star & 0 & \star & \star \\
\hline
\star & \star & 0 & I \otimes I & 0 & \star & \star & 0 & 0 \\
\star & \star & 0 & 0 & I \otimes I & \star & 0 & \star & 0 \\
\star & \star & 0 & 0 & 0 & \star & 0 & 0 & \star
\end{array}\right).$$

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After subtracting $I\otimes B_0^2B_\infty^2$ times the first row from the sixth row, the identity matrices which appear in the entries $(1,3),(7,4)$ and $(8,5)$ of the above matrix become the only non-zero entries of their respective columns. They may thus be used for the cancellation of the third, the fourth and the fifth columns against the first, the seventh and the eighth rows. We thus arrive at a 6×6 matrix equivalent to $\mathfrak{D}$, which is of the form

$$\begin{pmatrix}\star&\star&\star&0&\star&\star\\ \star&\star&\star&0&\star&\star\\ \star&\star&\star&X_1^1\otimes I&\star&\star\\ \star&\star&\star&0&\star&\star\\ \star&\star&\star&I\otimes B_0^2X_1^2&\star&\star\\ \star&\star&\star&0&\star&\star\end{pmatrix}.$$

Since the kernel of $\mathfrak{D}'$ includes a subspace which is isomorphic to the kernel corresponding to the fourth column, we find that $k(\mathfrak{D})=k(\mathfrak{D}')\geqslant k(X_1^1)k(B_0^2X_1^2)$.

For case (C), using Lemma 4.7, choose the standard block form of (14) for K_1. In particular, A_0^1,A_1^1 and D_∞^1 are all zero. The entries $(3,2)$ and $(5,4)$ of $\mathfrak{D}'$ are thus identity matrices, which may be used for cancellation. Add $B_\infty^1B_1^1\otimes B_0^2X_1^2$ times the second row of the resulting matrix to its third row, add $B_\infty^1B_1^1\otimes D_0^2X_1^2$ times the second row to the last row and note that $B_1^1D_1^1=0$ to arrive at the following matrix, which is equivalent to $\mathfrak{D}'$ and thus to $\mathfrak{D}$:

$$\begin{pmatrix}0 & B_1^1B_0^1\otimes I & I\otimes B_1^2B_0^2 & 0\\ I\otimes B_\infty^2B_1^2 & D_1^1B_0^1\otimes B_\infty^2A_1^2 & B_0^1B_\infty^1\otimes I & B_0^1A_\infty^1\otimes I\\ B_\infty^1B_1^1\otimes I & I\otimes B_0^2B_\infty^2 & D_0^1B_\infty^1\otimes B_0^2A_\infty^2 & D_0^1A_\infty^1\otimes B_0^2A_\infty^2\\ B_\infty^1B_1^1\otimes D_0^2X_1^2B_\infty^2B_1^2 & I\otimes D_0^2B_\infty^2 & D_0^1B_\infty^1\otimes D_0^2A_\infty^2 & \begin{matrix}I\otimes I+\\ D_0^1A_\infty^1\otimes D_0^2A_\infty^2\end{matrix}\end{pmatrix}. \tag{16}$$

Replacing the block forms of (14) for $\tau_0(K_1),\tau_1(K_1)$ and $\tau_\infty(K_1)$, we arrive at a matrix of the form

$$\left(\begin{array}{cc|ccc|c|ccc}0&0&0&0&I\otimes I&I\otimes B_1^2B_0^2&0&0&0\\ \hline \star&\star&\star&\star&0&\star&\star&\star&\star\\ \star&\star&\star&\star&0&\star&\star&\star&\star\\ \hline \star&\star&\star&\star&0&\star&\star&\star&\star\\ \star&\star&\star&\star&0&\star&\star&\star&\star\\ 0&X_0^1\otimes I&0&0&I\otimes B_0^2B_\infty^2&0&0&0&0\\ \hline \star&\star&\star&\star&0&\star&\star&\star&\star\\ \star&\star&\star&\star&0&\star&\star&\star&\star\\ \star&\star&\star&\star&0&\star&\star&\star&\star\end{array}\right),$$

which is in turn equivalent to a matrix of the form

$$\begin{pmatrix}\star&\star&\star&\star&\star&\star&\star&\star\\ \star&\star&\star&\star&\star&\star&\star&\star\\ \star&\star&\star&\star&\star&\star&\star&\star\\ \star&\star&\star&\star&\star&\star&\star&\star\\ 0&X_0^1\otimes I&0&0&I\otimes X_\infty^2B_0^2&0&0&0\\ \star&\star&\star&\star&\star&\star&\star&\star\\ \star&\star&\star&\star&\star&\star&\star&\star\\ \star&\star&\star&\star&\star&\star&\star&\star\end{pmatrix}.$$

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In particular, we conclude that $c(\mathfrak{D}) \geqslant c(X_0^1)c(X_\infty^2 B_0^2)$. This completes the proof of case (C) when $(\circ, \bullet, *) = (0, 1, \infty)$. $\square$

5.2 The special cases (S-1) and (S-2)

Lemma 5.5. *If (K_1, K_2) is a special pair of type (S-1) or (S-2), then one of the knots K_1 or K_2 is trivial.*

Proof. Suppose otherwise that (K_1, K_2) is a special pair of type (S-1) and that both K_1 and K_2 are non-trivial. After an admissible change of basis, assume that

$$\tau_0^1 = \left(\begin{array}{c|cc} 0 & I & 0 \\ \hline I & 0 & 0 \\ 0 & 0 & \star \end{array}\right), \quad \tau_1^1 = \left(\begin{array}{cc|c} 0 & 0 & I \\ 0 & \star & 0 \\ \hline I & 0 & 0 \end{array}\right) \quad \text{and} \quad \tau_\infty^1 = \left(\begin{array}{cc|cc} \star & \star & X_\infty^1 & \star \\ \star & \star & \star & \star \\ \hline \star & \star & \star & \star \\ \star & \star & \star & \star \end{array}\right). \tag{17}$$

In particular, A_0^1 and D_1^1 are zero. We may also assume that

$$\tau_0^2 = \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right), \quad \tau_1^2 = \left(\begin{array}{c|cc} \star & X_\infty^2 & \star \\ \hline \star & \star & \star \\ \star & \star & \star \end{array}\right) \quad \text{and} \quad \tau_\infty^2 = \left(\begin{array}{cc|c} 0 & 0 & I \\ 0 & \star & 0 \\ \hline I & 0 & 0 \end{array}\right). \tag{18}$$

In particular, A_0^2, D_0^2 and D_∞^2 are zero. The identity matrices which appear as entries $(3,2)$, $(5,4)$ and $(6,6)$ in $\mathfrak{D}'(K_1, K_2)$ may be used for cancellation to obtain the equivalent matrix

$$\begin{pmatrix} 0 & B_1^1 B_0^1 \otimes I & I \otimes B_1^2 B_0^2 \\ I \otimes B_\infty^2 B_1^2 & 0 & B_0^1 B_\infty^1 \otimes I \\ B_\infty^1 B_1^1 \otimes I & I \otimes B_0^2 B_\infty^2 & \begin{matrix} D_0^1 B_\infty^1 \otimes B_0^2 A_\infty^2 \\ +X_1^1 B_\infty^1 \otimes B_0^2 X_1^2 \\ +B_\infty^1 A_1^1 \otimes D_1^2 B_0^2 \end{matrix} \end{pmatrix}.$$

Subtracting $X_1^1 B_\infty^1 \otimes B_0^2 B_\infty^2$ times the first row from the third row, we arrive at the equivalent matrix

$$\begin{pmatrix} 0 & B_1^1 B_0^1 \otimes I & I \otimes B_1^2 B_0^2 \\ I \otimes B_\infty^2 B_1^2 & 0 & B_0^1 B_\infty^1 \otimes I \\ B_\infty^1 B_1^1 \otimes I & I \otimes B_0^2 B_\infty^2 & \begin{matrix} D_0^1 B_\infty^1 \otimes B_0^2 A_\infty^2 \\ +B_\infty^1 A_1^1 \otimes D_1^2 B_0^2 \end{matrix} \end{pmatrix}.$$

Replacing the block forms of (18) and (17), the above matrix takes the form

$$\left(\begin{array}{cc|cc|cccc} 0 & \star & I\otimes I & 0 & I\otimes X_\infty^2 & \star & \star & \star \\ 0 & \star & 0 & 0 & 0 & \star & \star & \star \\ \hline I\otimes X_\infty^2 & \star & 0 & 0 & X_\infty^1\otimes I & \star & \star & \star \\ 0 & \star & 0 & 0 & 0 & \star & \star & \star \\ \hline X_\infty^1\otimes I & \star & I\otimes I & 0 & 0 & \star & \star & \star \\ 0 & \star & 0 & 0 & 0 & \star & \star & \star \\ \star & \star & \star & I\otimes I & 0 & \star & \star & \star \\ 0 & \star & 0 & 0 & 0 & \star & \star & \star \end{array}\right).$$

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Subtract the first row from the fifth row and use the identity matrices which appear as $(1,3)$ and $(7,4)$ entries of the above matrix for cancellation to arrive at the following equivalent matrix:

$$\begin{pmatrix} 0 & \star & 0 & \star & \star & \star \\ I\otimes X^2_\infty & \star & X^1_\infty\otimes I & \star & \star & \star \\ 0 & \star & 0 & \star & \star & \star \\ X^1_\infty\otimes I & \star & I\otimes X^2_\infty & \star & \star & \star \\ 0 & \star & 0 & \star & \star & \star \\ 0 & \star & 0 & \star & \star & \star \end{pmatrix}.$$

From the above presentation, we conclude that

$$k(\mathfrak{D}) = k(\mathfrak{D}') \geqslant 2k(X^1_\infty)k(X^2_\infty) \geqslant 2.$$

This contradiction rules out the case (S-1). Ruling out the case (S-2) is similar. $\square$

6. Incompressible tori in homology spheres

6.1 The main theorem

Theorem 6.1. *Suppose that K_i is a non-trivial knot in the homology sphere Y_i for $i=1,2$. Let $Y=Y(K_1,K_2)$ denote the three-manifold obtained by splicing the complements of K_1 and K_2. Then the rank of $\widehat{\mathrm{HF}}(Y)$ is bigger than 1.*

Proof. Suppose otherwise that Y is an L-space. Thus, (K_1,K_2) is a special pair. By Proposition 5.3 and Lemma 5.5, we may assume that K_1 is full-rank. In particular, one of the cases (K) or (C) from Proposition 5.4 will happen. Note that in case (K) the kernel of $\mathfrak{D}$ is necessarily non-trivial by Lemma 4.6, while in case (C) the cokernel of $\mathfrak{D}$ is non-trivial.

Let us assume that (K) is the case. Thus, $c(\mathfrak{D})=0$ and

$$k(X^1_\bullet) = k(B^2_{\imath(*)}X^2_{\imath(\bullet)}) = 1.$$

Note that $\mathrm{Ker}(B^2_{\imath(*)}) \subset \mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)})$, which implies that either $B^2_{\imath(*)}$ is injective or we have $\mathrm{Ker}(B^2_{\imath(*)}) = \mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)})$. Let us first assume that the latter happens. It follows that

$$\mathrm{Ker}(B^2_{\imath(*)}) = \mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)}) = \mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)}X^2_{\imath(\bullet)}) = \mathrm{Ker}(0),$$

since $X^2_{\imath(\bullet)}X^2_{\imath(\bullet)} = 0$ by Lemma 4.6. Since $B^2_{\imath(*)} \neq 0$ for $*=0,1$, this cannot happen unless $*=\infty$ and $B^2_0=0$. By assumption, B^1_0 and B^1_1 are injective, while B^1_∞ is surjective, implying that $a^1_0 \geqslant a^1_\infty \geqslant a^1_1$. Since the parity of a^1_0 is different from the parity of a^1_1 and a^1_∞, we may further assume that $a^1_0 > a^1_\infty \geqslant a^1_1$. Furthermore, we may assume that $D^1_0 = D^1_1 = A^1_\infty = 0$. The kernel of $\mathfrak{D}'$ has a subspace, which corresponds to the fifth column of $\mathfrak{D}'$ and is isomorphic to $\mathrm{Ker}(B^1_0B^1_\infty \otimes I)$. The rank of this subspace is at least $(a^1_0 - a^1_\infty)a^2_1$. This implies that $a^2_1 = 1$ and, for some positive integer a,

$$a^1_0 - 1 = a^1_\infty = a^1_1 = a.$$

Since $a^1_\infty = a^1_1$, it follows that B^1_0 is invertible. Part (1) of Lemma 4.7 then implies that (after an admissible change of basis) we may assume that

$$\tau^1_0 = \left(\begin{array}{c|c} 0 & I \\ I & 0 \end{array}\right), \quad \tau^1_1 = \left(\begin{array}{cc|c} 0 & 0 & I \\ 0 & 1 & 0 \\ I & 0 & 0 \end{array}\right) \quad \text{and} \quad \tau^1_\infty = \left(\begin{array}{c|cc} 0 & X^1_1 & x \\ \star & d & \star \\ \star & d' & \star \end{array}\right) \tag{19}$$

for some $a \times 1$ matrix x, some $a \times a$ matrix d and some $1 \times a$ matrix d'. Proposition 5.4 then implies that

$$1 = k(\mathfrak{D}) \geqslant k(X_1^1)k(B_0^2 X_1^2) = k(X_1^1) \geqslant 1.$$

This means that X_1^1 is an $a \times a$ matrix of rank $a-1$, and there is a unique $1 \times a$ vector y with $yX_1^1 = 0$. Since B_∞^1 is full-rank, $yx \in \mathbb{F}$ cannot be 0, i.e. $yx = 1$. Note that the $(1, 2)$ entry of $(\tau_\infty^1)^2$ is zero, implying that

$$X_1^1 d + xd' = 0 \quad \Rightarrow \quad yX_1^1 d + yxd' = 0.$$

Since $yX_1^1 = 0$ and $yx = 1$, we conclude that $d' = 0$. We can now replace the block forms of (19) in the matrix of (15) (which is equivalent to $\mathfrak{D}'$) and obtain the matrix

$$\left(\begin{array}{c|c|cc|cc} \star & I \otimes I & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & \star & 0 & 0 \\ \hline \star & 0 & 0 & 0 & X_1^1 \otimes I & x \otimes I \\ \hline X_1^1 \otimes I & 0 & 0 & x \otimes I & 0 & 0 \\ \hline \star & 0 & I \otimes I & \star & 0 & 0 \\ 0 & 0 & 0 & \star_0 & 0 & 0 \end{array}\right).$$

If we add $\star_0 \cdot (y \otimes I)$ times the fourth row to the last row, the last row of the resulting equivalent matrix becomes zero. It follows that the cokernel of the above matrix has a subspace of size $a_\infty^2 > 0$. In particular, $c(\mathfrak{D}) > 0$. From this contradiction, we conclude that $B_{\imath(*)}^2$ is injective.

Let us first assume that B_0^1 is not invertible. Then $c_\bullet^1, c_\circ^1 \neq 0$. From the equalities $c_\bullet^1 c_{\imath(\bullet)}^2 = c_\circ^1 c_{\imath(\circ)}^2 = 0$, we conclude that $B_{\imath(\circ)}^2$ and $B_{\imath(\bullet)}^2$ are both surjective. Thus, K_2 is full-rank and, by part (C) of Proposition 5.4, $c(\mathfrak{D}'(K_2, K_1)) > 0$. Since $c(\mathfrak{D}'(K_2, K_1)) = c(\mathfrak{D}'(K_1, K_2))$, this is a contradiction, which implies that B_0^1 is invertible. Moreover, the argument implies that $0 \in \{\circ, \bullet\}$ and at least one of $c_{\imath(\circ)}^2$ and $c_{\imath(\bullet)}^2$ is trivial. It is easy to conclude from here that we are then either in case (S-1) or in case (S-2) of Proposition 5.3, which are both excluded by Lemma 5.5. The contradiction rules out case (K) of Proposition 5.4.

Now assume that (C) is the case. Thus, $k(\mathfrak{D}) = 0$ and

$$c(X_\circ^1) = c(X_{\imath(\circ)}^2 B_{\imath(*)}^2) = 1.$$

Note that $\mathrm{Coker}(B_{\imath(*)}^2)$ is a quotient of $\mathrm{Coker}(X_{\imath(\circ)}^2 B_{\imath(*)}^2)$, which implies that either $B_{\imath(*)}^2$ is surjective or $\mathrm{Coker}(B_{\imath(*)}^2) = \mathrm{Coker}(X_{\imath(\circ)}^2 B_{\imath(*)}^2)$. If the latter happens, similar to the previous case we find that $\mathrm{Coker}(B_{\imath(*)}^2) = \mathrm{Coker}(0)$, since $X_{\imath(\circ)}^2$ is nilpotent by Lemma 4.6. Since $B_{\imath(*)}^2 \neq 0$ for $* = 0, 1$, this cannot happen unless $* = \infty$ and $B_0^2 = 0$. Since B_0^1 and B_1^1 are surjective and B_∞^1 is injective, we may also assume that $A_0^1 = A_1^1 = D_\infty^1 = 0$. Furthermore, $a_0^1 < a_\infty^1 \leqslant a_1^1$. Then the only non-zero entry in the fourth row of $\mathfrak{D}'(K_1, K_2)$ is $B_\infty^1 B_1^1 \otimes I$, where the identity matrix is of size a_∞^2. Since B_∞^1 is a matrix of size $a_1^1 \times a_0^1$, we conclude that $1 = c(\mathfrak{D}') \geqslant (a_1^1 - a_0^1)a_\infty^2$, which means that $a_\infty^2 = 1$ and there is a positive integer a such that

$$a_0^1 = a_\infty^1 - 1 = a_1^1 - 1 = a.$$

In this case, we may apply part (2) of Lemma 4.7 to obtain the following block forms:

$$\tau_0^1 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \\ \hline 1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right), \quad \tau_1^1 = \left(\begin{array}{c|cc} 0 & 0 & I \\ \hline 0 & 1 & 0 \\ I & 0 & 0 \end{array}\right) \quad \text{and} \quad \tau_\infty^1 = \left(\begin{array}{cc|c} \star & \star & x \\ d' & d & X_0^1 \\ \hline \star & \star & 0 \end{array}\right) \tag{20}$$

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for some $1 \times a$ matrix x, some $a \times a$ matrix d and some $a \times 1$ matrix d'. The identity matrices appearing in the above block forms are all $a \times a$ matrices. Proposition 5.4 then implies that

$$1 = c(\mathfrak{D}) \geqslant c(X_0^1)c(X_\infty^2 B_0^2) = k(X_1^1) \geqslant 1.$$

This means that X_0^1 is an $a \times a$ matrix of rank $a-1$, and there is a unique $a \times 1$ vector y with $X_0^1 y = 0$. Since B_∞^1 is full-rank, $xy \in \mathbb{F}$ cannot be 0, i.e. $xy = 1$. Note that the $(2,3)$ entry of $(\tau_\infty^1)^2$ is zero, and we have

$$dX_0^1 + d'x = 0 \quad \Rightarrow \quad dX_0^1 y + d'xy = 0.$$

Since $X_0^1 y = 0$ and $xy = 1$, we conclude that $d' = 0$. We can now replace the block forms of (20) in the matrix of (16) (which is equivalent to $\mathfrak{D}'$) and obtain the matrix

$$\left(\begin{array}{cc|cc|c|cc}
0 & 0 & 0 & I \otimes I & 0 & 0 & 0 \\
\hline
B_\infty^2 B_1^2 & 0 & \star & 0 & x \otimes I & \star & \star \\
0 & \star & 0 & 0 & X_0^1 \otimes I & 0 & \star \\
\hline
0 & \star & 0 & 0 & 0 & 0 & 0 \\
0 & \star & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \star & 0 & 0 & \star & \star \\
0 & 0 & 0 & \star & 0 & \star & \star
\end{array}\right).$$

If we add $y \otimes B_\infty^2 B_1^2$ times the fifth column to the first column, the first column becomes zero, and we thus obtain a subspace of the kernel of the above matrix (and of the kernel of $\mathfrak{D}'$) which is of size $a_\infty^2 > 0$. In particular, $k(\mathfrak{D}) > 0$. From this contradiction, we conclude that $B_{\imath(*)}^2$ is injective.

Again, let us first assume that B_0^1 is not invertible. Then $k_\bullet^1, k_\circ^1 \neq 0$. Since $k_\bullet^1 k_{\imath(\bullet)}^2 = k_\circ^1 k_{\imath(\circ)}^2 = 0$, we conclude that $B_{\imath(\circ)}^2$ and $B_{\imath(\bullet)}^2$ are both injective. Thus, K_2 is full-rank and, by part (K) of Proposition 5.4, $k(\mathfrak{D}'(K_2, K_1)) > 0$. Since $k(\mathfrak{D}'(K_2, K_1)) = k(\mathfrak{D}'(K_1, K_2))$, this is a contradiction, which implies that B_0^1 is invertible. Moreover, the argument implies that $0 \in \{\circ, \bullet\}$ and at least one of $k_{\imath(\circ)}^2$ and $k_{\imath(\bullet)}^2$ is trivial. Again, it is implied that we are either in case (S-1) or in case (S-2) of Proposition 5.3, which are both excluded by Lemma 5.5. The contradiction rules out case (C) of Proposition 5.4. $\square$

Corollary 6.2. *If the homology sphere Y contains an incompressible torus, then*

$$\mathrm{rnk}(\widehat{\mathrm{HF}}(Y, \mathbb{F})) > 1.$$

Proof. If Y contains an incompressible torus T, T will be separating and there will be a pair of curves λ and μ on T such that λ is homologically trivial on one side of T and μ is homologically trivial on the other side of T. Since Y is a homology sphere, the intersection number of μ and λ is 1. Let U_1 and U_2 be the two components of $Y - T$ and let U_1 be the component containing a surface which bounds λ. Capping off $\mu \subset T = \partial U_1$ by a disc and then gluing a three-ball gives a three-manifold Y_1. The simple closed curve λ represents a knot $K_1 \subset Y_1$. Similarly, capping off $\lambda \subset T = \partial U_2$ by a disc and then gluing a three-ball gives a three-manifold Y_2 and μ represents a knot $K_2 \subset Y_2$. Both Y_1 and Y_2 are homology spheres and Y is obtained by splicing K_1 and K_2. Since T is incompressible, both K_1 and K_2 are non-trivial and Theorem 6.1 completes the proof of this corollary. $\square$

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6.2 Applications

We may use the relation between Khovanov homology of a knot inside the standard sphere and the Heegaard Floer homology of its branched double-cover, discovered by Ozsváth and Szabó [OS05], to show the non-triviality of Khovanov homology for certain classes of knots. We emphasize again that the results presented here are all special cases of the theorem of Kronheimer and Mrowka [KM11] that Khovanov homology is an unknot detector.

Definition 6.3. A prime knot $K \subset S^3$ is an *n-string composite* if there is an embedded 2-sphere intersecting the knot transversely which separates (S^3, K) into prime n-string tangles. A 2-string composite knot is called a *doubly composite knot.*

We refer the reader to [Ble84] for more on doubly composite and doubly prime knots, and only quote the following lemma from that paper.

Lemma 6.4. *A prime knot $K \subset S^3$ is a doubly composite knot if and only if the double cover $\Sigma(K)$ of S^3 branched over the knot K contains an incompressible torus T which is invariant under the non-trivial covering translation and meets the fixed point set of this map precisely in four points, and separates $\Sigma(K)$ into irreducible boundary irreducible pieces.*

Corollary 6.5. *If the prime knot $K \subset S^3$ is doubly composite, the rank of its reduced Khovanov homology group $\widetilde{\mathrm{Kh}}(K)$ is bigger than* 1.

Proof. If K is doubly composite, by Lemma 6.4 there exists an incompressible torus T inside the three-manifold $\Sigma(K)$. Thus, the rank of $\widehat{\mathrm{HF}}(\Sigma(K), \mathbb{F})$ is bigger than 1. By the main theorem of [OS05], there is a spectral sequence whose E^2-term consists of Khovanov's reduced homology $\widetilde{\mathrm{Kh}}(K)$ of the mirror of K with coefficients in $\mathbb{F}$ which converges to $\widehat{\mathrm{HF}}(\Sigma(K), \mathbb{F})$, and is of rank greater than 1 by Theorem 6.1. Thus, the rank of $\widetilde{\mathrm{Kh}}(K)$ is bigger than 1 as well. □

Furthermore, if K is a prime satellite knot, we will have an incompressible torus in the complement of K. This torus gives an incompressible torus in the double cover $\Sigma(K)$ of S^3 branched over the knot K. Thus, Heegaard Floer homology of $\Sigma(K)$ will be non-trivial. We thus have the following corollary.

Corollary 6.6. *If $K \subset S^3$ is a prime satellite knot, the rank of its reduced Khovanov homology group $\widetilde{\mathrm{Kh}}(K)$ is greater than* 1.

In fact, we may prove a slightly more general statement.

Proposition 6.7. *If the rank of the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ of a non-trivial knot $K \subset S^3$ is* 1*, the double cover $\Sigma(K)$ of S^3, branched over the knot K, is hyperbolic.*

Proof. Note that if a knot K is doubly composite Corollary 6.5 implied that the rank of $\widetilde{\mathrm{Kh}}(K)$ is bigger than 1. Thus, K has to be doubly prime. By Thurston's orbifold geometrization theorem (see [BP01] and [CHK00]), the branched double cover $\Sigma(K)$ is a geometric manifold and there are three possible cases.

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(1) $\Sigma(K)$ is a Lens space and thus admits a spherical structure. If $\widehat{\mathrm{HF}}(\Sigma(K))$ is one dimensional, $\Sigma(K)$ is forced to be the standard sphere and K is trivial. Thus, in this case, the rank of $\widetilde{\mathrm{Kh}}(K)$ is bigger than 1 only if K is trivial.

(2) $\Sigma(K)$ admits a Seifert fibration and K is a Montesinos knot with at most three rational tangles. If $\Sigma(K)$ is not a homology sphere, $\widetilde{\mathrm{Kh}}(K)$ is clearly different from $\mathbb{F}$ and, if it is a homology sphere which admits a Seifert fibration and $\widehat{\mathrm{HF}}(\Sigma(K)) = \mathbb{F}$, we know (see [Rus04] or [Eft09]) that $\Sigma(K)$ is either the standard sphere or the Poincaré sphere. Moreover, for $\Sigma(K)$ to be the Poincaré sphere we should have $K = T(3,5)$, i.e. K is the $(3,5)$-torus knot or equivalently the $(-2,3,5)$-pretzel knot, which is 10_{124} in Rolfsen's table (see [HW10] and [Rol76]). The homology $\widetilde{\mathrm{Kh}}(T(3,5))$ has rank 7 by direct computation [Shu].

(3) $\Sigma(K)$ admits a hyperbolic structure which is invariant under the deck transformation. Having ruled out the first two possibilities, the proof is complete. □

The knots K with the property that $\Sigma(K)$ admits a hyperbolic structure which is invariant under the involution of $\Sigma(K)$ are called *π-hyperbolic*. The hyperbolic structure comes from a hyperbolic structure on $S^3 - K$, which becomes a singular folding with angle π around K. Thus, in particular, π-hyperbolic knots are hyperbolic.

Suppose that K is not the unknot. By Proposition 6.7, if $\widetilde{\mathrm{Kh}}(K) = \mathbb{F}$, the branched double cover $\Sigma(K)$ is hyperbolic. Conjecture 1.2 then implies that $\widehat{\mathrm{HF}}(\Sigma(K))$ is non-trivial and, by the correspondence of [OS05],

$$1 = \mathrm{rnk}(\widetilde{\mathrm{Kh}}(K)) \geqslant \mathrm{rnk}(\widehat{\mathrm{HF}}(\Sigma(K))) > 1.$$

In particular, if Conjecture 1.2 is true then for every non-trivial knot K the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ is non-trivial (i.e. different from $\mathbb{F}$).

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