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# Trifactorisable groups

## Elizabeth Pennington

The group $G$ is called trifactorisable if $G$ has three subgroups, $A$, $B$, and $C$ such that $G = AB = BC = CA$. Obviously the structure of the group $G$ will be restricted by the structure of these subgroups. In this paper it will be shown that a finite group $G$ is $\pi$-separable if and only if it satisfies $D_\pi$ and has a trifactorisation with two factors $\pi$-closed and the third, $C$ say, $\pi$-separable. In this case we show that the $\pi$- and $\pi'$-lengths of $G$ can be at most one more than those of $C$, and so it is this factor which "controls" the structure of $G$. Similar results are proved for $\pi$-solubility and solubility.

Kegel in his paper [3] introduced the notion of trifactorisable groups as groups $G$ having subgroups $A$, $B$ and $C$ such that $G = AB = BC = CA$. Obviously the structure of the group $G$ will be restricted by the structure of these subgroups. In this paper it will be shown that a finite group $G$ satisfying $D_\pi$ is $\pi$-separable if and only if it has such a trifactorisation in which two factors are $\pi$-closed and the third factor, $C$ say, is $\pi$-separable. In this case we show that the $\pi$- and the $\pi'$-lengths of $G$ can be at most one more than that of $C$, and so it is this factor which controls the structure of $G$.

This work rests heavily on the following result due to Wielandt [4], 15.7, p. 70.

*If $A$ and $B$ are subgroups of the finite group $G$ such that*

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$AB^g = B^gA$ *for all element* $g$ *of* $G$ *then*

$$[A, B] \triangleleft A^{AB} \cap B^{AB} \triangleleft\triangleleft G .$$

Throughout this paper all groups considered will be finite. $\pi$ will denote a set of primes and $\pi'$ its complement. $G_\pi$ denotes a Hall $\pi$-subgroup of $G$ . The group $G$ is $\pi$-closed if the $\pi$-elements of $G$ generate a normal $\pi$-group. The group $G$ satisfies $E_\pi$ if $G$ has at least one Hall $\pi$-subgroup; it satisfies $D_\pi$ if there exists precisely one conjugacy class of Hall $\pi$-subgroups and if every $\pi$-subgroup of $G$ is contained in a Hall $\pi$-subgroup of $G$ . $O_\pi(G)$ denotes the largest normal $\pi$-subgroup of $G$ .

For any group $G$ the upper $\pi$-series is formed as follows:

$$P_1 = O_{\pi'}(G) , \quad P_i = O_\sigma^-(G/P_{i-1}) ,$$

where

$$\sigma = \begin{cases} \pi & \text{if } i \text{ is even,} \\ \pi' & \text{if } i \text{ is odd,} \end{cases}$$

and $O^-(G/H)$ denotes the inverse image in $G$ of the group $O_\sigma(G/H)$ with respect to the projection $G \to G/H$ . The terms of the upper $\pi'$-series will be denoted by $Q_i$ .

$G$ is $\pi$-separable if the upper $\pi$-series reaches $G$ , in which case the $\pi$-length of $G$ , denoted $l_\pi(G)$ , is the number of non-trivial $\pi$-factors in the series.

$G$ is $\pi$-soluble if and only if it is $p$-separable for all primes $p$ in $\pi$ .

Wielandt's result is used to prove part (2) of the following lemma.

LEMMA. *Let* $G = AB$ *be a finite group satisfying* $D_\pi$ *and suppose both* $A$ *and* $B$ *satisfy* $E_\pi$ . *Then:*

(1) *there exist Hall* $\pi$-*subgroups of* $A$ *and* $B$ *such that*

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$A_\pi B_\pi = B_\pi A_\pi$ *is a Hall* $\pi$*-subgroup of* $G$ ;

(2) *if* $A$ *and* $B$ *are* $\pi$*-closed then* $[A_\pi, B_\pi] \subseteq O_\pi(G)$ ;

(3) *if there is a subgroup* $C$ *satisfying* $E_\pi$ *such that* $G = AB = BC = CA$ *then there is a* $g \in G$ *and Hall* $\pi$*-subgroups of* $A$, $B$ *and* $C$ *such that* $A_\pi B_\pi = B_\pi C_\pi^g = C_\pi^g A_\pi$ *is a Hall* $\pi$*-subgroup of* $G$ *and* $G = AB = BC^g = C^g A$ .

Proof. (1) See [2], VI.4.6, p. 676.

(2) $A_\pi B_\pi = B_\pi A_\pi$ by (1). Take any $g = ba \in G$ ; then

$$A_\pi B_\pi^g = A_\pi B_\pi^{ba} = (A_\pi B_\pi)^a = (B_\pi A_\pi)^a = B_\pi^g A_\pi .$$

So Wielandt's Theorem applies and $[A_\pi, B_\pi] \lhd\lhd G$ . But $[A_\pi, B_\pi]$ is a $\pi$-group and so is contained in $O_\pi(G)$ , as the normal closure of a subnormal $\pi$-group is a $\pi$-group.

(3) For any $g = ca \in G$ , $AC^g = (AC)^a = G^a = G$ and so $G = AB = BC^g = C^g A$ for all $g \in G$ . By (1), $A_\pi B_\pi = G_\pi$ . But $C_\pi^g \subseteq G_\pi$ for some $C_\pi$ and some $g \in G$ , so $A_\pi C_\pi^g \subseteq G_\pi$ , and comparing orders gives equality. Similarily, $B_\pi C_\pi^g = G_\pi$ .

Any $\pi$-separable group has a trifactorisation $G = AB = BC = CA$ with two factors, $A$ and $B$ say, $\pi$-closed and the third $\pi$-separable. Take $A$ to be a Hall $\pi$-subgroup of $G$ , $B$ a Hall $\pi'$-subgroup of $G$ and $C = G$ . The following theorem shows that the converse is also true.

THEOREM. *Let* $G = AB = BC = CA$ *be a finite group satisfying* $D_\pi$ *with* $A$ *and* $B$ $\pi$*-closed subgroups and* $C$ $\pi$*-separable. Then* $G$ *is* $\pi$*-separable and* $O_\pi(C) \subseteq O_\pi(G)$ *and* $O_{\pi'}(C) \subseteq O_{\pi,\pi'}(G)$ .

Now assume if possible that there are trifactorised groups with the above structure not satisfying the theorem, and let $G$ be such a group having minimal order.

Suppose $O_\pi(G) \neq \langle 1 \rangle$ . Let denote the natural homomorphism $G \to G/O_\pi(G) = \bar{G}$ . The group $\bar{G}$ has the same structure as $G$ but is of smaller order, hence $\bar{G}$ is $\pi$-separable. But $\bar{G} = G/O_\pi(G)$ and $O_\pi(G)$ are $\pi$-separable, so $G$ is $\pi$-separable. But

$$O_\pi(C) \subseteq O_\pi^-(\bar{C}) \subseteq O_\pi^-(\bar{G}) = O_\pi(G)$$

and

$$O_{\pi'}(C) \subseteq O_{\pi'}^-(\bar{C}) \subseteq O_{\pi,\pi'}^-(\bar{G}) = O_{\pi,\pi'}(G)$$

contradicting the choice of $G$ . So $O_\pi(G) = \langle 1 \rangle$ .

Suppose now that $G$ has two distinct minimal normal subgroups, $M$ and $N$ . Both factor groups $G/M$ and $G/N$ are $\pi$-separable by the minimality of $G$ and hence so is their direct product. $G$ embeds into $G/M \times G/N$ , and so $G$ is $\pi$-separable. Further one has

$$O_\pi(C) \subseteq O_\pi^-(G/M) \cap O_\pi^-(G/N) = X \lhd G$$

and

$$O_{\pi'}(C) \subseteq O_{\pi,\pi'}^-(G/M) \cap O_{\pi,\pi'}^-(G/N) = Y \lhd G .$$

Let $Y_\pi$ be a Hall $\pi$-subgroup of $Y$ , then $Y_\pi \subseteq O_{\pi,\pi'}^-(G/M)$ and so $Y_\pi \subseteq O_\pi^-(G/M)$ . Similarly, $Y_\pi \subseteq O_\pi^-(G/N)$ and so $Y_\pi \subseteq X$ . But

$$[XM, O_{\pi'}(G)] \subseteq M \text{ and } [XN, O_{\pi'}(G)] \subseteq N ,$$

so $[X, O_{\pi'}(G)] \subseteq M \cap N = \langle 1 \rangle$ and $X$ centralises $O_{\pi'}(G)$ . But $G$ is $\pi$-separable and $O_\pi(G) = \langle 1 \rangle$ , so $O_{\pi'}(G)$ contains its centraliser (see [1], 6.3.2, p. 228), that is, $Y_\pi \subseteq X \subseteq O_{\pi'}(G)$ . Since $O_{\pi'}(G)$ does not contain any non-trivial $\pi$-subgroups, one gets $O_\pi(C) = \langle 1 \rangle = Y_\pi$ and $O_{\pi'}(C) \subseteq Y = Y_{\pi'} \subseteq O_{\pi'}(G)$ . This contradicts the choice of $G$ . Thus the group has a unique minimal normal subgroup $M$ .

Suppose that $A_\pi = \langle 1 \rangle$ . Then by the lemma, $G_\pi = B_\pi = C_\pi$ and so $G_\pi^G = B_\pi^{BC} = C_\pi^C \subseteq C$ , and $C$ contains a non-trivial normal subgroup of $G$ , that is, $C \supseteq M$ . Thus $M$ is $\pi$-separable and so is $G$ . Also one has

$$O_\pi(C) \subseteq O_\pi^-(C/M) \subseteq O_\pi^-(G/M) \subseteq C \subseteq N_G(O_\pi(C)) \ .$$

Thus

$$O_\pi(C) \triangleleft O_\pi^-(G/M) \triangleleft G \text{ and } O_\pi(C) \subseteq O_\pi(G) = \langle 1 \rangle \ .$$

As $G$ is $\pi$-separable, one gets $G = G_\pi G_{\pi'}$ , and since $G_\pi \subseteq C \subseteq N_G(O_{\pi'}(C))$ , one has

$$O_{\pi'}(C)^G = O_{\pi'}(C)^{G_{\pi'}} \subseteq G_{\pi'} \ ,$$

and so $O_{\pi'}(C) \subseteq O_{\pi'}(G)$ . This is again a contradiction and so $A_\pi \neq \langle 1 \rangle$ . Similarly, $B_\pi \neq \langle 1 \rangle$ .

By the lemma, $[A_\pi, B_\pi] \subseteq O_\pi(G) = \langle 1 \rangle$ . Let $g = ab \in G$ , then

$$\left[A_\pi^g, B_\pi\right] = \left[A_\pi^{ab}, B_\pi\right] = [A_\pi, B_\pi]^b = \langle 1 \rangle \ .$$

All conjugates commute, so $\left[A_\pi^G, B_\pi^G\right] = \langle 1 \rangle$ . But $A_\pi^G$ and $B_\pi^G$ are both non-trivial normal subgroups of $G$ and so contain $M$ , so $M$ is an abelian $\pi'$-group and centralises every Hall $\pi$-subgroup of $G$ . Further, $G$ is $\pi$-separable. Now $O_\pi(C) \subseteq O_\pi^-(G/M) = PM$ where $P$ is a Hall $\pi$-subgroup of $O_\pi^-(G/M)$ and $P \triangleleft PM \triangleleft G$ , so $P = \langle 1 \rangle = O_\pi(C)$ and $O_\pi(G/M)$ is trivial. But

$$O_{\pi'}(C) \subseteq O_{\pi,\pi'}^-(G/M) = O_{\pi'}^-(G/M) \subseteq O_{\pi'}(G) \ .$$

This final contradiction completes the proof.

This theorem has as an obvious corollary, Satz 1 of [3].

**COROLLARY 1.** *Let $G$ be a finite group satisfying $D_\pi$ . Then $G$ is $\pi$-closed if and only if there are subgroups, $A$, $B$ , and $C$ of $G$ , all $\pi$-closed and satisfying $G = AB = BC = CA$ .*

That, in the situation of the theorem, $O_{\pi'}(C)$ is not necessarily contained in $O_{\pi'}(G)$ is shown in the following example.

Let $G$ be any $\pi$-separable group with $O_{\pi'}(G) = \langle 1 \rangle$. Take as $A$ a Hall $\pi$-subgroup of $G$, as $B = O_{\pi}(G)G_{\pi'}$, where $G_{\pi'}$ is any Hall $\pi'$-subgroup of $G$ and as $C$ any compliment to $O_{\pi}(G)$ in $G$. Then $A$ and $B$ are $\pi$-closed but $O_{\pi'}(C) \nsubseteq O_{\pi'}(G) = \langle 1 \rangle$. This example shows that the theorem cannot be extended to characterise $\pi$-soluble groups (even if $C$ is $\pi$-soluble) without added restrictions, for take $O_{\pi}(G)$ non-soluble in the above. Further, in the example, the $\pi$-length of $G$ is greater by one than that of $C$. That this is the maximal difference possible is shown by the following corollary.

COROLLARY 2. *Let $G$ be a finite $\pi$-separable group with subgroups $A$, $B$, and $C$ of $G$ such that $G = AB = BC = CA$ with $A$ and $B$ $\pi$-closed; then:*

(1) *if $O_{\pi}(G) = \langle 1 \rangle$, then $P_i(C) = P_i(G) \cap C$ for any natural number $i$, and $l_{\pi}(C) = l_{\pi}(G)$. Anyway, $l_{\pi}(G) \le l_{\pi}(C) + 1$;*

(2) *$Q_i(C) = Q_i(G) \cap C$ for all natural numbers $i$ and $l_{\pi'}(G) \le l_{\pi'}(C) + 1$;*

(3) *if $A$ and $B$ are also $\pi'$-closed, then $l_{\pi}(C) = l_{\pi}(G)$ and $l_{\pi'}(C) = l_{\pi'}(G)$.*

Proof. (1) Suppose $O_{\pi}(G) = \langle 1 \rangle$. By the theorem, one has

$$P_1(C) = O_{\pi'}(C) \subseteq O_{\pi,\pi'}(G) = O_{\pi'}(G) = P_1(G) .$$

Obviously, $P_1(G) \cap C \subseteq P_1(C)$ and so $P_1(C) = P_1(G) \cap C$.

Assume now that $P_i(G) \cap C = P_i(C)$ for all $i \le n$. Let $\sigma = \pi$ if $i$ is even and $\sigma = \pi'$ if $i$ is odd. Then

$$\frac{\left(P_{n+1}(G)\cap C\right)P_n(C)}{P_n(C)} = \frac{P_{n+1}(G)\cap C}{\left(P_{n+1}(G)\cap C\right)\cap P_n(G)} \cong \frac{\left(P_{n+1}(G)\cap C\right)P_n(G)}{P_n(G)} \subseteq \frac{P_{n+1}(G)}{P_n(G)} ,$$

which is a $\sigma$-group. So $\dfrac{\left(P_{n+1}(G)\cap C\right)P_n(C)}{P_n(C)}$ is a normal $\sigma$-subgroup of

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$C/P_n(C)$ and so $P_{n+1}(G) \cap C \subseteq P_{n+1}(C)$ . But

$$P_{n+1}(C) = O_\sigma^-\big(C/P_n(C)\big) = O_\sigma^-\big(C/P_n(G)\cap C\big) = O_\sigma^-\big(CP_n(G)/P_n(G)\big)$$
$$\subseteq O_\sigma^-\big(GP_n(G)/P_n(G)\big) = P_{n+1}(G) ,$$

by the theorem and discussion above. So $P_i(G) \cap C = P_i(C)$ for all natural numbers $i$ ..

Suppose now $l_\pi(C) = k$ , then $C = P_{2k+1}(C) = P_{2k+1}(G) \cap C$ . Let $^-$ denote the homomorphism $G \to G/P_{2k-1}(G)$ . Then

$$\overline{C} = \frac{CP_{2k-1}(G)}{P_{2k-1}(G)} \cong \frac{C}{P_{2k-1}(G)\cap C} = \frac{C}{P_{2k-1}(C)}$$

and this group is $\pi$-closed. So $\overline{G} = \overline{A}\overline{B} = \overline{B}\overline{C} = \overline{C}\overline{A}$ is $\pi$-closed by Corollary 1, and

$$l_\pi(G) = 1 + l_\pi\big(P_{2k-1}(G)\big) = 1 + (k-1) = k = l_\pi(C) .$$

If $O_\pi(G) \neq \langle 1 \rangle$ , consider $G/O_{\pi',\pi}(G)$ . Then $O_\pi\big(G/O_{\pi',\pi}(G)\big) = \langle 1 \rangle$ , and so

$$l_\pi(G) = l_\pi\big(G/O_{\pi',\pi}(G)\big) + 1 = l_\pi\big(CO_{\pi',\pi}(G)/O_{\pi',\pi}(G)\big) + 1 \le l_\pi(C) + 1 .$$

(2) By the analogous argument for the $\pi'$-series, one has $Q_i(G) \cap C = Q_i(C)$ for all natural numbers $i$ .

Suppose now $l_{\pi'}(C) = j$ ; then $C = Q_{2j+1}(C) = Q_{2j+1}(G) \cap C$ . Let $^-$ denote the homomorphism $G \to G/Q_{2j}(G)$ . Then $\overline{C} = C/Q_{2j}(C)$ is $\pi$-closed and hence so is $\overline{G}$ . So

$$l_{\pi'}(G) \le 1 + l_{\pi'}\big(Q_{2j}(G)\big) = 1 + j = l_{\pi'}(C) + 1 .$$

(3) If $A$ and $B$ are $\pi'$-closed then $C = Q_{2j+1}(C)$ and $C/Q_{2j-1}(C)$ is a $\pi$-closed group. So as above, $G/Q_{2j-1}(G)$ is $\pi'$-closed and $G/Q_{2j}(G)$ is a $\pi'$-group, and $l_{\pi'}(G) = j = l_{\pi'}(C)$ .

Similarly, $l_\pi(G) = l_\pi(C)$ .

For any group $G$ , the Fitting subgroup $F(G)$ is the largest normal nilpotent subgroup of $G$. The Fitting series of $G$ is defined by

$$F_1(G) = F(G) , \quad F_i(G) = F^{-}\left(G/F_{i-1}(G)\right) \text{ for } i > 1 .$$

If $G$ is soluble, and $n$ is the least integer such that $G = F_n(G)$, then $n$ is called the Fitting length of $G$.

COROLLARY 3. (1) *Let the finite group $G$ be of the form $G = AB = BC = CA$ where $A$ and $B$ are $\pi$-closed with nilpotent Hall $\pi$-subgroups and $C$ is $\pi$-soluble; then $G$ is $\pi$-soluble.*

(2) *If $A$ and $B$ are nilpotent and $C$ is soluble, then $G$ is soluble and $F_i(C) = F_i(G) \cap C$. Further the Fitting length of $G$ is equal to that of $C$.*

Proof. (1) Let $p$ be any prime in $\pi$. Then $A$ and $B$ are both $p$-closed and $C$ is $p$-separable. Hence, by the theorem, $G$ is $p$-separable. As this is true for all primes $p \in \pi$ the group $G$ is $\pi$-soluble.

(2) The solubility of $G$ follows from (1).

Let $p$ be any prime dividing the order of $G$; then $A$ and $B$ are $p$-closed and by the theorem, $O_p(C) \subseteq O_p(G)$. Thus $F_1(C) \subseteq F_1(G)$. But obviously $F_1(G) \cap C \subseteq F_1(C)$ and so $F_1(C) = F_1(G) \cap C$. Suppose now that $F_i(C) = F_i(G) \cap C$ for all natural numbers $i \leq n$. Then

$$F_{n+1}(C) = F^{-}\left(C/F_n(C)\right) = F^{-}\left(C/F_n(G) \cap C\right) \cong F^{-}\left(CF_n(G)/F_n(G)\right)$$
$$\subseteq F^{-}\left(GF_n(G)/F_n(G)\right) = F_{n+1}(G) .$$

But

$$\frac{\left(F_{n+1}(G) \cap C\right)F_n(C)}{F_n(C)} \cong \frac{\left(F_{n+1}(G) \cap C\right)F_n(G)}{F_n(G)} \subseteq \frac{F_{n+1}(G)}{F_n(G)} ,$$

which is a nilpotent group. So $F_{n+1}(G) \cap C \subseteq F_{n+1}(C) \subseteq F_{n+1}(G) \cap C$.

Let the Fitting length of $C$ be $k$. Then $F_k(C) = C = F_k(G) \cap C$. Let $^{-}$ denote the homomorphism $G \to G/F_{k-1}(G)$. Then $\overline{C} = CF_{k-1}(G)/F_{k-1}(G) = C/F_{k-1}(C)$ is nilpotent, so $\overline{G}$ is nilpotent by

Corollary 1. Then $G = F_k(G)$ , and the Fitting length of $G$ is equal to the Fitting length of $C$ .

It would be convenient if the two parts of this corollary could be combined into a single statement involving some form of series for the $\pi$-soluble case, say defining the $\pi$-Fitting subgroup to be $F_\pi(G) = O_{\pi'}(G) \times F\left(O_\pi(G)\right)$ , which reduced to the Fitting series in the soluble case. It seems however as if this is impossible as the hypothesis of the first part of the corollary is not strong enough to ensure that $O_{\pi'}(G)$ contains $O_{\pi'}(C)$ . This is shown in the above example by taking $\pi$ to be the single prime $p$ .

Finally, it may be pointed out that the arguments presented here shorten the arguments in [3] considerably.

## References

[1] Daniel Gorenstein, *Finite groups* (Harper and Row, New York, Evanston, London, 1968).

[2] B. Huppert, *Endliche Gruppen I* (Die Grundlehren der mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[3] Otto H. Kegel, "Zur Struktur mehrfach faktorisierter endlicher Gruppen", *Math. Z.* 87 (1965), 42-48.

[4] Helmut Wielandt, *Topics in the theory of composite groups* (Lecture Notes, Mathematics Department, University of Wisconsin, Madison, 1967).

Department of Pure Mathematics,
Queen Mary College,
University of London,
London,
England.

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