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A CLASS OF REGULARITIES FOR RINGS

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Abstract

In this paper a general concept of regularity for rings is defined. It is shown that every regularity determines in a natural way a subradical and a radical for rings. A wide class of regularities is constructed: the class of polynomial regularities. All well-known regularities, such as the Perlis–Jacobson regularity, the von Neumann regularity and many others, belong to this class. Special attention is paid to regularities which are elementary in the sense that the so-called unic and nullic polynomial regularities can be thought of as intersections of the elementary ones.

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1. Introduction

In ring theory many so-called regularities appear. The oldest one seems to be the von Neumann regularity. Von Neumann (1936) defined a ring R (with identity) to be regular if for each element r of R there exists an element s of R such that $r = rsr$. Brown and McCoy (1950) generalized to rings without identity, and they succeeded in proving that every ring contains a greatest regular ideal. In the meantime Perlis (1942) had introduced the concept of quasi-regularity for algebras with identity. He defined an algebra A with identity to be quasi-regular if for every element r of A there exists an element s of R such that $r+s+rs = 0$. Jacobson (1945) generalized this concept to arbitrary rings without identity and he showed that every ring R contains a greatest quasi-regular ideal, called later on the Jacobson radical of R. In Section 2.5 many more examples of 'classical' regularities are given. In 1950 Brown and McCoy attempted to define a general concept of regularity for rings. At that time their theory was general enough. All regularities introduced up to then were regularities in the sense of it. However, in 1971 a wide

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class of regularities was introduced by McKnight and others, called (p,q)-regularities, and it was noted by Goulding and Ortiz (1971) that some of these (p,q)-regularities fail to satisfy the set of axioms of the theory of Brown and McCoy.

After these introductory historical remarks we now come to a brief description of the substance of this paper. In Chapter 2 we introduce a general concept of regularities for rings. It is shown that each regularity determines a subradical and a radical for rings. Eventually the subradical and the radical determined by a regularity may coincide; a condition under which this happens is given. Furthermore, an appropriate notion of equivalence for regularities is defined. At the end of Chapter 2 we list all well-known regularities and we establish that each of them is a regularity in the sense of our theory. In Chapter 3 the so-called polynomial regularities are introduced, and it appears that all well-known regularities are polynomial regularities. It is shown that under certain conditions a polynomial regularity may be regarded as the intersection of some 'elementary' regularities and a monomial regularity. Finally, the elementary and monomial regularities are investigated.

2. A general type of regularity for rings

2.1. Definitions

Let there be assigned to each ring R a mapping $F_R\colon R \to S(R,+)$, where $S(R,+)$ denotes the set consisting of all subgroups of the additive group $(R,+)$ of R. The class $\mathscr{F}$, consisting of all mappings F_R, will be called a *regularity for rings* if the following three conditions are satisfied:

(C1) *if* $\alpha\colon R \to S$ *is a ring epimorphism and* $r \in R$, *then* $F_S(\alpha r) = \alpha F_R(r)$;

(C2) *if* A *is an ideal of* R *and* $a \in A$, *then* $F_A(a) \subset F_R(a)$;

(C3) *if* $r, s \in R$ *and* $s \in F_R(r)$, *then* $F_R(r+s) \subset F_R(r)$.

If $\mathscr{F}$ is a regularity, an element r of R will be called F_R*-regular* if $r \in F_R(r)$. R will be called F*-regular* if each element of R is F_R-regular. The class of all F-regular rings will be denoted by $\mathbf{F}$. An ideal A of R will be called F_R-regular if each element of A is F_R-regular, and F-regular if each element of A is F_A-regular, that is if $A \in \mathbf{F}$. From (C2) it follows that each F-regular ideal of R is F_R-regular.

2.2. The radical and the subradical determined by a regularity

Throughout this section the symbol $\mathscr{F}$ will be used for an arbitrary, but fixed regularity for rings.

LEMMA 1. *If* $r, s \in R$, *then* $F_R(r+s) \subset F_R(r) + \langle s \rangle$, *where* $\langle s \rangle$ *denotes the ideal of* R *generated be the element* s.

PROOF. Define $S = R/\langle s\rangle$ and let α denote the canonical ring epimorphism of R onto S. Then $\alpha(r+s) = \alpha r$. Hence $F_S[\alpha(r+s)] = F_S(\alpha r)$. Using (C1) we obtain $\alpha F_R(r+s) = \alpha F_R(r)$, whence $F_R(r+s) \subset F_R(r) + \langle s\rangle$, proving the lemma.

LEMMA 2. *Let A be an F_R-regular ideal and b an F_R-regular element of R. Then, if $a \in A$, $a+b$ is F_R-regular.*

PROOF. Putting $c = a+b$, we have $c-a \in F_R(c-a)$, since b is F_R-regular. Applying Lemma 1 we obtain that $c-a \in F_R(c)+A$, whence $c \in F_R(c)+A$. Thus we may write $c = -d+a'$, with $d \in F_R(c)$ and $a' \in A$. Since A is F_R-regular we have $c+d \in F_R(c+d)$ Using (C3) we find $F_R(c+d) \subset F_R(c)$, since $d \in F_R(c)$. Therefore $c+d \in F_R(c)$, whence $c \in F_R(c)$, which was to be proved.

COROLLARY 1. *If the ideals A and B of R are F_R-regular, then so is $A+B$.*

THEOREM 1. *Every ring R contains a greatest F_R-regular ideal. This ideal will be* denoted by $\bar{F}(R)$ *and called the F-subradical of R.*

PROOF. The set of all F_R-regular ideals of R is directed as a consequence of Corollary 1 and $\bar{F}(R)$ is their set-theoretic union.

THEOREM 2. *Every ring R contains a greatest F-regular ideal. This ideal will be denoted by $F(R)$ and called the F-radical of R. One has $F(R) \subset \bar{F}(R)$.*

PROOF. For the first part of the theorem it suffices to show that the sum $C = A+B$ of two F-regular ideals A and B of R is F-regular as well. But this follows by noting that A and B are F_C-regular, according to (C2), and by applying Corollary 1. The inclusion $F(R) \subset \bar{F}(R)$ is immediate since every F-regular ideal is F_R-regular.

REMARKS. The use of the words 'radical' and 'subradical' suggest that the functions F and $\bar{F}$ have radical-like properties. This is the case indeed. To clarify this we insert some radical theoretical material. Let $\bar{\rho}$ be a function assigning to every ring R an ideal $\rho(R)$ of R. Then $\bar{\rho}$ is called a *subradical* if the following three conditions are satisfied:

(A) if α: $R \to S$ is a ring homomorphism, then $\alpha(\bar{\rho}(R)) \subset \bar{\rho}(\alpha(R))$;

(B) if A is an ideal of R such that $\bar{\rho}(A) = A$, then $A \subset \bar{\rho}(R)$;

(C) $\bar{\rho}(R/\bar{\rho}(R)) = 0$ for every ring R.

If, moreover, the function $\bar{\rho}$ is idempotent, that is satisfies

(D) $\bar{\rho}(\bar{\rho}(R)) = \bar{\rho}(R)$ for every ring R, then $\bar{\rho}$ is called a *radical*. For any subradical $\bar{\rho}$ the class ρ, consisting of all rings R such that $\bar{\rho}(R) = R$, is a radical class in the well-known Kurosh–Amitsur sense, whereas the corresponding radical ρ coincides with $\bar{\rho}$ if and only if $\bar{\rho}$ is a radical. In general one has $\rho(R) \subset \bar{\rho}(R)$ for each ring R. It may easily be verified that for any regularity $\mathscr{F}$ the induced function $\bar{F}$ is a subradical. Having done this, it is obvious that F is the radical determined by the subradical $\bar{F}$ in the just described way.

2.3. Coincidence of the radical and the subradical

We proceed by giving a condition under which the radical F and the subradical $\bar{F}$ coincide. Before doing so, let us give a simple example to show that the inclusion $F(R) \subset \bar{F}(R)$ may be proper.

EXAMPLE 1. For each ring R and for each element r of R define $F_R(r) = R^2$. One may easily verify that the conditions (C1), (C2) and (C3) are satisfied. So $\{F_R | R \text{ is a ring}\}$ is a regularity. Clearly we have for any ring R that $\bar{F}(R) = R^2$. Consequently, R is F-regular if and only if $R^2 = R$. Therefore, $F(R)$ is the greatest idempotent ideal of R. By taking for R the ring $2Z$ of the even integers we obtain $\bar{F}(R) = 4Z$ and $F(R) = 0$.

The radical F is called *hereditary* if each ideal of an F-regular ring is F-regular. This is the case if and only if $F(A) = A \cap F(R)$ holds for every ideal A of every ring R, according to a well-known radical theoretical result (Divinsky (1965)). The subradical $\bar{F}$ will be called hereditary if $\bar{F}(A) = A \cap \bar{F}(R)$ holds for every ideal A of every ring R. Now we may state

THEOREM 3. *The following three statements are equivalent*:

(1) *the subradical $\bar{F}$ is hereditary*;

(2) *the radical F is hereditary and coincides with $\bar{F}$*;

(3) *if A is an ideal of R such that $A \subset \bar{F}(R)$, then $\bar{F}(A) = A$.*

PROOF. The implications (1) $\Rightarrow$(3) and (2) $\Rightarrow$(1) are trivial. So we only need to show that (3) implies (2). To do so, let R be any ring. Then $\bar{F}(\bar{F}(R)) = \bar{F}(R)$, since $\bar{F}(R) \subset \bar{F}(R)$, showing that $\bar{F}(R)$ is F-regular. By Theorem 2 this implies that $\bar{F}(R) \subset F(R)$. The converse inclusion holds as well. Therefore, $\bar{F}(R) = F(R)$. Hence, F coincides with $\bar{F}$. Now suppose that R is F-regular and that A is an ideal of R. Then $\bar{F}(R) = R$. Hence $A \subset \bar{F}(R)$. Using (3) it follows that $\bar{F}(A) = A$. So A is F-regular. Thus F is hereditary. This proves the theorem.

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2.4. Equivalence of regularities

Two regularities $\mathscr{F}$ and $\mathscr{G}$ will be called *equal* if $F_R(r) = G_R(r)$ holds for every element r of every ring R. Since different regularities may determine the same radical or the same subradical we are led to the following:

DEFINITION 1. Two regularities $\mathscr{F}$ and $\mathscr{G}$ will be called *strongly equivalent* if $r \in F_R(r)$ holds if and only if $r \in G_R(r)$ for every element r of every ring R; *equivalent* if $\mathscr{F}$ and $\mathscr{G}$ determine the same subradical; and *weakly equivalent* if they determine the same radical.

Obviously equality of $\mathscr{F}$ and $\mathscr{G}$ implies strong equivalence, strong equivalence implies equivalence and equivalence implies weak equivalence. The converse implications do not hold, as can be shown by means of appropriate counterexamples. The proof of the next theorem is straightforward and will therefore be left out.

THEOREM 4. *Let $\mathscr{F}$ be a regularity. Define $G_R(r) = F_R(-r)$ for every element r of every ring R. Then $\mathscr{G} = \{G_R \mid R$ is a ring$\}$ is a regularity. Moreover, $\mathscr{F}$ and $\mathscr{G}$ are equivalent.*

2.5. A survey of well-known regularities

In the ring theoretical literature many so-called regularities appear. In this section we shall list those regularities which are known to us. One may easily verify that each of them satisfies the conditions (C1), (C2) and (C3), that is each of these regularities is a regularity in the general sense of Section 2.1. We shall not carry out this verification however; in the next chapter we introduce a class of regularities of a particular type, the so-called polynomial regularities. It will appear then that all well-known regularities are polynomial regularities.

1. $F_R(r) = Rr$. This is the D-regularity, more precisely left D-regularity, introduced by Divinsky (1958), See also Baer (1943) and Roos (1976).

2. $F_R(r) = rRr$. This is the well-known regularity of von Neumann (1936). See also Brown and McCoy (1950).

3. $F(r) = RrR$. This is the λ-regularity, introduced by de la Rosa (1970). See also Roos (1976).

4. $F_R(r) = R(1+r)$. An element r of R is F_R-regular in this sense if $r+s+sr = 0$ for some element s of R. Hence F-regularity coincides with left quasi-regularity as defined by Perlis (1942) and Jacobson (1945).

5. $F_R(r) = Rr(1+r)$. In this case F-regularity coincides with the left pseudo-regularity defined by Divinsky (1955).

6. $F_R(r) = Rr^2$. F-regular rings in this sense are the so-called strongly regular rings as introduced by Arens and Kaplansky (1948). This regularity is studied by Kando (1952), Lajos and Szász (1970) and Sogawa (1971).

7. $F_R(r) = RrRrR$. Now F-regularity coincides with f-regularity as defined by Blair (1955).

8. $F_R(r) = Rr+RrR$. This regularity is introduced by Szász (1973). See also Roos (1974).

9. $F_R(r) = R(1+r)+R(1+r)R$. In this case the F-regular rings constitute the radical class of Brown and McCoy (1947, 1948).

10. $F_R(r) = Rr+rR+RrR$. This regularity is studied by Szász (1974).

11. $F_R(r) = p(r)Rq(r)$, where p and q are arbitrary, but fixed integral polynomials. This regularity is treated by Goulding and Ortiz (1971), Musser (1971), Ortiz (1971) and McKnight and Musser (1972).

3. Polynomial regularities

In this chapter we develop a method to produce regularities. The regularities which arise by using this method will be called polynomial regularities. Each of the well-known regularities listed in Section 2.5 will prove to be a polynomial regularity.

3.1. P-regularities

We start this section by stating the main theorem, namely:

THEOREM 5. *Let $f_1, f_2, \ldots, f_n$ be a set of at least two integral polynomials. For each ring R define the mapping F_R as follows:*

$$F_R(r) = f_1(r)Rf_2(r)R \ldots Rf_n(r),$$

for every element r of R. Then $\mathscr{F} = \{F_R \mid R \text{ is a ring}\}$ is a regularity.

For the proof of Theorem 5 we need three lemmas.

LEMMA 3. *Let R be a ring, S a subring and r an element of R such that $rS \subset S$ and $Sr \subset S$. If $f \in \mathbf{Z}[x]$ and $s \in S$, then $f(r+s) = f(r)+s'$ for some element s' of S.*

PROOF. Let f be the polynomial $a_0+a_1x+\ldots+a_nx^n$ ($a_i \in \mathbf{Z}$, $0 \leqslant i \leqslant n$). Then

$$\begin{aligned} f(r+s) &= a_0+a_1(r+s)+\ldots+a_n(r+s)^n \\ &= a_0+a_1r+\ldots+a_nr^n+\ldots a_1s+\ldots = f(r)+a_1s+\ldots. \end{aligned}$$

The remaining terms in the latter expression are integral multiples of finite products of r and s. Each of these terms contains the element s as a factor. Since $s \in S$ and S is closed under left and right multiplication by r, it follows that each of these terms belongs to S. Also $a_1s \in S$, whence the lemma follows.

LEMMA 4. *Let $\mathscr{F}$ and $\mathscr{G}$ be regularities such that $F_R(r)$ and $G_R(r)$ are ideals of R, for every element r of every ring R. If $f \in \mathbf{Z}[x]$, then*

$$H_R(r) = F_R(r) f(r) G_R(r)$$

determines a regularity.

PROOF. One may easily verify that $\mathscr{H} = \{H_R \mid R \text{ is a ring}\}$ satisfies (C1) and (C2), by using the fact that $\mathscr{F}$ and $\mathscr{G}$ do so. We only show that $\mathscr{H}$ satisfies (C3). Let $s \in H_R(r)$. Since $F_R(r)$ and $G_R(r)$ are ideals we have $H_R(r) \subset F_R(r)$ and $H_R(r) \subset G_R(r)$, whence $s \in F_R(r)$ and $s \in G_R(r)$. Consequently

$$F_R(r+s) \subset F_R(r) \quad \text{and} \quad G_R(r+s) \subset G_R(r).$$

So we may write

$$H_R(r+s) = F_R(r+s) f(r+s) G_R(r+s) \subset F_R(r) f(r+s) G_R(r).$$

By Lemma 3 it follows that $f(r+s) \in f(r) + H_R(r)$. Hence we have

$$F_R(r) f(r+s) G_R(r) \subset F_R(r) [f(r) + H_R(r)] G_R(r).$$

Using the definition of $H_R(r)$ and the fact that $H_R(r)$ is an ideal of R, it now easily follows that the right-hand member of this inclusion is contained in $H_R(r)$, whence $H_R(r+s) \subset H_R(r)$, which was to be proved.

By noting that $F_R(r) = R$ determines a regularity we obtain

COROLLARY 2. *Let $f \in \mathbf{Z}[x]$. Then $F_R(r) = Rf(r)R$ determines a regularity.*

LEMMA 5. *Let $\mathscr{F}$ be a regularity such that $F_R(r)$ is an ideal of R, for every element r of every ring R. If $f, g \in \mathbf{Z}[x]$, then*

$$H_R(r) = f(r) F_R(r) g(r)$$

determines a regularity.

The proof of Lemma 5 is quite similar to that of Lemma 4; therefore we omit it. Furthermore, it will be clear that it is now easy to prove Theorem 5 by using induction on n and by applying the Lemmas 4 and 5 repeatedly. This is left to the reader.

DEFINITION 2. Each regularity which is of the type described in Theorem 5, will be called a *polynomial regularity*, or shortly a p-regularity.

REMARK 1. Returning to the list of well-known regularities in Section 2.5, one sees that eight of them are p-regularities, namely the first seven ones and the last one. The remaining three seem to arise by forming sums of p-regularities in an appropriate manner. For this reason the next section is devoted to the investigation of what we shall call the 'summability' of regularities.

3.2. P'-regularities

DEFINITION 3. Let $\{\mathscr{F}^{(i)} | i \in I\}$ denote a family of regularities. If

$$S_R(r) = \sum_{i \in I} F_R(r)$$

determines a regularity, this family will be called *summable* and $\mathscr{S} = \{S_R | R \text{ is a ring}\}$ will be called the *sum* of the regularities $\mathscr{F}^{(i)}$ $(i \in I)$.

Not every family of regularities is summable. The crucial point is condition (C3). The class $\mathscr{S} = \{S_R | R \text{ is a ring}\}$, as defined above, satisfies (C1) and (C2), but may fail to satisfy (C3). Our main result in this connection is

THEOREM 6. *Any family of p-regularities $\mathscr{F}^{(i)}$ $(i \in I)$ with the property that $S_R(r)$, as defined in Definition* 3, *is an ideal of R for every element r of every ring R, is summable.*

PROOF. As we noted before it suffices to prove that $\mathscr{S}$ satisfies (C3). Therefore, let $s \in S_R(r)$. Then we need to show that $S_R(r+s) \subset S_R(r)$. We shall do this by showing that $F_R^{(i)}(r+s) \subset S_R(r)$ holds for every element i of I. Take i fixed. Since $\mathscr{F}^{(i)}$ is a p-regularity, there exist integral polynomials $f_1, f_2, \ldots, f_n$ such that $F_R^{(i)}(r) = f_1(r)Rf_2(r)R \ldots Rf_n(r)$. Hence we have

$$F_R^{(i)}(r+s) = f_1(r+s)Rf_2(r+s)R \ldots Rf_n(r+s).$$

Using Lemma 3 we obtain that $f_i(r+s) \in f_i(r) + S_R(r)$ for each i, $1 \leqslant i \leqslant n$. By substituting this, it readily follows that $F_R^{(i)}(r+s) \subset F_R^{(i)}(r) + S_R(r)$, whence $F_R^{(i)}(r+s) \subset S_R(r)$, which was to be shown.

COROLLARY 3. *Any p-regularity $\mathscr{F}$ gives rise to another regularity, namely the regularity $\mathscr{F}'$ determined by $F'_R(r) = \langle F_R(r) \rangle$. The regularity $\mathscr{F}'$ will be said to be generated by the p-regularity $\mathscr{F}$.*

DEFINITION 4. A regularity which is the sum of a summable family of p-regularities will be called an *sp-regularity*. An *sp*-regularity which is generated by a p-regularity will be said to be a *p′-regularity*.

REMARK 2. Regularities 8, 9 and 10 in the list of well-known regularities are *sp*-regularities. Regularities 8 and 9 are p'-regularities. They are generated by regularities 1 and 4 respectively.

3.3. Intersection theorems

We shall say that the integral polynomial g is *unic* if $g(0) = 1$, and *nullic* if $g = x^k f$ for some natural number k $(\geqslant 1)$ and some unic polynomial f. A p-regularity will be called unic if all the polynomials concerned are unic, and nullic if at least one of them is nullic and the remaining are unic. In the same way one may define unic and nullic *sp*-regularities. From now on we shall restrict our investigations to *p*- and *p'*-regularities which are either unic or nullic. Note that each of the well-known regularities is unic or nullic.

For the sake of brevity we introduce the following notations. Let $\mathscr{F}$ be the *p*-regularity determined by $F_R(r) = g_1(r)\,Rg_2(r)\,R\ldots Rg_n(r)$. Then we shall denote $\mathscr{F}$ by $\{g_1\,Rg_2\,R\ldots Rg_n\}$ and the *F*-subradical of any ring S by $(g_1\,Sg_2\,S\ldots Sg_n)$. Similarly, the *p'*-regularity $\mathscr{F}'$ generated by $\mathscr{F}$ will be denoted by $\{\langle g_1\,Rg_2\,R\ldots Rg_n\rangle\}$ and the *F'*-subradical of any ring S by $(\langle g_1\,Sg_2\,S\ldots Sg_n\rangle)$.

If each of the polynomials g_i is unic or nullic, then we may write $g_i = x^{k_i} f_i$, with $k_i \geqq 0$ and f_i unic, $1 \leqq i \leqq n$, and we are able to prove

THEOREM 7. *For any ring R the following holds*:

(1) $(g_1\,Rg_2\,R\ldots Rg_n) = (g_1\,R)\cap(Rg_2\,R)\cap\ldots\cap(Rg_n)\cap(x^{k_1}\,Rx^{k_2}\,R\ldots Rx^{k_n})$;

(2) $(\langle g_1\,Rg_2\,R\ldots Rg_n\rangle) = (\langle g_1\,R\rangle)\cap(Rg_2\,R)\cap\ldots\cap(\langle Rg_n\rangle)$
$\cap(\langle x^{k_1}\,Rx^{k_2}\,R\ldots Rx^{k_n}\rangle)$.

For the proof the following lemma is needed.

LEMMA 6. *Let $g = x^k f$ ($k \geqq 0, f$ unic), R any ring and $r \in R$. Then we have*

(1) *$r \in Rg(r)$ if and only if $Rr^k = Rg(r)$ and $r \in Rr^k$*;

(2) *$r \in \langle Rg(r)\rangle$ if and only if $\langle Rr^k\rangle = \langle Rg(r)\rangle$ and $r \in \langle Rr^k\rangle$*;

(3) *$r \in Rg(r)\,R$ if and only if $Rr^k R = Rg(r)\,R$ and $r \in Rr^k R$.*

PROOF. The 'if' part of each of the statements in the lemma is obvious. Thus we only need to prove the 'only if' parts. This can be done as follows:

(1) If $k > 0$, then $r \in Rg(r)$ implies that $Rr \subset Rg(r) \subset Rr^k \subset Rr$, whence $Rr^k = Rg(r)$ and $r \in Rr^k$. If $k = 0$, we have $g = 1 + a_1 x + \ldots + a_n x^n$ for suitable integers a_i. Suppose that $r \in Rg(r)$. Then, if $s \in R$, we may write:

$$s = s(1 + a_1 r + \ldots + a_n r^n) - s(a_1 r + \ldots + a_n r^n) \in Rg(r) + Rr \subset Rg(r),$$

whence it follows that $R = Rg(r)$, which was to be proved.

(2) If $k > 0$, then $r \in \langle Rg(r)\rangle$ implies that $\langle Rr\rangle \subset \langle Rg(r)\rangle \subset \langle Rr^k\rangle \subset \langle Rr\rangle$, whence $\langle Rr^k\rangle = \langle Rg(r)\rangle$ and $r \in \langle Rr^k\rangle$. If $k = 0$, firstly let $s,t \in R$, while $r \in \langle Rg(r)\rangle$. Then we may write:

$$st = sg(r)\,t - s(a_1 r + \ldots + a_n r^n)\,t \in Rg(r)\,R + RrR \subset R\langle Rg(r)\rangle\,R \subset Rg(r)\,R,$$

from which it follows that $Rg(r)\,R = R^2$. Now let $u \in R$. Then again we have that $u \in Rg(r) + Rr$. But $Rg(r) + R^2 = Rg(r) + Rg(r)\,R = \langle Rg(r)\rangle$. Hence $u \in \langle Rg(r)\rangle$. Consequently, $R = \langle Rg(r)\rangle$, which was to be proved.

(3) This statement follows from Statement (2), as one easily may verify.

PROOF OF THEOREM 7. In each of the two statements in the theorem the inclusion '$\subset$' is obvious. The converse inclusions may be proved as follows.

(1): Let r belong to the right-hand member of the equality. Then we have $r \in g_1(r)R$, $r \in Rg_i(r)R$ $(2 \leqslant i \leqslant n-1)$, $r \in Rg_n(r)$ and $r \in r^{k_1}Rr^{k_2}R \dots Rr^{k_n}$. Now Lemma 6 yields that $r^{k_1}R = g_1(r)R$, $Rr^{k_i}R = Rg_i(r)R$ $(2 \leqslant i \leqslant n-1)$ and $Rr^{k_n} = Rg_n(r)$. By simply substituting these equalities we obtain:

$$r^{k_1}Rr^{k_2}R \dots Rr^{k_n} = g_1(r)Rg_2(r)R \dots Rg_n(r).$$

Hence $r \in g_1(r)Rg_2(r)R \dots Rg_n(r)$. By Theorem 1 this implies that the desired inclusion holds.

(2): Let $r \in R$. Then

$$\langle g_1(r)Rg_2(r)R \dots Rg_n(r)\rangle = \langle g_1(r)R\rangle g_2(r)R \dots Rg_{n-1}(r)\langle Rg_n(r)\rangle.$$

Having noticed this, the proof becomes quite similar to that of part (1); it will therefore be omitted.

For any integral polynomial g, let us call the regularities $\{Rg\}$, $\{gR\}$, $\{\langle Rg\rangle\}$, $\{\langle gR\rangle\}$ and $\{RgR\}$ the *elementary regularities determined by* g. Then we may say, according to Theorem 7, that every unic p-regularity (p'-regularity) is the intersection of some elementary unic p-regularities (p'-regularities) and a unic m-regularity (m'-regularity), and every nullic p-regularity (p'-regularity) is the intersection of some elementary p-regularities (p'-regularities), each of them being either unic or nullic, and a nullic m-regularity (m'-regularity). Here the m stands for 'monomial' in stead of 'polynomial'.

3.4. Elementary regularities

In this section we trace all equivalences in the set of elementary regularities determined by the polynomials $x^k f$, with $k = 0, 1, 2, \dots$ and f a fixed unic polynomial.

3.4.1. The unic case: $k=0$

THEOREM 8. *For every ring* R *we have*

(1) $(Rf) = (fR)$;

(2) $(\langle Rf\rangle) = (\langle fR\rangle)$;

(3) $(RfR) = (\langle Rf\rangle) \cap R^2$.

PROOF. We shall prove each of the three statements in the theorem in detail.

(1) Let $r \in (Rf)$. Then $r = sf(r)$ for some element s of R. We shall show that $r = f(r)s$. This will imply that $r \in f(r)R$, whence $(Rf) \subset (fR)$ follows by Theorem 1.

The converse inclusion can be proved in the same way. Thus, putting

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$u = sf(r) - f(r)s$, Statement (1) will follow if we are able to show that $u = 0$. To do so, firstly note that $uf(r) = (r - f(r)s)f(r) = rf(r) - f(r)sf(r) = rf(r) - f(r)r = 0$. Using this, we obtain that $u^2 = u(r - f(r)s) = ur$, whence $u^m \cdot u = u \cdot r^m$ holds for each natural number m.

Consequently, $f(u)u = uf(r)$. Hence it follows that $f(u)u = 0$. At the other side, we have as an obvious consequence of the definition of u that u belongs to (Rf). Hence $u \in Rf(u)$, whence $u = tf(u)$ for some element t of R. Therefore, we may write $u^2 = tf(u) \cdot u = t \cdot f(u)u = t \cdot 0 = 0$. Hence $u^m = 0$ holds for each natural number m, with $m \geqslant 2$. Since $f(u)u = 0$ and f is unic, this implies that $u = 0$. This proves Statement (1).

(2) Let $r \in (\langle Rf \rangle)$. Then $r \in \langle Rf(r) \rangle$, whence $r = sf(r) + t$ for some elements s of R and t of $Rf(r)R$. We shall show that $r = f(r)s + t'$ for some element t' of $Rf(r)R$. This will imply that $r \in \langle f(r)R \rangle$, whence the inclusion $(\langle Rf \rangle) \subset (\langle fR \rangle)$ will follow by Theorem 1. Since the opposite inclusion can be proved in the same way, this will suffice for the proof of Statement (2). To begin with, let us observe that $r - f(r)s \in Rf(r)R$ if and only if $sf(r) - f(r)s \in Rf(r)R$. Therefore, putting $v = sf(r) - f(r)s$, we only need to show that $v \in Rf(r)R$. Now $r \in \langle Rf(r) \rangle$ implies that $R = \langle Rf(r) \rangle$, by Lemma 6(2). So it follows that

$$R^2 = \langle Rf(r) \rangle R = (Rf(r) + Rf(r)R)R = Rf(r)R.$$

But it is an obvious consequence of the definition of the element v that $v \in R^2$. Thus we may conclude that $v \in Rf(r)R$, whence the statement follows.

(3) The inclusion '$\subset$' in Statement (3) is obvious. To prove the converse inclusion, suppose that $r \in (\langle Rf \rangle) \cap R^2$. Then (*) $r \in \langle Rf(r) \rangle$ and (**) $r \in R^2$. By Lemma 6(2), (*) implies that $R = Rf(r) + Rf(r)R$. Substituting this in (**) we obtain that $r \in (Rf(r) + Rf(r)R)R = Rf(r)R$. By Theorem 1 this suffices for the proof of the inclusion '$\supset$'.

3.4.2. The nullic case: $k = 1$

THEOREM 9. *For every ring R we have*:

(1) $(Rxf) = (Rf) \cap (Rx)$;

(2) $(RxfR) = (\langle Rxf \rangle) \cap (RxR) = (\langle Rxf \rangle) \cap (\langle xfR \rangle)$.

The method of the proof of Theorem 9 is almost the same as that of Theorem 8(3); therefore it will be left out. One should note that Theorem 9 does not contain the statements $(Rxf) = (xfR)$ and $(\langle Rxf \rangle) = (\langle xfR \rangle)$. The reason is that these equalities do not hold in general. It can also be shown that we do not always have

$$(\langle Rxf \rangle) = (\langle Rf \rangle) \cap (\langle Rx \rangle).$$

For examples the reader may be referred to Roos (1975).

3.4.3. The nullic case: $k \geqslant 2$

THEOREM 10. *For every ring R we have*:
(1) $(Rx^k f) = (Rf) \cap (Rx^2)$;
(2) $(\langle Rx^k f\rangle) = (\langle Rf\rangle) \cap (\langle Rx^2\rangle)$;
(3) $(Rx^k fR) = (\langle Rx^k f\rangle)$.

PROOF. We shall give proofs of Statements (1) and (3). The proof of Statement (2) is similar to that of (1); it will be left to the reader.

(1) The method of the proof of Theorem 8(3) may be used to show that $(Rx^k f) = (Rf) \cap (Rx^k)$. Hence it suffices to show that $(Rx^k) = (Rx^2)$. The inclusion '$\subset$' is obvious. To prove the converse inclusion, let $r \in (Rx^2)$. Then also $r^2 \in (Rx^2)$. Hence $r^2 \in Rr^4$, whence $Rr^2 \subset Rr^4 \subset Rr^3$. Consequently, $Rr^2 = Rr^3$. From this one easily deduces that $Rr^2 = Rr^k$. Since $r \in Rr^2$ it follows that $r \in Rr^k$, whence the inclusion '$\supset$' follows from Theorem 1.

(3) The inclusion '$\subset$' is trivial. Let $r \in (\langle Rx^k f\rangle)$. Then $r \in Rx^k f(r) + Rr^k f(r) R$. This implies that $r \in RrR$, since $k \geqslant 2$. But

$$RrR \subset R(Rr^k f(r) + Rr^k f(r)\, R)\, R \subset Rr^k f(r)\, R,$$

whence $r \in Rr^k f(r)\, R$, proving the statement.

By right—left dualizing Statement (3) we obtain that $(Rx^k fR) = (\langle x^k fR\rangle)$. Hence we have $(\langle Rx^k f\rangle) = (\langle x^k fR\rangle)$. It is much less obvious that the regularities $\{Rx^k f\}$ and $\{x^k fR\}$ are equivalent. In view of Theorems 8(1) and 10(1) this will be the case if $(Rx^2) = (x^2 R)$. In this connection the following lemma is of great interest.

LEMMA 7. *Let $r \in (Rx^2)$ and $r = sr^2$, with $s \in R$. Then $r = rsr = r^2 s$. Consequently $(Rx^2) = (x^2 R)$.*

PROOF. Define $u = r - rsr$. One may easily verify that $u^2 = 0$. Since $u \in (Rx^2)$, we have $u \in Ru^2$. Hence $u = 0$. So $r = rsr$. In just the same way one proves that $rsr = r^2 s$. Hence $r \in r^2 R$. By Theorem 1 it follows from this that $(Rx^2) \subset (x^2 R.)$ The converse inclusion may be proved similarly. Hence $(Rx^2) = (x^2 R)$.

3.4.4. The lattice of elementary regularities

Let $\mathscr{F}$ and $\mathscr{G}$ denote any two regularities. Define $\mathscr{F} \prec \mathscr{G}$ if and only if $\bar{F}(R) \subset \bar{G}(R)$ holds for every ring R. Obviously, the relation $\prec$ is reflexive and transitive. Moreover, $\mathscr{F}$ and $\mathscr{G}$ are equivalent if and only if $\mathscr{F} \prec \mathscr{G}$ and $\mathscr{G} \prec \mathscr{F}$. Hence, $\prec$ determines a partial ordering of the family whose elements are the classes of mutually equivalent regularities. Identifying equivalent regularities, we thus may say that $\prec$ is a partial ordering of the family consisting of all regularities. Now let f be any unic integral polynomial. Then $\prec$ yields a partial ordering of the set of all elementary regularities determined by the polynomials $x^k f$, $k = 0, 1, 2, \ldots$.

Using the equivalences derived in the previous sections, one may draw the corresponding diagram. This is done in Fig. 1.

From the regularities appearing in this diagram only the following four determine hereditary subradicals: $\{Rf\}$, $\{\langle Rf\rangle\}$, $\{Rx^2f\}$ and $\{Rx^2fR\}$. For each of the remaining six regularities the radical and the subradical do not coincide. These results can be obtained with the help of Theorem 3. For the proofs we refer to Roos (1965).

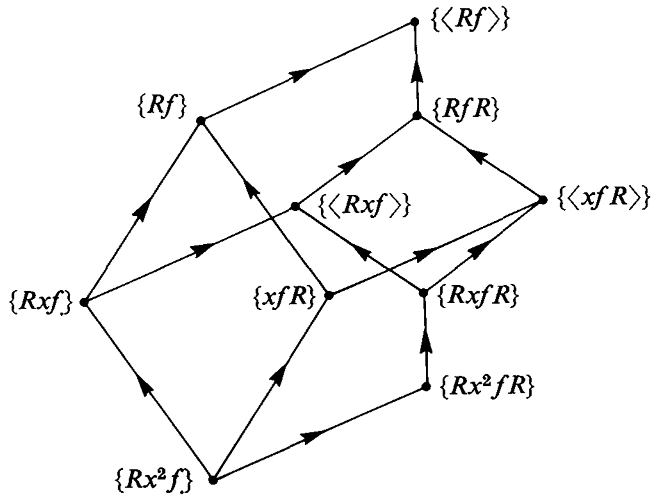

FIG. 1. The diagram of elementary regularities.

3.5. Monomial regularities

In this section it will be shown that there exist, up to equivalence, eleven nullic m-regularities and five nullic m'-regularities, three of which are at the same time a nullic m-regularity.

3.5.1. Nullic m'-regularities

Throughout this section $\mathscr{F}$ will denote any nullic m'-regularity. By definition, $\mathscr{F}$ is of the form $\{\langle x^{k_1}Rx^{k_2}R\dots Rx^{k_n}\rangle\}$, with $k_1+k_2+\dots+k_n\geqslant 1$.

THEOREM 11. *If $k_i\geqslant 2$ for some i, $1\leqslant i\leqslant n$, then $\bar{F}(R)=(Rx^2R)$ for each ring R.*

PROOF. If $1<i<n$, then the inclusion '$\subset$' is trivial. If $i=n$, then we have

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$\bar{F}(R) \subset (\langle Rx^2 \rangle)$. From Theorem 10(3) it follows that $(\langle Rx^2 \rangle) = (Rx^2 R)$ by substituting $f = 1$. Therefore $\bar{F}(R) \subset (Rx^2 R)$. Similarly if $i = 1$. So it remains to show that the opposite inclusion holds. To do so, define $m = k_1 + k_2 + \ldots + k_n + n - 1$. Then $(Rx^m R) = (Rx^{k_1} \cdot x \cdot x^{k_2} \cdot x \ldots x \cdot x^{k_n} R) \subset \bar{F}(R)$. Using that $m \geqslant 2$ one may easily deduce from Theorem 10 that $(Rx^2 R) = (Rx^m R)$. Hence the theorem is proved.

THEOREM 12. *If $k_i \leqslant 1$ for every i, $1 \leqslant i \leqslant n$, and $\Sigma_i k_i \geqslant 2$, then we have for each ring R: $\bar{F}(R) = (RxRxR)$.*

PROOF. If $k_1 = k_n = 0$, then the inclusion '$\subset$' is obvious. If $k_1 = 0$ and $k_n = 1$, then we have that $\bar{F}(R) \subset (\langle RxRx \rangle)$. So the inclusion '$\subset$' will hold if

$$(\langle RxRx \rangle) \subset (RxRxR).$$

That this is the case can be shown as follows. Let $r \in (\langle RxRx \rangle)$. Then $r \in \langle RrRr \rangle$. But $\langle RrRr \rangle = Rr(Rr + RrR)$, as easily may be verified. Hence it follows that $r \in R \cdot Rr(Rr + RrR) \cdot (Rr + RrR) \subset RrRrR$, whence $(\langle RxRx \rangle) \subset (RxRxR)$.

Note that in fact $(\langle RxRx \rangle) = (RxRxR)$.

In just the same way one proves that also $(\langle xRxR \rangle) = (RxRxR)$ and $(\langle xRx \rangle) = (RxRxR)$, from which one may deduce the inclusion '$\subset$' in the cases $k_1 = 1$, $k_n = 0$ and $k_1 = k_n = 1$. So it remains to prove the opposite inclusion. Suppose that $r \in (RxRxR)$. Then $r \in RrRrR$. Hence $RrR \subset RrRrR$, whence we have $RrR = RrR \cdot rR$. Consequently $RrR = RrR(rR)^{n-1}$. But

$$RrR(rR)^{n-1} \subset Rr^{k_1} Rr^{k_2} R \ldots Rr^{k_n} R \subset \langle r^{k_1} Rr^{k_2} R \ldots Rr^{k_n} \rangle.$$

So it follows that $r \in \langle r^{k_1} Rr^{k_2} R \ldots Rr^{k_n} \rangle$, completing the proof.

THEOREM 13. *If $\sum_i k_i = 1$, then we have for each ring R:*

(1) $\bar{F}(R) = (RxR)$ *if* $k_1 = k_n = 0$;

(2) $\bar{F}(R) = (\langle Rx \rangle)$ *if* $k_n = 1$;

(3) $\bar{F}(R) = (\langle xR \rangle)$ *if* $k_1 = 1$;

(4) $(RxR) = (\langle Rx \rangle) \cap (\langle xR \rangle)$.

PROOF. The proof of Statements (1), (2) and (3) uses the same arguments as the proof of Theorem 12; we therefore omit it. Statement (4) follows from Theorem 9(2) by merely substituting $f = 1$.

3.5.2. Nullic *m*-regularities

Throughout this section $\mathscr{F}$ will denote any nullic m-regularity. Thus $\mathscr{F}$ is of the form $\{x^{k_1} Rx^{k_2} R \ldots Rx^{k_n}\}$, with $k_1 + k_2 + \ldots + k_n \geqslant 1$.

THEOREM 14. *If $k_1 \geqslant 2$ or $k_n \geqslant 2$, then $\bar{F}(R) = (Rx^2)$ for each ring R.*

PROOF. One may easily deduce from Lemma 8 that $(x^p Rx^q) = (Rx^2)$ for any two non-negative integers p and q, with $p \geqslant 2$ or $q \geqslant 2$. Having done this, the inclusion '$\subset$' becomes obvious, and the opposite inclusion can be proved by taking $p = k_1$ and $q = k_2 + k_3 + \ldots + k_n + n - 2$.

For our convenience we define the non-negative integer $k = \max\{k_i | 1 < i < n\}$.

THEOREM 15. *If $k \geqslant 2$, then for each ring R the following holds*:

(1) $\bar{F}(R) = (Rx^2 Rx)$ *if* $k_1 = 0$ *and* $k_n = 1$;
(2) $\bar{F}(R) = (xRx^2 R)$ *if* $k_1 = 1$ *and* $k_n = 0$;
(3) $\bar{F}(R) = (Rx^2)$ *if* $k_1 = k_n = 1$.

PROOF. Firstly we show that $\bar{F}(R) = (x^{k_1} Rx^2 Rx^{k_n})$. Let $r \in (x^{k_1} Rx^2 Rx^{k_n})$. Then we have $r \in r^{k_1} Rr^2 Rr^{k_n}$ and $r \in (Rx^2 R)$. From Theorem 11 we deduce that

$$(Rx^{k_2} R \ldots Rx^{k_{n-1}} R) = (Rx^2 R).$$

So it follows that $r \in Rr^{k_1} R \ldots Rr^{k_{n-1}} R$. This clearly implies that

$$Rr^2 R \subset Rr^{k_2} R \ldots Rr^{k_{n-1}} R,$$

whence we are allowed to conclude that $r \in r^{k_1} Rr^{k_2} R \ldots Rr^{k_n}$, since $r \in r^{k_1} Rr \ldots Rr^{k_n}$. Hence $\bar{F}(R) \supset (x^{k_1} Rx^2 Rx^{k_n})$. The opposite inclusion being trivial, we even have equality. Now the Statements (1) and (2) easily follow, and Statement (3) becomes clear on account of the following lemma.

LEMMA 8. *For each ring R: $(xRx^2 Rx) = (Rx^2)$.*

PROOF. The inclusion '$\supset$' may be deduced from Lemma 7. The proof of the converse inclusion is as follows. Let $r \in (xRx^2 Rx|$. Then $r = rsr$ for some element s of R. Define $t = r - sr^2$. By a simple calculation one may verify that $t^2 = 0$. Clearly $t \in (xRx^2 Rx)$. Hence $t \in tRt^2 Rt$, whence $t = 0$. Consequently, $r = sr^2$ and thus we conclude that $r \in Rr^2$. This proves the lemma.

THEOREM 16. *If $k = 1$, then for each ring R the following holds*:

(1) $\bar{F}(R) = (RxRx)$ *if* $k_1 = 0$ *and* $k_n = 1$;
(2) $\bar{F}(R) = (xRxR)$ *if* $k_1 = 1$ *and* $k_n = 0$;
(3) $\bar{F}(R) = (xRx)$ *if* $k_1 = k_n = 1$.

The proof of this theorem does not use new arguments and will therefore be left out. The same holds for the next theorem.

THEOREM 17. *If $k = 0$, then for each ring R the following holds:*

(1) $\bar{F}(R) = (Rx)$ *if* $k_1 = 0$ *and* $k_n = 1$;

(2) $\bar{F}(R) = (xR)$ *if* $k_1 = 1$ *and* $k_n = 0$;

(3) $\bar{F}(R) = (xRx)$ *if* $k_1 = k_n = 1$.

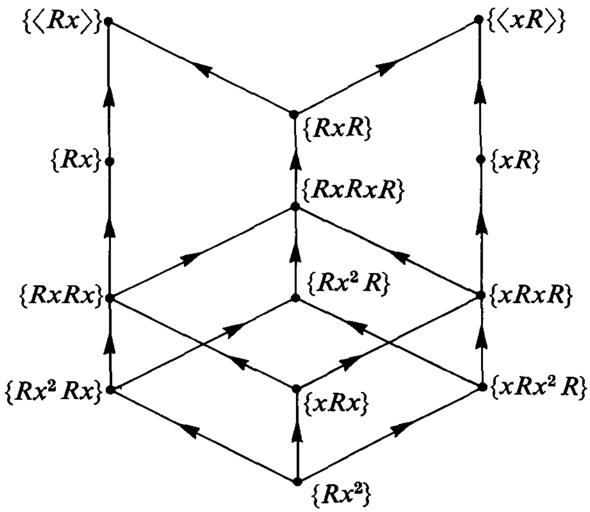

FIG. 2. The diagram of nullic m- and m'-regularities.

3.5.3. The diagram of nullic m- and m'-regularities

Since we have determined, up to equivalence, all nullic m- and all nullic m'-regularities, we now are able to draw Fig. 2. This diagram shows how these regularities are ordered according to the partial ordering defined in Section 3.4.4. With the help of Theorem 3 it is easy to show that those of the regularities $\mathscr{F}$ appearing in the diagram determine a hereditary subradical for which $\mathscr{F} \prec \{RxRxR\}$.

For each of the remaining regularities it is possible to show that the induced radical and subradical do not coincide, by means of appropriate counterexamples. For the details the reader may be referred to Roos (1975).

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