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EPSILON-STRONGLY GROUPOID-GRADED RINGS, THE PICARD INVERSE CATEGORY AND COHOMOLOGY

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Abstract. We introduce the class of partially invertible modules and show that it is an inverse category which we call the Picard inverse category. We use this category to generalize the classical construction of crossed products to, what we call, generalized epsilon-crossed products and show that these coincide with the class of epsilon-strongly groupoid-graded rings. We then use generalized epsilon-crossed groupoid products to obtain a generalization, from the group-graded situation to the groupoid-graded case, of the bijection from a certain second cohomology group, defined by the grading and the functor from the groupoid in question to the Picard inverse category, to the collection of equivalence classes of rings epsilon-strongly graded by the groupoid.

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1. Introduction. Almost 40 years ago, Năstăsescu and Van Oystaeyen [15] proved an elegant result that relates the collection of group-graded equivalence classes of strongly graded rings to certain second cohomology groups. Namely, let S be a ring. We always assume that S is associative and equipped with a multiplicative identity 1_S. The ring S is called *graded* by a group G (or *G-graded*) if there is a set $\{S_g\}_{g\in G}$ of additive subgroups of S such that $S = \oplus_{g\in G} S_g$, and for all $g, h \in G$ the inclusion $S_g S_h \subseteq S_{gh}$ holds. The ring S is called *strongly graded* if for all $g, h \in G$ the equality $S_g S_h = S_{gh}$ holds. Given two G-graded rings S and T, a ring homomorphism $f : S \to T$ is called *graded* if for all $g, h \in G$ the inclusion $f(S_g) \subseteq T_g$ holds. Now, by [15, Proposition I.3.13], the collection of strongly graded rings can be parameterized by the so-called *generalized crossed products* (A, F, f), for rings A and group homomorphisms F from G to the Picard group $\mathrm{Pic}(A)$ of A. Indeed, for each $g \in G$, we put $F(g) = [P_g]$ (the A-bimodule isomorphism class of P_g) and we assume that $F(e) = [A]$, where e denotes the identity element of G. The map f is a factor set associated with F. By this we mean a collection of A-bimodule isomorphisms $f_{g,h} : P_g \otimes_A P_h \to P_{gh}$ chosen so that the following diagram commutes:

$$\begin{array}{ccc} P_g \otimes_A P_h \otimes_A P_p & \xrightarrow{\mathrm{id}_{P_g} \otimes f_{h,p}} & P_g \otimes_A P_{hp} \\ {\scriptstyle f_{g,h} \otimes \mathrm{id}_{P_p}} \big\downarrow & & \big\downarrow {\scriptstyle f_{g,hp}} \\ P_{gh} \otimes_A P_p & \xrightarrow{f_{gh,p}} & P_{ghp} \end{array}$$

for all $g, h, p \in G$. The multiplication in $(A, F, f) = \oplus_{g \in G} P_g$ is defined by the biadditive extension of the relation $x \cdot y = f_{g,h}(x \otimes y)$ for all $x \in P_g$, all $y \in P_h$, and all $g, h \in G$. Let (A, F) denote the collection of all generalized crossed products of the form (A, F, f), where f is a factor set associated with F. Given D and D' in (A, F) put $D \approx D'$ if there is an isomorphism of graded rings $D \to D'$ which is simultaneously an A-bimodule isomorphism. Then $\approx$ is an equivalence relation on (A, F) and we can define $C(A, F) = (A, F)/\approx$. Let $U(Z(A))$ denote the unit group of the center $Z(A) = \{a \in A \mid \forall b \in A,\ ab = ba\}$ of A. The classical cohomology groups $H^n(G, U(Z(A)))$, for $n \geq 0$, can then be defined, and in particular the corresponding second cohomology group.

THEOREM 1 (Năstăsescu and Van Oystaeyen [15]). *If A is a ring, $F : G \to \mathrm{Pic}(A)$ is a group homomorphism and f is a factor set associated with F, then the map $H^2(G, U(Z(A))) \to C(A, F)$, defined by $[q] \mapsto qf$, is bijective.*

Many natural examples of rings, such as rings of matrices, crossed product algebras defined by separable extensions, and groupoid rings are not, in a natural way, graded by groups, but instead by groupoids (see, e.g. [13, 14] or Section 6 of the present article). Let G be a groupoid, that is, a category where each morphism is an isomorphism. The classes of objects and morphisms of G are denoted by G_0 and G_1, respectively. If $g \in G_1$, then the domain and codomain of g are denoted by $d(g)$ and $c(g)$, respectively. We let G_2 denote the set of all pairs $(g, h) \in G_1 \times G_1$ that are composable, that is, such that $d(g) = c(h)$. From now on, assume that G is small, that is, such that G_1 is a set, and let S be a ring which is *graded* by G. Recall from [13, 14] that this means that there is a set $\{S_g\}_{g \in G_1}$ of additive subgroups of S such that $S = \oplus_{g \in G_1} S_g$ and, for all $g, h \in G_1$, $S_g S_h \subseteq S_{gh}$, if $(g, h) \in G_2$, and $S_g S_h = \{0\}$, otherwise. In that case, S is called *strongly graded* if for all $(g, h) \in G_2$ the equality $S_g S_h = S_{gh}$ holds. By [14, Proposition 4.1], the collection of strongly groupoid-graded rings can be parameterized by generalized groupoid-crossed products (A, F, f), for rings A and groupoid homomorphisms F from G to the Picard groupoid PIC. Here A denotes the direct product of the rings $\{F(e)\}_{e \in G_0}$. Westman [20] (see also [19]) has developed a cohomology theory for groupoids which extends the classical group cohomology theory. In particular, the corresponding second cohomology group $H^2(G, Z(A))$ can be defined. The set $C(A, F)$ is defined analogously to the group-graded case.

THEOREM 2 (Lundström [14]). *If G is a groupoid, $F : G \to \mathrm{PIC}$ is a functor and f is a factor set associated with F, then the map $H^2(G, U(Z(A))) \to C(A, F)$, defined by $[q] \mapsto qf$, is bijective.*

In [17], the authors of the present article introduced the class of epsilon-strongly group-graded rings. This class properly contains both the class of strongly graded rings and the class of unital partial crossed products. The main goal of the present article is to show a simultaneous generalization (see Theorem 3) of Theorem 1 and Theorem 2 that holds for an even wider family of rings, namely the class of epsilon-strongly groupoid-graded rings. Indeed, let S be a ring which is graded by a small groupoid G. We say that S is *epsilon-strongly graded by G* if for each $g \in G_1$, $S_g S_{g^{-1}}$ is a unital ideal of $S_{c(g)}$ such that

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for all $(g, h) \in G_2$ the equalities $S_g S_h = S_g S_{g^{-1}} S_{gh} = S_{gh} S_{h^{-1}} S_h$ hold. Let ϵ_g denote the multiplicative identity element of $S_g S_{g^{-1}}$ and put $R = \oplus_{e \in G_0} S_e$. In this context, it turns out that the multiplication map $S_g \otimes_R S_h \to S_{gh}$, for $(g, h) \in G_2$, is injective with image equal to $\epsilon_g S_{gh}$. In particular, this implies that the R-isomorphism classes of the modules $[S_g]$ do not form a groupoid. Instead they form an inverse category PIC_{cat} which we call *the Picard inverse category*. The collection of epsilon-strongly groupoid-graded rings can then be parameterized by generalized epsilon-crossed groupoid products (A, F, f), for rings A and partial functors of inverse categories $F : G \to \mathrm{PIC}_{cat}$. Here A denotes the direct product of the rings $\{F(e)\}_{e \in G_0}$ and f is a partial factor set.

On the other hand, in [9], the concept of a partial action was introduced as an efficient tool to study C^*-algebras, permitting to characterize various important classes of them as crossed products by partial actions. The study of partial actions and partial representations of groups on algebras was initiated in [4] and extended to the groupoid situation in [2]. Recently, Dokuchaev and Khrypchenko [5] have developed a cohomology theory for partial actions of groups on monoids which we extend to the groupoid setting. In particular, we define the corresponding second cohomology group $H^2(G, Z(A))$. The set $C(A, F)$ is defined analogously to the group-graded case, and we obtain the following theorem, which is the main result of this article.

THEOREM 3. *If G is a groupoid, $F : G \to \mathrm{PIC}_{cat}$ is a partial functor of inverse categories and f is a partial factor set associated with F, then the map $H^2(G, U(Z(A))) \to C(A, F)$, defined by $[q] \mapsto qf$, is bijective.*

Here is a detailed outline of this article. In Section 2, we state our conventions on categories. In particular, we recall the definition of inverse categories. In that section, we also show that the collection of partial bijections between sets forms an inverse category which we denote by BIJ_{cat} (see Proposition 13). Inside this category sits the well-known groupoid BIJ of bijections between sets. In Section 3, we show (see Proposition 17) that the collection of partial (commutative) ring isomorphisms ISO_{cat} (ISOC_{cat}) is an inverse subcategory of BIJ_{cat}. This category contains, in turn, the well-known groupoid ISO (ISOC) having all (commutative) rings as objects and ring isomorphisms as morphisms. We also show a result concerning central idempotents in rings that we need in subsequent sections. In Section 4, we recall the definition of (pre-)equivalence data and some properties of such systems, and we also introduce the Picard inverse category PIC_{cat} (see Definition 27 and Theorem 28). In Section 5, we define epsilon-strongly groupoid-graded rings (see Definition 34). In Section 6, we provide examples of epsilon-strong groupoid gradings on partial skew groupoid rings, Leavitt path algebras, and Morita rings. In Section 7, we define epsilon-strongly groupoid-graded modules (see Definition 46) and we provide a characterization of them (see Proposition 47) that generalizes a result previously obtained for strongly group-graded modules. At the end of the section, after putting $R = \oplus_{e \in G_0} S_e$, we show (see Proposition 48) that the multiplication maps $m_{g,h} : S_g \otimes_R S_h \to \epsilon_g S_{gh} = S_{gh} \epsilon_{h^{-1}}$, for $(g, h) \in G_2$, are R-bimodule isomorphisms. In particular, this implies that for every $g \in G_1$, the sextuple

$$\left(\epsilon_g R, \epsilon_{g^{-1}} R, S_g, S_{g^{-1}}, m_{g,g^{-1}}, m_{g^{-1},g}\right)$$

is a set of equivalence data. In Section 8, we introduce generalized epsilon-crossed groupoid products (see Definition 53), and we show that they parameterize the family of epsilon-strongly groupoid-graded rings (see Proposition 55 and Proposition 56). In Section 9, we extend the construction of a partial cohomology theory for partial actions of groups on commutative monoids, from [5], to partial actions of groupoids. In Section 10, we use the results in the previous sections to prove Theorem 3.

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2. Preliminaries on categories. In this section, we state our conventions on categories. In particular, we recall the definition of inverse categories. We introduce the new notion of a partial functor of inverse categories (see Definition 4) and we show that the composition of two partial functors of inverse categories is again a partial functor of inverse categories (see Proposition 9). In this section, we also show that the collection of partial bijections between sets forms an inverse category which we denote by BIJ_{cat} (see Proposition 13). Suppose that G is a category. The classes of objects and morphisms of G are denoted by G_0 and G_1, respectively. Recall that G is called *small* if G_1 is a set. If $g \in G_1$, then the domain and codomain of g are denoted by $d(g)$ and $c(g)$, respectively. If $n \geq 2$, then we let G_n denote the set of all $(g_1, \ldots, g_n) \in \times_{i=1}^{n} G_1$ that are composable, that is, such that for every $i \in \{1, \ldots, n-1\}$ the relation $d(g_i) = c(g_{i+1})$ holds. The category G is called a *groupoid* if to each $g \in G_1$ there is a unique $h \in G_1$ such that $(g, h) \in G_2$, $(h, g) \in G_2$, $gh = c(g)$, and $hg = d(g)$. In that case, we write $h = g^{-1}$. A subcategory H of a groupoid G is called a *subgroupoid* if the restriction of the map $(\cdot)^{-1}$ on G_1 to H_1 coincides with the map $(\cdot)^{-1}$ on H_1.

Let G be an inverse category. Recall that this means that there to each $g \in G_1$ is a unique $h \in G_1$ such that $(g, h) \in G_2$, $(h, g) \in G_2$, $ghg = g$, and $hgh = h$. In that case, we write $h = g^*$. A subcategory H of G is called an *inverse subcategory* if the restriction of the map $*$ on G_1 to H_1 coincides with the map $*$ on H_1. Note that if G is a groupoid, then G is an inverse category if we for each $g \in G_1$ put $g^* = g^{-1}$. It turns out that, for our purposes, the usual notion of functor is too restrictive when considered for inverse categories. Therefore, we make the following weakening.

Definition 4. Suppose that G and H are inverse categories. A *partial functor of inverse categories* $F : G \to H$ is a pair of maps (F_0, F_1), where $F_0 : G_0 \to H_0$ and $F_1 : G_1 \to H_1$, that satisfy the following axioms:

(I1) If $g : a \to b$ in G_1, then $F_1(g) : F_0(a) \to F_0(b)$.
(I2) If $a \in G_0$, then $F_1(\mathrm{id}_a) = \mathrm{id}_{F_0(a)}$.
(I3) If $(g, h) \in G_2$, then

$$F_1(g)F_1(h) = F_1(g)F_1(g^*)F_1(gh) = F_1(gh)F_1(h^*)F_1(h).$$

By abuse of notation, we will write F for both F_0 and F_1 in the sequel.

Remark 5. Note that if we replace axiom (I3) in Definition 4 by

- If $(g, h) \in G_2$, then $F_1(g)F_1(h) = F_1(gh)$,

then F is an ordinary functor.

Proposition 6. *If G and H are inverse categories and $F : G \to H$ is an ordinary functor, then F is a partial functor of inverse categories.*

Proof. Take $(g, h) \in G_2$. Then, since F is an ordinary functor, we get that $F(g)F(h) = F(gh) = F(gg^*gh) = F(g)F(g^*)F(gh)$ and $F(g)F(h) = F(gh) = F(ghh^*h) = F(gh)F(h^*)F(h)$. □

Proposition 7. *If $F : G \to H$ is a partial functor of inverse categories, then for every $g \in G_1$ the relation $F(g^*) = F(g)^*$ holds.*

Proof. Take a morphism $g : a \to b$ in G_1. If we put $h = \mathrm{id}_a$ in (I3), then we get that $F(g)F(\mathrm{id}_a) = F(g)F(g^*)F(g)$. From (I2), it follows that this relation can be written as

$$F(g) = F(g)F(g^*)F(g). \tag{1}$$

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By replacing g by g^* in (1), we get that

$$F(g^*) = F(g^*)F(g)F(g^*). \tag{2}$$

Equations (1) and (2) show that $F(g^*)$ satisfies the axioms for $F(g)^*$. Since $F(g)^*$ is uniquely defined, we thus get that $F(g^*) = F(g)^*$. □

LEMMA 8. *Suppose that $F : G \to H$ is a partial functor of inverse categories. If $(g, h) \in G_2$ is chosen so that $gh = h$ ($gh = g$), then $F(g)F(h) = F(h)$ ($F(g)F(h) = F(g)$).*

Proof. Suppose that $(g, h) \in G_2$ satisfies $gh = h$. From (I3) and Proposition 7, we get that $F(g)F(h) = F(gh)F(h^*)F(h) = F(h)F(h)^*F(h) = F(h)$. Now suppose that $(g, h) \in G_2$ satisfies $gh = g$. From (I3) and Proposition 7, we get that $F(g)F(h) = F(g)F(g^*)F(gh) = F(g)F(g)^*F(g) = F(g)$. □

PROPOSITION 9. *If $F : G \to G'$ and $F' : G' \to G''$ are partial functors of inverse categories, then $F' \circ F : G \to G''$ is a partial functor of inverse categories.*

Proof. Put $F'' = F' \circ F$. It is easy to see that (I1) and (I2) hold for F''. Now we show (I3). To this end, take $(g, h) \in G_2$. We wish to show that

$$F''(g)F''(h) = F''(g)F''(g^*)F''(gh) \tag{3}$$

and

$$F''(g)F''(h) = F''(gh)F''(h^*)F''(h). \tag{4}$$

First we show (3). Using (I3) for F' and F, we get that the left side of (3) equals

$$\begin{aligned} F'(F(g))F'(F(h)) &= F'(F(g))F'(F(g)^*)F'(F(g)F(h)) \\ &= F'(F(g))F'(F(g)^*)F'(F(g)F(g)^*F(gh)). \end{aligned}$$

Using Lemma 8 for the right side of (3), we get

$$F'(F(g))F'(F(g)^*)F'(F(gh)) = F'(F(g))[F'(F(g)^*)F'(F(g)F(g)^*)]F'(F(gh)).$$

Using (I3) and Lemma 8, the last expression equals

$$\begin{aligned} &F'(F(g))F'(F(g)^*)[F'(F(g)F(g)^*)F'(F(g)F(g)^*)F'(F(gh))] \\ &= F'(F(g))F'(F(g)^*)F'(F(g)F(g)^*F(gh)), \end{aligned}$$

which shows (3). Now we show (4).
Using (I3) for F' and F, we get that the left side of (4) equals

$$\begin{aligned} F'(F(g))F'(F(h)) &= F'(F(g)F(h))F'(F(h)^*)F'(F(h)) \\ &= F'(F(gh)F(h^*)F(h))F'(F(h)^*)F'(F(h)). \end{aligned}$$

The right side of (4) equals

$$F'(F(gh))F'(F(h^*))F'(F(h)). \tag{5}$$

Using Lemma 8, we get that (5) equals

$$F'(F(gh))F'(F(h^*)F(h))F'(F(h^*))F'(F(h)).$$

Using (I3) and Lemma 8, the last expression equals

$$F'(F(gh)F(h^*)F(h))F'(F(h^*)F(h))F'(F(h^*)F(h))F'(F(h^*))F'(F(h)),$$

which equals $F'(F(gh)F(h^*)F(h))F'(F(h^*))F'(F(h))$ showing (4). □

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PROPOSITION 10. *If $F : G \to H$ is a partial functor of inverse categories, where H is a groupoid, then F is an ordinary functor.*

Proof. Take $(g, h) \in G_2$. From Proposition 7, it follows that $F(g)F(h) = F(g)F(g^*)F(gh) = F(g)F(g)^*F(gh) = F(g)F(g)^{-1}F(gh) = F(gh)$. □

DEFINITION 11. Let BIJ denote the groupoid having the collection of all sets as objects and bijections between sets as morphisms.

DEFINITION 12. Let A and B be sets. By a *partial bijection from B to A*, we mean a choice of subsets $Y \subseteq B$ and $X \subseteq A$ and a bijection $f : Y \to X$. We will indicate this by writing ${}_A^X f_B^Y$. We will now define, what we call, *the category of partial bijections* BIJ_{cat}. The class of objects in BIJ_{cat} consists of all sets. The class of morphisms in BIJ_{cat} consists of all partial bijections ${}_A^X f_B^Y$. The domain and codomain of such a morphism are B and A, respectively. The identity morphism at A is defined to be ${}_A^A(\mathrm{id}_A)_A^A$. The empty partial bijection from B to A is denoted by ${}_A 0_B$. Suppose that ${}_A^X f_B^Y$ and ${}_C^{X'} g_D^{Y'}$ are morphisms in BIJ_{cat}. If $B = C$, then the composition of these morphisms is defined to be

$$ {}_A^{f(Y\cap X')} f|_{Y\cap X'} \circ g|_{g^{-1}(Y\cap X')}{}_D^{g^{-1}(Y\cap X')}. $$

Otherwise, the composition is defined to be ${}_A 0_D$. We also define a map $* : (\mathrm{BIJ}_{cat})_1 \to (\mathrm{BIJ}_{cat})_1$ by $\left({}_A^X f_B^Y\right)^* = {}_B^Y f^{-1}{}_A^X$.

PROPOSITION 13. BIJ_{cat} *is an inverse category.*

Proof. Suppose that $\alpha = {}_A^X f_B^Y$, $\beta = {}_B^{X'} g_C^{Y'}$, and $\gamma = {}_C^{X''} h_D^{Y''}$ are morphisms in BIJ_{cat}. First we check the axioms for identity elements:

$$ {}_A^A(\mathrm{id}_A)_A^A \alpha = {}_A^{\mathrm{id}_A(A\cap X)} \left(\mathrm{id}_A|_{A\cap X} \circ f|_{f^{-1}(A\cap X)}\right)_B^{f^{-1}(A\cap X)} = {}_A^X(\mathrm{id}_X \circ f)_B^Y = \alpha $$

and

$$ \alpha_B^B(\mathrm{id}_B)_B^B = {}_A^{f(Y\cap B)} \left(f|_{Y\cap B} \circ \mathrm{id}_B|_{\mathrm{id}_B^{-1}(Y\cap B)}\right)_B^{\mathrm{id}_B^{-1}(Y\cap B)} = {}_A^X (f \circ \mathrm{id}_Y)_B^Y = \alpha. $$

Now we prove associativity. We get that

$$ (\alpha\beta)\gamma = {}_A^{X_1} \left(f|_{Y\cap X'} \circ g|_{g^{-1}(Y\cap X')}\right)|_{g^{-1}(Y\cap X')\cap X''} \circ h|_{Y_1}{}_C^{Y_1}, $$

where

$$ X_1 = \left(f|_{Y\cap X'} \circ g|_{g^{-1}(Y\cap X')}\right)\left(g^{-1}(Y\cap X') \cap X''\right) $$

and

$$ Y_1 = h^{-1}(g^{-1}(Y\cap X') \cap X''). $$

We also get that

$$ \alpha(\beta\gamma) = {}_A^{X_2} f|_{Y\cap g(Y'\cap X'')} \circ (g|_{Y'\cap X''} \circ h|_{h^{-1}(Y'\cap X'')})|_{Y_2}{}_D^{Y_2}, $$

where

$$ X_2 = f(Y \cap g(Y' \cap X'')) $$

and

$$ Y_2 = (g|_{Y'\cap X''} \circ h|_{h^{-1}(Y'\cap X'')})^{-1}(Y \cap g(Y' \cap X'')). $$

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Since composition of functions is associative, we only need to show that $X_1 = X_2$ and $Y_1 = Y_2$.

First we show that $X_1 \subseteq X_2$. Take $a \in X_1$. Then $a = f(g(b))$ for some $b \in X''$ such that $g(b) \in Y \cap X'$. Since $g : Y' \to X'$, we also get that $b \in Y'$. Thus $g(b) \in Y \cap g(Y' \cap X'')$, and hence $a = f(g(b)) \in X_2$. Next we show that $X_2 \subseteq X_1$. Take $c \in X_2$. Then $c = f(g(d))$, for some $d \in Y' \cap X''$ such that $g(d) \in Y$. Since $g : Y' \to X'$, we get that $g(d) \in Y \cap X'$. Thus $d \in g^{-1}(Y \cap X') \cap X''$, and hence $c \in X_1$.

Now we show that $Y_1 \subseteq Y_2$. Take $a \in Y_1$. Then $h(a) \in g^{-1}(Y \cap X') \cap X'' \subseteq Y' \cap X''$. Thus $g(h(a)) \in g(Y' \cap X'')$. Also $g(h(a)) \in g(g^{-1}(Y \cap X')) \subseteq Y$. Hence $g(h(a)) \in Y \cap g(Y' \cap X'')$. So we get that $a \in Y_2$. Next we show that $Y_2 \subseteq Y_1$:

$$\begin{aligned} Y_2 &= \left(h^{-1}|_{h^{-1}(Y' \cap X'')} \circ g|_{Y' \cap X''}^{-1}\right) \left(Y \cap g(Y' \cap X'')\right) \\ &= h^{-1}|_{h^{-1}(Y' \cap X'')} \left(g^{-1}(Y \cap g(Y' \cap X'')) \cap Y' \cap X''\right) \\ &= h^{-1}\left(g^{-1}(Y \cap g(Y' \cap X'')) \cap Y' \cap X''\right) \cap h^{-1}\left(Y' \cap X''\right). \end{aligned}$$

Since $g(Y' \cap X'') \subseteq X'$ and $Y' \cap X'' \subseteq X''$, we thus get that $Y_2 \subseteq Y_1$.

Finally, we show that BIJ_{cat} is an inverse category. To this end, first note that

$$\alpha\alpha^* = {}_A^X f_B^Y \left({}_A^X f_B^Y\right)^* = {}_A^X f_B^Y {}_B^Y f^{-1}{}_A^X = {}_A^X \mathrm{id}_X{}_A^X \tag{6}$$

and

$$\alpha^*\alpha = \left({}_A^X f_B^Y\right)^* {}_A^X f_B^Y = {}_B^Y f^{-1}{}_A^X {}_A^X f_B^Y = {}_B^Y \mathrm{id}_Y{}_B^Y. \tag{7}$$

Thus, it follows that

$$\alpha\alpha^*\alpha = {}_A^X \mathrm{id}_X{}_A^X {}_A^X f_B^Y = {}_A^X f_B^Y = \alpha$$

and

$$\alpha^*\alpha\alpha^* = {}_B^Y \mathrm{id}_Y{}_B^Y {}_B^Y f^{-1}{}_A^X = {}_B^Y f^{-1}{}_A^X = \alpha^*.$$

Next suppose that

$$\alpha\beta\alpha = \alpha \tag{8}$$

and

$$\beta\alpha\beta = \beta, \tag{9}$$

where $\beta = {}_B^{X'} g_A^{Y'}$. From (6) and (8), it follows that

$$\begin{aligned} {}_B^Y f^{-1}{}_A^X &= \alpha^* = \alpha^*\alpha\alpha^* = \alpha^*\alpha\beta\alpha\alpha^* = {}_B^Y \mathrm{id}_Y{}_B^Y {}_B^{X'} g_A^{Y'} {}_A^X \mathrm{id}_X{}_A^X \\ &= {}_B^{g(g^{-1}(Y \cap X') \cap X)} g|_{g^{-1}(Y \cap X') \cap X}{}_A^{g^{-1}(Y \cap X') \cap X}. \end{aligned}$$

Thus, we get that $Y = g(g^{-1}(Y \cap X') \cap X) \subseteq X'$ and $X = g^{-1}(Y \cap X') \cap X \subseteq Y'$. Analogously, from (9), it follows that $X' \subseteq Y$ and $Y' \subseteq X$. Thus $X' = Y$ and $Y' = X$, and hence it follows that $\beta = \alpha^*$. □

3. Preliminaries on rings. In this section, we introduce the category of partial (commutative) ring isomorphisms ISO_{cat} (ISOC_{cat}). This category contains the well-known groupoid ISO (ISOC) having all (commutative) rings as objects and ring isomorphisms as

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morphisms. We also show a result (see Proposition 18), concerning central idempotents in rings, that we need in subsequent sections. Let A be a ring. We always assume that A is associative and equipped with a multiplicative identity element 1_A.

DEFINITION 14. Let ISO (ISOC) denote the category having all (commutative) rings as objects and ring isomorphisms as morphisms.

REMARK 15. Recall that an ideal I of a ring A is said to be a *unital ideal* if I, viewed as a ring in itself, is unital. In this case, the multiplicative identity element of I is denoted by 1_I and lies in the center of A. Indeed, let $a \in A$ be arbitrary. Since I is an ideal of A, it follows that $1_I a, a1_I \in I$. Thus, $1_I a = 1_I a 1_I = a 1_I$. Furthermore, if I and J are unital ideals of a ring A, then $I \cap J = IJ$.

Now we will define two ring versions of the inverse category BIJ_{cat} that we defined in the previous section.

DEFINITION 16. Let ISO_{cat} (ISOC_{cat}) denote the subcategory of BIJ_{cat} having (commutative) rings as objects and, as morphisms, all ${}_A^I f_B^J$ in BIJ_{cat} such that I and J are unital ideals of A and B, respectively and $f : J \to I$ is a ring isomorphism. Note that the composition of two morphisms, ${}_A^I f_B^J$ and ${}_B^{I'} g_C^{J'}$, in these categories equals ${}_A^{f(JI')} f|_{JI'} \circ g|_{g^{-1}(JI')}{}_C^{g^{-1}(JI')}$. Define a map $* : (\mathrm{ISO}_{cat})_1 \to (\mathrm{ISO}_{cat})_1$ by restriction of the map $*$ defined on $(\mathrm{BIJ}_{cat})_1$. This restricts, in turn, to a map $* : (\mathrm{ISOC}_{cat})_1 \to (\mathrm{ISOC}_{cat})_1$.

The following is clear.

PROPOSITION 17. ISO_{cat} *and* ISOC_{cat} *are inverse subcategories of* BIJ_{cat}.

The *center of* A, denoted by $Z(A)$, is the subring of A consisting of all elements $a \in A$ with the property that for all $b \in A$ the equality $ab = ba$ holds. We let $\mathrm{idem}(A)$ denote the set of all central idempotents of A and we let $\mathrm{ideal}(A)$ denote the set of all unital ideals of A.

PROPOSITION 18. *Let A be a ring. The map $\theta : \mathrm{idem}(A) \to \mathrm{ideal}(A)$, defined by $\theta(x) = Ax$, for $x \in \mathrm{idem}(A)$, is an isomorphism of multiplicative monoids. For all $x \in \mathrm{idem}(A)$ the equality $Z(Ax) = Z(A)x$ holds.*

Proof. It is clear that θ is a homomorphism of multiplicative monoids. Take $x, y \in \mathrm{idem}(A)$ such that $\theta(x) = \theta(y)$. Then $Ax = Ay$. Since both x and y are multiplicative identity elements for the same monoid, it follows that $x = y$. Thus θ is injective. Now we show that θ is surjective. Take $I \in \mathrm{ideal}(A)$. Recall that $1_I \in Z(A)$, by Remark 15. By the idempotency of 1_I, we get that $\theta(1_I) = A1_I = I$. Thus the surjectivity of θ follows. For the last part, take $x \in \mathrm{idem}(A)$. The inclusion $Z(A)x \subseteq Z(Ax)$ clearly holds. Take $y \in Z(Ax)$. Then $y = ax$ for some $a \in A$. Clearly, $yx = ax^2 = ax = y$, since x is idempotent. Thus, it suffices to show that $y \in Z(A)$. Take $b \in A$. Then, since $x \in Z(A)$ and $y \in Z(Ax)$, we get that $yb = yxb = ybx = bxy = byx = by$. □

4. The Picard inverse category. In this section, we recall the definition of (pre-)equivalence data and some properties of such systems. Then we introduce the Picard inverse category PIC_{cat} (see Definition 27 and Theorem 28). From [3, Definition (3.2)], we recall the following.

DEFINITION 19. A *set of pre-equivalence data* $(I, J, P, Q, \alpha, \beta)$ consists of rings I and J, an I–J-bimodule P, a J–I-bimodule Q, an I-bimodule homomorphism

$$\alpha : P \otimes_J Q \to I, \tag{10}$$

and an I-bimodule homomorphism

$$\beta : Q \otimes_I P \to J, \tag{11}$$

such that the following two diagrams commute:

$$\begin{array}{ccc} P \otimes_J Q \otimes_I P & \xrightarrow{\alpha\otimes \mathrm{id}_P} & I \otimes_I P \\ {\scriptstyle \mathrm{id}_P\otimes\beta}\downarrow & & \downarrow \\ P \otimes_J J & \longrightarrow & P \end{array} \qquad \begin{array}{ccc} Q \otimes_I P \otimes_J Q & \xrightarrow{\beta\otimes \mathrm{id}_Q} & J \otimes_J Q \\ {\scriptstyle \mathrm{id}_Q\otimes\alpha}\downarrow & & \downarrow \\ Q \otimes_I I & \longrightarrow & Q, \end{array} \tag{12}$$

where the unlabelled arrows are the multiplication maps. We shall call it a *set of equivalence data* if α and β are isomorphisms.

Now we gather some well-known properties concerning pre-equivalence data that we need in the sequel.

PROPOSITION 20. *If $(I, J, P, Q, \alpha, \beta)$ is a set of pre-equivalence data such that α (or β) is surjective, then the following assertions hold:*

(a) α (or β) is an isomorphism;
(b) P and Q are generators as I-modules (or J-modules); and
(c) P and Q are finitely generated and projective J-modules (I-modules).

Proof. See [3, Theorem (3.4)]. □

PROPOSITION 21. *If $(I, J, P, Q, \alpha, \beta)$ is a set of equivalence data, then the ring homomorphisms*

$$\mathrm{End}_J(P) \leftarrow I \to \mathrm{End}_J(Q)^{\mathrm{op}}$$

and

$$\mathrm{End}_I(P)^{\mathrm{op}} \leftarrow J \to \mathrm{End}_I(Q),$$

induced by the bimodule structure on P and Q, are isomorphisms. These isomorphisms restrict to ring isomorphisms

$$\mathrm{End}_{I-J}(P) \leftarrow Z(I) \to \mathrm{End}_{J-I}(Q)$$

and

$$\mathrm{End}_{I-J}(P) \leftarrow Z(J) \to \mathrm{End}_{J-I}(Q),$$

which in turn restrict to group isomorphisms

$$\mathrm{Aut}_{I-J}(P) \leftarrow U(Z(I)) \to \mathrm{Aut}_{J-I}(Q)$$

and

$$\mathrm{Aut}_{I-J}(P) \leftarrow U(Z(J)) \to \mathrm{Aut}_{J-I}(Q).$$

Proof. See [3, Theorem (3.5)]. □

REMARK 22. Let $(I, J, P, Q, \alpha, \beta)$ be a set of equivalence data. It follows from Proposition 21 that there is a unique ring isomorphism $\gamma_P : Z(J) \to Z(I)$ with the property that for all $p \in P$ and all $b \in Z(J)$ the equality $\gamma_P(b)p = pb$ holds.

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DEFINITION 23. A *set of partial equivalence data*

$$(A, B, I, J, P, Q, \alpha, \beta)$$

consists of rings A and B, unital ideals I and J of, respectively, A and B such that $(I, J, P, Q, \alpha, \beta)$ is a set of equivalence data, P is an I–J-bimodule, and Q is a J–I-bimodule.

REMARK 24. Note that, with the notation and assumptions of Definition 23, P (resp. Q) extends uniquely to an A–B-bimodule (resp. B–A-bimodule). Thus, we may interchangeably think of P (resp. Q) as an A–B-bimodule or an I–J-bimodule (resp. B–A-bimodule or J–I-bimodule).

PROPOSITION 25. *Suppose that*

$$(A, B, I, J, P, Q, \alpha, \beta) \tag{13}$$

and

$$(B, C, I', J', P', Q', \alpha', \beta') \tag{14}$$

are sets of partial equivalence data. Then

$$(A, C, I'', J'', P'', Q'', \alpha'', \beta'')$$

is a set of partial equivalence data, where

$$I'' = \gamma_P(1_J 1_{I'})A, \quad J'' = \gamma_{P'}^{-1}(1_J 1_{I'})C, \quad P'' = P \otimes_B P', \quad Q'' = Q' \otimes_B Q, \tag{15}$$

and for $p \in P$, $p' \in P'$, $q \in Q$, $q' \in Q'$, we put

$$\alpha''(p \otimes p' \otimes q' \otimes q) = \alpha(p\alpha'(p' \otimes q') \otimes q)$$

and

$$\beta''(q' \otimes q \otimes p \otimes p') = \beta'(q' \otimes \beta(q \otimes p)p').$$

Proof. We begin by noticing that $P' \otimes_C Q' \cong P' \otimes_{J'} Q'$. Indeed, J' is a unital ideal of C, and hence $J' = 1_{J'}C$. Moreover, P' is a right J'-module and Q' is a left J'-module. Thus, $P' \otimes_C Q' \ni p' \otimes_C q' \mapsto p' \otimes_{J'} q' \in P' \otimes_{J'} Q'$ is a well-defined isomorphism of I'-bimodules (and B-bimodules).

The map $\alpha'' : P'' \otimes_C Q'' \to I''$ is an isomorphism of A-bimodules since it is the composition of the following chain of A-bimodule isomorphisms:

$$\begin{aligned} P \otimes_B P' \otimes_C Q' \otimes_B Q &\cong P \otimes_B P' \otimes_{J'} Q' \otimes_B Q \cong P \otimes_B I' \otimes_B Q \\ &= P \otimes_B 1_{I'}B \otimes_B Q = P1_{I'} \otimes_B B \otimes_B Q \\ &\cong P1_J 1_{I'} \otimes_B Q \\ &\cong \gamma_P(1_J 1_{I'})P \otimes_B Q \cong \gamma_P(1_J 1_{I'})I = \gamma_P(1_J 1_{I'})A. \end{aligned}$$

Analogously, the map $\beta'' : Q'' \otimes_A P'' \to J''$ is an isomorphism of C-bimodules since it is the composition of the following chain of C-bimodule isomorphisms:

$$\begin{aligned} Q' \otimes_B Q \otimes_A P \otimes_B P' &\cong Q' \otimes_B J \otimes_B P' = Q' \otimes_B 1_J B \otimes_B P' \\ &= Q' \otimes_B B \otimes_B 1_J P' \cong Q' \otimes_B 1_J 1_{I'} P' \\ &\cong Q' \otimes_B P' \gamma_{P'}^{-1}(1_J 1_{I'}) \cong J' \gamma_{P'}^{-1}(1_J 1_{I'}) = \gamma_{P'}^{-1}(1_J 1_{I'})C. \end{aligned}$$

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From Proposition 18, it follows that $I'' = \gamma_P(1_J 1_{I'})A$ and $J'' = \gamma_{P'}^{-1}(1_J 1_{I'})C$ are well defined. Now we verify the diagrams in (12). By abuse of notation, we let m denote all of the various multiplication maps. Take $p_1, p_2 \in P$, $p'_1, p'_2 \in P'$, $q_1, q_2 \in Q$, and $q'_1, q'_2 \in Q'$. Put $p''_1 = p_1 \otimes p'_1$, $p''_2 = p_2 \otimes p'_2$, $q''_1 = q'_1 \otimes q_1$, and $q''_2 = q'_2 \otimes q_2$. Then, by making use of (12) for (13) and (14), we get that

$$\begin{aligned}
(m \circ (\alpha'' \otimes \mathrm{id}_{P''}))(p''_1 \otimes q''_1 \otimes p''_2) &= (m \circ (\alpha'' \otimes \mathrm{id}_{P''}))(p_1 \otimes p'_1 \otimes q'_1 \otimes q_1 \otimes p_2 \otimes p'_2) \\
&= \alpha(p_1 \alpha'(p'_1 \otimes q'_1) \otimes q_1) p_2 \otimes p'_2 \\
&= (m \circ (\alpha \otimes \mathrm{id}_P))(p_1 \alpha'(p'_1 \otimes q'_1) \otimes q_1 \otimes p_2) \otimes p'_2 \\
&= (m \circ (\mathrm{id}_P \otimes \beta))(p_1 \alpha'(p'_1 \otimes q'_1) \otimes q_1 \otimes p_2) \otimes p'_2 \\
&= p_1 \alpha'(p'_1 \otimes q'_1) \beta(q_1 \otimes p_2) \otimes p'_2 \\
&= p_1 \otimes \alpha'(p'_1 \otimes q'_1) \beta(q_1 \otimes p_2) p'_2 \\
&= p_1 \otimes (m \circ (\alpha' \otimes \mathrm{id}_{P'}))(p'_1 \otimes q'_1 \otimes \beta(q_1 \otimes p_2) p'_2) \\
&= p_1 \otimes (m \circ (\mathrm{id}_{P'} \otimes \beta'))(p'_1 \otimes q'_1 \otimes \beta(q_1 \otimes p_2) p'_2) \\
&= p_1 \otimes p'_1 \beta'(q'_1 \otimes \beta(q_1 \otimes p_2) p'_2) \\
&= (m \circ (\mathrm{id}_{P''} \otimes \beta''))(p''_1 \otimes q''_1 \otimes p''_2)
\end{aligned}$$

and

$$\begin{aligned}
(m \circ (\mathrm{id}_{Q''} \otimes \alpha''))(q''_1 \otimes p''_1 \otimes q''_2) &= q'_1 \otimes q_1 \alpha(p_1 \alpha'(p'_1 \otimes q'_2) \otimes q_2) \\
&= q'_1 \otimes (m \circ (\mathrm{id}_Q \otimes \alpha))(q_1 \otimes p_1 \alpha'(p'_1 \otimes q'_2) \otimes q_2) \\
&= q_1 \otimes (m \circ (\beta \otimes \mathrm{id}_Q))(q_1 \otimes p_1 \alpha'(p'_1 \otimes q'_2) \otimes q_2) \\
&= q'_1 \otimes \beta(q_1 \otimes p_1 \alpha'(p'_1 \otimes q'_2)) q_2 \\
&= q'_1 \otimes \beta(q_1 \otimes p_1) \alpha'(p'_1 \otimes q'_2) q_2 \\
&= q'_1 \beta(q_1 \otimes p_1) \alpha'(p'_1 \otimes q'_2) \otimes q_2 \\
&= q'_1 \alpha'(\beta(q_1 \otimes p_1) p'_1 \otimes q'_2) \otimes q_2 \\
&= (m \circ (\mathrm{id}_{Q'} \otimes \alpha'))(q'_1 \otimes \beta(q_1 \otimes p_1) p'_1 \otimes q'_2) \otimes q_2 \\
&= (m \circ (\beta' \otimes \mathrm{id}_{Q'}))(q'_1 \otimes \beta(q_1 \otimes p_1) p'_1 \otimes q'_2) \otimes q_2 \\
&= \beta'(q'_1 \otimes \beta(q_1 \otimes p_1) p'_1) q'_2 \otimes q_2 \\
&= (m \circ (\beta'' \otimes \mathrm{id}_{Q''}))(q''_1 \otimes p''_1 \otimes q''_2),
\end{aligned}$$

which finishes the proof. □

To motivate the approach taken later, we now recall the definition of the Picard groupoid PIC.

DEFINITION 26. Let PIC denote the category having as objects all unital rings. A morphism in PIC from B to A is the collection of all A–B-bimodule isomorphism classes $[P]$, for invertible A–B-bimodules P. Given two such classes $[P]$ and $[Q]$, where $d([P]) = B = c([Q])$, we put $[P][Q] = [P \otimes_B Q]$. Then PIC is a groupoid. Indeed, if P is an invertible A–B-bimodule, then there is an invertible B–A-bimodule Q such that $P \otimes_B Q \cong A$ (as A-bimodules) and $Q \otimes_A P \cong B$ (as B-bimodules). Thus, if we put $[P]^{-1} = [Q]$, then, clearly $[P][P]^{-1} = [A]$ and $[P]^{-1}[P] = [B]$.

DEFINITION 27. Let P be an A–B-bimodule and suppose that I and J are unital ideals of, respectively, A and B, making P an I–J-bimodule. We will indicate this by writing ${}_A^I P_B^J$. We say that ${}_A^I P_B^J$ is *partially invertible* if there is ${}_B^J Q_A^I$ and maps α and β such that

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$(A, B, I, J, P, Q, \alpha, \beta)$ is a set of partial equivalence data. Let PART denote the collection of all partially invertible bimodules ${}_A^I P_B^J$. Define an equivalence relation $\sim$ on PART by saying that ${}_A^I P_B^J \sim {}_{A'}^{I'} P'^{J'}_{B'}$ if

$$(A, B, I, J) = (A', B', I'J') \quad \text{and} \quad P \cong P' \quad \text{as } I\text{–}J\text{-bimodules.}$$

The equivalence class of ${}_A^I P_B^J$ in PART will be denoted by $\left[{}_A^I P_B^J\right]$. The class of objects in PIC_{cat} consists of all rings. The class of morphisms in PIC_{cat} consists of all equivalence classes $\left[{}_A^I P_B^J\right]$ of partially invertible modules ${}_A^I P_B^J$. Define the domain and codomain of a morphism $\left[{}_A^I P_B^J\right]$ in PIC_{cat} by the relations $d(\left[{}_A^I P_B^J\right]) = B$ and $c(\left[{}_A^I P_B^J\right]) = A$, respectively. Given a ring A, the identity morphism at A is defined to be the morphism $\left[{}_A^A A_A^A\right]$. Given two morphisms $\left[{}_A^I P_B^J\right]$ and $\left[{}_B^{I'} P'^{J'}_C\right]$ in PIC_{cat} put $\left[{}_A^I P_B^J\right]\left[{}_B^{I'} P'^{J'}_C\right] = \left[{}_A^{I''} P''^{J''}_C\right]$, where I'', P'', and J'' are defined in Proposition 25. It is clear that the morphisms of the form $\left[{}_A^A A_A^A\right]$, for rings A, satisfy the axioms for identity morphisms of PIC_{cat}. If $\left[{}_A^I P_B^J\right] \in (\mathrm{PIC}_{cat})_1$, then there is ${}_B^J Q_A^I$ and maps α and β such that $(A, B, I, J, P, Q, \alpha, \beta)$ is a set of partial equivalence data. Put $\left[{}_A^I P_B^J\right]^* = \left[{}_B^J Q_A^I\right]$.

THEOREM 28. PIC_{cat} *is an inverse category.*

Proof. First we show that the partial composition in $(\mathrm{PIC}_{cat})_1$ is associative. Suppose that $[{}_A^{I_1} P_1{}_B^{J_1}]$, $[{}_B^{I_2} P_2{}_C^{J_2}]$, and $[{}_C^{I_3} P_3{}_D^{J_3}]$ are morphisms in PIC_{cat}. We need to show that

$$\left(\left[{}_A^{I_1} P_1{}_B^{J_1}\right]\left[{}_B^{I_2} P_2{}_C^{J_2}\right]\right)\left[{}_C^{I_3} P_3{}_D^{J_3}\right] = \left[{}_A^{I_1} P_1{}_B^{J_1}\right]\left(\left[{}_B^{I_2} P_2{}_C^{J_2}\right]\left[{}_C^{I_3} P_3{}_D^{J_3}\right]\right). \tag{16}$$

By repeated application of the composition in PIC_{cat}, it follows that (16) is equivalent to showing the equalities

$$\gamma_{P_1 \otimes_B P_2}(\gamma_{P_2}^{-1}(1_{J_1} 1_{I_2}) 1_{I_3}) = \gamma_{P_1}(1_{J_1} \gamma_{P_2}(1_{J_2} 1_{I_3})) \tag{17}$$

and

$$\gamma_{P_3}^{-1}(\gamma_{P_2}^{-1}(1_{J_1} 1_{I_2}) 1_{I_3}) = \gamma_{P_2 \otimes_C P_3}^{-1}(1_{J_1} \gamma_{P_2}(1_{J_2} 1_{I_3})). \tag{18}$$

First we show (17). Take $p_1 \in P_1$ and $p_2 \in P_2$. Then

$$\begin{aligned} p_1 \otimes p_2 \gamma_{P_2}^{-1}(1_{J_1} 1_{I_2}) 1_{I_3} &= p_1 \otimes p_2 \gamma_{P_2}^{-1}(1_{J_1} 1_{I_2}) 1_{J_2} 1_{I_3} = p_1 \otimes 1_{J_1} 1_{I_2} p_2 1_{J_2} 1_{I_3} \\ &= p_1 \otimes 1_{J_1} p_2 1_{J_2} 1_{I_3} = p_1 1_{J_1} \otimes p_2 1_{J_2} 1_{I_3} \\ &= p_1 1_{J_1} \otimes \gamma_{P_2}(1_{J_2} 1_{I_3}) p_2 = p_1 1_{J_1} \gamma_{P_2}(1_{J_2} 1_{I_3}) \otimes p_2 \\ &= \gamma_{P_1}(1_{J_1} \gamma_{P_2}(1_{J_2} 1_{I_3})) p_1 \otimes p_2. \end{aligned}$$

Now we show (18). Take $p_2 \in P_2$ and $p_3 \in P_3$. Then

$$\begin{aligned} 1_{J_1} \gamma_{P_2}(1_{J_2} 1_{I_3}) p_2 \otimes p_3 &= 1_{J_1} 1_{I_2} \gamma_{P_2}(1_{J_2} 1_{I_3}) p_2 \otimes p_3 = 1_{J_1} 1_{I_2} p_2 1_{J_2} 1_{I_3} \otimes p_3 \\ &= 1_{J_1} 1_{I_2} p_2 1_{I_3} \otimes p_3 = p_2 \gamma_{P_2}^{-1}(1_{J_1} 1_{I_2}) 1_{I_3} \otimes p_3 \\ &= p_2 \otimes \gamma_{P_2}^{-1}(1_{J_1} 1_{I_2}) 1_{I_3} p_3 = p_2 \otimes p_3 \gamma_{P_3}^{-1}(\gamma_{P_2}^{-1}(1_{J_1} 1_{I_2}) 1_{I_3}), \end{aligned}$$

as desired. Next we show the axioms for $*$. Take $g = \left[{}_A^I P_B^J\right] \in (\mathrm{PIC}_{cat})_1$. Then there is ${}_B^J Q_A^I$ and maps α and β such that $(A, B, I, J, P, Q, \alpha, \beta)$ is a set of partial equivalence data. Put $g^* = \left[{}_B^J Q_A^I\right]$. Then

$$gg^* = [{}^I_A P^J_B][{}^J_B Q^I_A] = \left[{}^{\gamma_P(1_J 1_J)A}_{A}(P \otimes_B Q)^{\gamma_Q^{-1}(1_J 1_J)A}_{A}\right]$$
$$= \left[{}^{1_I A}_{A}(P \otimes_B Q)^{1_I A}_{A}\right] = \left[{}^I_A I^I_A\right].$$

Using this we get that

$$gg^*g = \left[{}^I_A I^I_A\right]\left[{}^I_A P^J_B\right] = \left[{}^{\gamma_I(1_I 1_I)A}_{A}(I \otimes_A P)^{\gamma_P^{-1}(1_I 1_I)B}_{B}\right] = \left[{}^{1_I A}_{A} P^{1_J B}_{B}\right] = \left[{}^I_A P^J_B\right] = g$$

and

$$g^*gg^* = \left[{}^J_B Q^I_A\right]\left[{}^I_A I^I_A\right] = \left[{}^{\gamma_Q(1_I 1_I)B}_{B}(Q \otimes_A I)^{\gamma_I^{-1}(1_I 1_I)A}_{A}\right] = \left[{}^{1_J B}_{B} Q^{1_I A}_{A}\right] = \left[{}^J_B Q^I_A\right] = g^*.$$

Now we show uniqueness of g^*. To this end, first note that

$$g^*g = \left[{}^J_B Q^I_A\right]\left[{}^I_A P^J_B\right] = \left[{}^{\gamma_Q(1_I 1_I)B}_{B}(Q \otimes_A P)^{\gamma_P^{-1}(1_I 1_I)B}_{B}\right] = \left[{}^{1_J B}_{B} J^{1_J B}_{B}\right] = \left[{}^J_B J^J_B\right].$$

Next, suppose that

$$ghg = g \quad \text{and} \quad hgh = h \tag{19}$$

for some $h = \left[{}^K_B M^L_A\right]$ with $h^* = \left[{}^L_A N^K_B\right]$. From the first equality in (19), it follows that $g^*ghgg^* = g^*gg^*$, and thus that $\left[{}^J_B J^J_B\right]\left[{}^K_B M^L_A\right]\left[{}^I_A I^I_A\right] = \left[{}^J_B Q^I_A\right]$. Rewriting the last equality we get that

$$\left[{}^{\gamma_J(1_J\gamma_M(1_L 1_I))B}_{B}(J \otimes_B M \otimes_A I)^{\gamma^{-1}_{M\otimes_A I}(1_J\gamma_M(1_L 1_I))A}_{A}\right] = [{}^J_B Q^I_A] \tag{20}$$

and thus that

$$\gamma_J(1_J\gamma_M(1_L 1_I))B = J \tag{21}$$

and

$$\gamma^{-1}_{M\otimes_A I}(1_J\gamma_M(1_L 1_I))A = I. \tag{22}$$

Since γ_J is the identity map $Z(J) \to Z(J)$, (21) implies that $1_J\gamma_M(1_L 1_I)B = J$, and hence, in particular, that $J \subseteq K$. Using that γ_I equals the identity map on $Z(I)$, it follows that $\gamma^{-1}_{M\otimes_A I} : Z(\gamma_M(1_L 1_I)) \to Z(1_L 1_I)$. From (22), it therefore, in particular, follows that $I \subseteq L$. From the second equality in (19), it follows, by symmetry, that $J \subseteq K$ and $L \subseteq I$. Thus $J = K$ and $L = I$, and hence from (20), it follows that $h = g^*$. □

EXAMPLE 29 (**The Picard semigroup of a commutative ring).** Let R be a unital commutative ring and let M be a finitely generated (central) R-bimodule of rank less than or equal to one, that is, $\mathbf{rk}(M_{\mathfrak{p}}) \leq 1$, for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Let $M^* = \mathrm{Hom}_R(M, R)$ be the dual of M. Then by [8, Proposition 3.8, Lemma 3.9], there exists $e \in \mathrm{idem}(R)$ and an R-bimodule isomorphism $\alpha : M \otimes_R M^* \to Re$, given by $\alpha(m \otimes f) := f(m)$, for all $f \in M^*$ and $m \in M$. Moreover, by [8, Lemma 3.10], the isomorphisms classes of M and M^* are elements of the Picard group $\mathrm{Pic}(Re)$. In particular, both M and M^* are unital Re-bimodules. From this, we get that $(R, R, Re, Re, M, M^*, \alpha, \alpha)$ is a set of partial equivalence data. By [8, Proposition 3.8], the inverse subcategory of PIC_{cat}, whose only object is R and whose morphisms are of the form $[{}^{Re}_R M^{Re}_R]$, is a commutative inverse semigroup, denoted by $\mathrm{PicS}(R)$. It was defined in [8, Definition 3.1] and is called *the Picard semigroup of* R.

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DEFINITION 30. Now we will define a partial functor of inverse categories $L: \mathrm{PIC}_{cat} \to \mathrm{ISOC}_{cat}$. If A is a ring, then put $L(A) = Z(A)$. If $\left[{}_A^I P_B^J\right]$ is a morphism in PIC_{cat}, then put $L\left(\left[{}_A^I P_B^J\right]\right) = {}_{Z(A)}^{Z(I)}\gamma_P{}_{Z(B)}^{Z(J)}$, where $\gamma_P : Z(J) \to Z(I)$ is the ring isomorphism defined in Remark 22.

PROPOSITION 31. *The map $L: \mathrm{PIC}_{cat} \to \mathrm{ISOC}_{cat}$ is a functor and hence, by Proposition 6, a partial functor of inverse categories.*

Proof. Take morphisms $g = \left[{}_A^I P_B^J\right]$ and $h = \left[{}_B^{I'} P_C^{J'}\right]$ in PIC_{cat}. Then

$$L(gh) = {}_{Z(A)}^{\gamma_P(1_J 1_{I'})Z(A)}\gamma_{P\otimes_B P'}{}_{Z(C)}^{\gamma_{P'}^{-1}(1_J 1_{I'})Z(C)}$$

and

$$L(g)L(h) = {}_{Z(A)}^{\gamma_P(Z(J)Z(I'))}\gamma_P\big|_{Z(J)Z(I')} \circ \gamma_{P'}\big|_{\gamma_{P'}^{-1}(Z(J)Z(I'))}{}_{Z(C)}^{\gamma_{P'}^{-1}(Z(J)Z(I'))}.$$

Note that

$$\gamma_P(Z(J)Z(I')) = \gamma_P(1_J 1_{I'})Z(A)$$

and

$$\gamma_{P'}^{-1}(Z(J)Z(I')) = \gamma_{P'}^{-1}(1_J 1_{I'})Z(C).$$

Put $\gamma_1 = \gamma_P|_{Z(J)Z(I')}$ and $\gamma_2 = \gamma_{P'}|_{\gamma_{P'}^{-1}(Z(J)Z(I'))}$. If $a \in \gamma_{P'}^{-1}(1_J 1_{I'})Z(C)$, $p \in P$ and $p' \in P'$, then

$$\gamma_{P\otimes_B P'}(a)p \otimes p' = p \otimes p'a = p \otimes \gamma_2(a)p' = p\gamma_2(a) \otimes p' = (\gamma_1 \circ \gamma_2)(a)p \otimes p'.$$

From Remark 22, it therefore follows that $\gamma_{P\otimes_B P'} = \gamma_1 \circ \gamma_2$. □

5. Epsilon-strongly groupoid-graded rings. In this section, we recall the definition of groupoid-graded rings and some of their properties. Then we define epsilon-strongly groupoid-graded rings (see Definition 34) and provide a characterization of them which generalizes an analogous result for group-graded rings (see Proposition 37). Throughout this section, S denotes a ring which is graded by a small groupoid G. Recall from [**13**, **14**] that this means that there is a set of additive subgroups $\{S_g\}_{g\in G}$ of S such that $S = \oplus_{g\in G} S_g$ and, for all $g, h \in G_1$, $S_g S_h \subseteq S_{gh}$, if $(g, h) \in G_2$, and $S_g S_h = \{0\}$, if $(g, h) \notin G_2$. In that case, S is called *strongly graded* if for all $(g, h) \in G_2$ the equality $S_g S_h = S_{gh}$ holds. Given two G-graded rings S and T, a ring homomorphism $f: S \to T$ is called *graded* if for all $g \in G_1$ the inclusion $f(S_g) \subseteq T_g$ holds. We put $R = \oplus_{e\in G_0} S_e$. From the next result, it follows that we may always assume that G_0 is finite.

PROPOSITION 32. *With the above notation, we get that $1_S \in R$. If we put $G'_0 = \{e \in G_0 \mid (1_S)_e \neq 0\}$ and $G'_1 = \{g \in G_1 \mid (1_S)_{d(g)}, (1_S)_{c(g)} \neq 0\}$, then G' is a subgroupoid of G such that G'_0 is finite and $S = \oplus_{g\in G'_1} S_g$.*

Proof. This follows from [**13**, Proposition 2.1.1]. □

PROPOSITION 33. *The ring S is strongly graded if and only if for all $g \in G_1$ the inclusion $1_{S_{c(g)}} \in S_g S_{g^{-1}}$ holds.*

Proof. This follows from [**14**, Lemma 3.2]. □

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DEFINITION 34. The ring S is said to be *epsilon-strongly graded by* G if, for each $g \in G_1$, $S_g S_{g^{-1}}$ is a unital ideal of $S_{c(g)}$ such that for all $(g, h) \in G_2$ the equalities $S_g S_h = S_g S_{g^{-1}} S_{gh} = S_{gh} S_{h^{-1}} S_h$ hold.

REMARK 35. It follows from Propositions 32 and 33 that if S is strongly graded, then S is epsilon-strongly graded.

REMARK 36. Suppose that S is epsilon-strongly graded by G and $g \in G_1$. Then by the definition of R, the $S_{c(g)}$-ideal $S_g S_{g^{-1}}$ is a unital ideal of R. Moreover, if ϵ_g is its multiplicative identity element, then for $r \in R$, we get that $\epsilon_g r, r\epsilon_g \in S_g S_{g^{-1}}$. Therefore, $\epsilon_g r = (\epsilon_g r)\epsilon_g = \epsilon_g(r\epsilon_g) = r\epsilon_g$, which shows that $\epsilon_g \in Z(R)$, and $S_g S_{g^{-1}} = \epsilon_g S_{c(g)} = \epsilon_g R$.

We now wish to show an epsilon-analogue of Proposition 33.

PROPOSITION 37. *The ring* S *is epsilon-strongly graded by* G *if and only if for each* $g \in G_1$ *there is* $\epsilon_g \in S_g S_{g^{-1}}$ *such that for each* $s \in S_g$ *the equalities* $\epsilon_g s = s = s\epsilon_{g^{-1}}$ *hold.*

Proof. First we show the "only if" statement. Suppose that S is epsilon-strongly graded. Take $g \in G_1$. Let ϵ_g denote $1_{S_g S_{g^{-1}}}$. Take $s_g \in S_g$. From Proposition 32, it follows that $S_g S_{g^{-1}} S_g = S_g$. Therefore, there is a positive integer n and $a_g^{(i)}, c_g^{(i)} \in S_g$, and $b_{g^{-1}}^{(i)} \in S_{g^{-1}}$, for $i \in \{1, \ldots, n\}$, such that $s_g = \sum_{i=1}^n a_g^{(i)} b_{g^{-1}}^{(i)} c_g^{(i)}$. Since $\epsilon_g = 1_{S_g S_{g^{-1}}}$ and $\epsilon_{g^{-1}} = 1_{S_{g^{-1}} S_g}$, we get that

$$\epsilon_g s_g = \sum_{i=1}^n (\epsilon_g a_g^{(i)} b_{g^{-1}}^{(i)}) c_g^{(i)} = \sum_{i=1}^n a_g^{(i)} b_{g^{-1}}^{(i)} c_g^{(i)} = s_g$$

and

$$s_g \epsilon_{g^{-1}} = \sum_{i=1}^n a_g^{(i)} (b_{g^{-1}}^{(i)} c_g^{(i)} \epsilon_{g^{-1}}) = \sum_{i=1}^n a_g^{(i)} b_{g^{-1}}^{(i)} c_g^{(i)} = s_g.$$

Now we show the "if" statement. Suppose that to each $g \in G_1$ there is $\epsilon_g \in S_g S_{g^{-1}}$ such that for each $s \in S_g$ the equalities $\epsilon_g s = s = s\epsilon_{g^{-1}}$ hold. Take $(g, h) \in G_2$. Then, it follows that

$$S_g S_h = \epsilon_g S_g S_h \subseteq S_g S_{g^{-1}} S_g S_h \subseteq S_g S_{g^{-1}} S_{gh} \subseteq S_g S_{d(g)h} = S_g S_{c(h)h} = S_g S_h$$

and

$$S_g S_h = S_g S_h \epsilon_{h^{-1}} \subseteq S_g S_h S_{h^{-1}} S_h \subseteq S_{gh} S_{h^{-1}} S_h = S_{gc(h)} S_h = S_{gd(g)} S_h = S_g S_h.$$

Moreover, it is clear that ϵ_g is the multiplicative identity element of $S_g S_{g^{-1}}$. □

6. Examples of epsilon-strongly groupoid-graded rings.

In this section, we present some examples of epsilon-strongly groupoid-graded rings.

6.1. Partial skew groupoid rings.

The notion of a partial action of a groupoid on a ring, as well as the construction of the corresponding partial skew groupoid ring, is due to Bagio and Paques [2].

Let G be a groupoid and suppose that B is a ring which is the product of a collection of rings $\{B_e\}_{e \in G_0}$.

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DEFINITION 38. A *partial groupoid action of* G *on* B is a collection of maps $\{\theta_g\}_{g\in G_1}$, where, for each $g \in G_1$, B_g is an ideal of $B_{c(g)}$, $B_{c(g)}$ is an ideal of B, and $\theta_g : B_{g^{-1}} \to B_g$ is a ring isomorphism satisfying the following three axioms:

(G1) if $e \in G_0$, then $\theta_e = \mathrm{id}_{B_e}$;
(G2) if $(g, h) \in G_2$, then $\theta_h^{-1}(B_{g^{-1}} \cap B_h) = B_{(gh)^{-1}}$;
(G3) if $(g, h) \in G_2$ and $x \in \theta_h^{-1}(B_{g^{-1}} \cap B_h)$, then $\theta_g(\theta_h(x)) = \theta_{gh}(x)$.

Note that conditions (G2) and (G3) are equivalent to the fact that θ_{gh} is an extension of $\theta_g \circ \theta_h$. We say that θ is *global* if $\theta_{gh} = \theta_g \circ \theta_h$, for each $(g, h) \in G_2$.

DEFINITION 39. Let $\{\theta_g\}_{g\in G_1}$ be a partial groupoid action of G on B. Suppose that for each $g \in G_1$, B_g is unital, that is, B_g is generated by an idempotent 1_g which is central in $B_{c(g)}$, and θ_g is a monoid isomorphism. In that case, we say that $\{\theta_g\}_{g\in G_1}$ is a *unital partial groupoid action of* G *on* B.

The *partial skew groupoid ring* $B \star_\theta G$, associated with a unital partial groupoid action $\{\theta_g\}_{g\in G_1}$ of G on B, is the set of all finite formal sums $\sum_{g\in G_1} b_g\delta_g$, where $b_g \in B_g$, with addition defined componentwise and multiplication determined by the rule

$$(b_g\delta_g)(b'_h\delta_h) = b_g\alpha_g(b'_h 1_{g^{-1}})\delta_{gh} \tag{23}$$

if $(g, h) \in G^2$, and $(b_g\delta_g)(b'_h\delta_h) = 0$, otherwise.

There is a natural G-grading on $B \star_\theta G$. Indeed, if we put $S_g = B_g\delta_g$ for each $g \in G_1$, then $B \star_\theta G = \oplus_{g\in G_1} S_g$ is G-graded. For each $g \in G$, the idempotent $1_g\delta_{c(g)}$ satisfies the conditions of Proposition 37. Thus, $B \star_\theta G$ is an epsilon-strongly G-graded ring. Moreover, by [**1**, Proposition 2.5], one has that $B \star_\theta G$ is strongly G-graded if and only if θ is global.

6.2. Leavitt path algebras. Let $E = (E^0, E^1, r, s)$ be a directed graph, consisting of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. If both E^0 and E^1 are finite sets, then we say that E is *finite*. A *path* μ in E is a sequence of edges $\mu = \mu_1 \dots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i \in \{1, \dots, n-1\}$. In such a case, $s(\mu) := s(\mu_1)$ is the *source* of μ, $r(\mu) := r(\mu_n)$ is the *range* of μ, and n is the *length* of μ.

DEFINITION 40 ([**11**]). Let E be any directed graph and let K be a unital ring. The *Leavitt path algebra of* E *with coefficients in* K, denoted by $L_K(E)$, is the algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of elements $\{f \mid f \in E^1\} \cup \{f^* \mid f \in E^1\}$, which satisfy the following relations:

(1) $s(f)f = fr(f) = f$, for all $f \in E^1$;
(2) $r(f)f^* = f^*s(f) = f^*$, for all $f \in E^1$;
(3) $f^*f' = \delta_{f,f'}r(f)$, for all $f, f' \in E^1$; and
(4) $v = \sum_{\{f\in E^1 \mid s(f)=v\}} ff^*$, for every $v \in E^0$ for which $s^{-1}(v)$ is non-empty and finite.

Here the ring K commutes with the generators.

REMARK 41. (a) Every path $\mu = \mu_1 \dots \mu_n$ may be viewed as an element of $L_K(E)$. Given such an element μ, we put $\mu^* := \mu_n^* \dots \mu_1^* \in L_K(E)$. The element μ^* may also be thought of as a *ghost path* in E, as opposed to μ which is a *real path*.

(b) Note that every element $x \in L_K(E)$ may be written on the form $x = \sum_{i=1}^n k_i\alpha_i\beta_i^*$, for suitable $k_i \in K$ and (real) paths α_i and β_i satisfying $r(\alpha_i) = r(\beta_i)$, for $i \in \{1, \dots, n\}$.

6.2.1. The groupoid

Based on E, we define a groupoid G in the following way. The objects of G are the vertices of E, that is, $G_0 = E^0$. An ordered pair of vertices, (u, v), is an arrow in G with $d(u, v) = v$ and $c(u, v) = u$ if there is a path

$$\mu = \mu_1\mu_2 \dots \mu_n,$$

such that $u = s(\mu) = s(\mu_1)$ and $v = r(\mu) = r(\mu_n)$, where $\mu_i \in E^1 \cup (E^1)^* \cup E^0$ for each $i \in \{1, 2, \dots, n\}$ and $r(\mu_i) = s(\mu_{i+1})$ for each $i \in \{1, 2, \dots, n-1\}$.

Two arrows (u, v) and (v', w) and composable if only if $v' = v$. In that case, their composition is defined to be equal to

$$(u, v)(v, w) = (u, w).$$

6.2.2. The grading

Let W denote the set of finite real paths in E, and consider W as a subset of the ring $L_K(E)$.

LEMMA 42. *If we, for each $(u, v) \in G_1$, put*

$$S_{(u,v)} = \mathrm{span}_K\{\alpha\beta^* \mid \alpha, \beta \in W, \ \textit{such that } s(\alpha) = u, r(\alpha) = r(\beta), s(\beta) = v\},$$

then this turns $S = L_K(E) = \bigoplus_{(u,v)\in G_1} S_{(u,v)}$ into a G-graded ring.

Proof. Clearly, $L_K(E) = \sum_{(u,v)\in G_1} S_{(u,v)}$, and this sum is in fact direct. Indeed, take a non-zero $x \in S_{(u,v)} \cap S_{(a,b)}$. Then $x = ux$ and $x = ax$, and hence $0 \neq x = ux = u(ax) = (ua)x$. In particular, $ua \neq 0$ which means that $u = a$. Similarly, we may conclude that $v = b$. That is, $(u, v) = (a, b)$.

Let $(u, v), (v', w) \in G_1$ be arbitrary. If $v' = v$, then we get that $S_{(u,v)}S_{(v,w)} \subseteq S_{(u,w)}$. On the other hand, if $v' \neq v$, then $S_{(u,v)}S_{(v',w)} = \{0\}$. This shows that $L_K(E)$ is G-graded. □

THEOREM 43. *If E is a finite graph, then the Leavitt path algebra $S = L_K(E)$ is epsilon-strongly G-graded.*

Proof. Let $(u, v) \in G_1$ be arbitrary.

If $u \notin S_{(u,v)}S_{(v,u)}$, then we shall be interested in the following set:

$$P_{(u,v)} = \{\alpha \mid \alpha\beta^* \in S_{(u,v)}\}.$$

For $\alpha_i, \alpha_j \in P_{(u,v)}$, we write $\alpha_i \leq \alpha_j$ if α_i is an initial subpath of α_j. Clearly, $\leq$ is a partial order on $P_{(u,v)}$. Moreover, using that E is finite, it is not difficult to see that there can only be a finite number of minimal elements of $P_{(u,v)}$ with respect to $\leq$. We collect all such minimal elements in the set $M_{(u,v)} = \{\alpha_1, \dots, \alpha_k\}$.

We are now ready to define $\epsilon_{(u,v)}$ in the following way:

$$\epsilon_{(u,v)} = \begin{cases} u & \text{if } u \in S_{(u,v)}S_{(v,u)} \\ \sum_{\alpha_j \in M_{(u,v)}} \alpha_j\alpha_j^* & \text{otherwise.} \end{cases}$$

Note that, whenever $\alpha\beta^* \in S_{(u,v)}$ is a non-zero monomial, that is, $r(\alpha) = r(\beta)$, we get that $\alpha\alpha^* = \alpha r(\beta)\alpha^* = \alpha\beta^*\beta\alpha^* \in S_{(u,v)}S_{(v,u)}$. In particular, $\alpha_j\alpha_j^* \in S_{(u,v)}S_{(v,u)}$ for each $\alpha_j \in M_{(u,v)}$. Hence, by construction, $\epsilon_{(u,v)} \in S_{(u,v)}S_{(v,u)}$. Moreover, $(\epsilon_{(u,v)})^* = \epsilon_{(u,v)}$.

Take any monomial $\gamma\delta^* \in S_{(u,v)}$. First we show that $\epsilon_{(u,v)}\gamma\delta^* = \gamma\delta^*$.

Case 1: $(u \in S_{(u,v)}S_{(v,u)})$

Clearly, $\epsilon_{(u,v)}\gamma\delta^* = u\gamma\delta^* = \gamma\delta^*$.

Case 2: $(u \notin S_{(u,v)}S_{(v,u)})$

Note that there is some $\alpha' \in M_{(u,v)}$ such that $\gamma = \alpha'\gamma'$. Thus,

$$\begin{aligned}\epsilon_{(u,v)}\gamma\delta^* &= \left(\alpha'\alpha'^* + \sum_{\alpha_j \in M_{(u,v)} \setminus \{\alpha'\}} \alpha_j\alpha_j^*\right)\gamma\delta^* \\ &= \alpha'\alpha'^*\alpha'\gamma'\delta^* + 0 = \alpha'\gamma'\delta^* = \gamma\delta^*.\end{aligned}$$

It remains to show that $\gamma\delta^*\epsilon_{(v,u)} = \gamma\delta^*$. Note that

$$\gamma\delta^* \in S_{(u,v)} \iff \delta\gamma^* \in S_{(v,u)}.$$

It follows, from Case 1 and Case 2, that $\epsilon_{(v,u)}\delta\gamma^* = \delta\gamma^*$. Using this we get that

$$\gamma\delta^*\epsilon_{(v,u)} = \gamma\delta^*(\epsilon_{(v,u)})^* = (\epsilon_{(v,u)}\delta\gamma^*)^* = (\delta\gamma^*)^* = \gamma\delta^*.$$

This concludes the proof. □

In general, $L_K(E)$ need not be strongly G-graded, as the following example shows.

EXAMPLE 44. Let K be a unital ring and consider the Leavitt path algebra $L_K(E)$ associated with the following graph E:

$$\bullet_{v_1} \xleftarrow{\ f_1\ } \bullet_{v_2} \xrightarrow{\ f_2\ } \bullet_{v_3}$$

A few short calculations reveal that

- $S_{(v_2,v_1)}S_{(v_1,v_2)} = Kf_1f_1^*$
- $S_{(v_1,v_2)}S_{(v_2,v_1)} = Kv_1$
- $S_{(v_3,v_2)}S_{(v_2,v_3)} = Kv_3$
- $S_{(v_2,v_3)}S_{(v_3,v_2)} = Kf_2f_2^*$
- $S_{(v_3,v_1)}S_{(v_1,v_3)} = \{0\}$
- $S_{(v_1,v_3)}S_{(v_3,v_1)} = \{0\}$

and we may choose

- $\epsilon_{(v_2,v_1)} = f_1f_1^*$
- $\epsilon_{(v_1,v_2)} = v_1$
- $\epsilon_{(v_3,v_2)} = v_3$
- $\epsilon_{(v_2,v_3)} = f_2f_2^*$
- $\epsilon_{(v_1,v_3)} = \epsilon_{(v_3,v_1)} = 0$.

Clearly, (v_1, v_3) and (v_3, v_1) are composable, but $\{0\} = S_{(v_1,v_3)}S_{(v_3,v_1)} \neq S_{(v_1,v_1)}$. Thus, $L_K(E)$ is not strongly G-graded.

REMARK 45. Gonçalves and Yoneda [10] have shown that each Leavitt path algebra may be viewed as a partial skew groupoid ring. Their observation gives rise to another example of an epsilon-strong groupoid grading on a Leavitt path algebra.

6.3. Morita rings. Let $(A, B, {}_AM_B, {}_BN_A, \varphi, \phi)$ be a strict Morita context. It consists of unital rings A and B, an A–B-bimodule M, a B–A-bimodule N, an A–A-bimodule epimorphism $\varphi : M \otimes_B N \to A$, and a B–B-bimodule epimorphism $\phi : N \otimes_A M \to B$.

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The associated *Morita ring* is the set

$$S = \begin{pmatrix} A & M \\ N & B \end{pmatrix},$$

with the natural addition and with a multiplication defined by

$$\begin{pmatrix} a_1 & m_1 \\ n_1 & b_1 \end{pmatrix} * \begin{pmatrix} a_2 & m_2 \\ n_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + \varphi(m_1 \otimes n_2) & a_1m_2 + m_1b_2 \\ n_1a_2 + b_1n_2 & \phi(n_1 \otimes m_2) + b_1b_2 \end{pmatrix},$$

for $a_1, a_2 \in A$, $b_1, b_2 \in B$, $m_1, m_2 \in M$, and $n_1, n_2 \in N$. Let G be a group and I a non-empty set. Then the set $I \times G \times I$ considered as morphisms, where the composition is given by the rule

$$(i, g, j)(j, h, k) = (i, gh, k),$$

for all $i, j, k \in I$ and $g, h \in G$, is a groupoid. Using this groupoid and taking $I = \{1, 2\}$ and G the infinite cyclic group generated by g, we can define a grading on S by putting

$$S_{(1,e,1)} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad S_{(2,e,2)} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix},$$

$$S_{(1,g,2)} = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}, \quad S_{(2,g^{-1},1)} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix},$$

and $S_{(i,h,j)} = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right\}$, in any other case. Then for $h \in G \setminus \{e, g, g^{-1}\}$, we have that

$$S_{(1,h^{-1},1)} S_{(1,h,1)} \neq S_{(1,e,1)},$$

and thus S is not strongly graded. However, S is epsilon-strongly graded. Indeed, it is easy to see that

$$S_{(1,g,2)} S_{(2,g^{-1},1)} = \begin{pmatrix} \operatorname{im}(\varphi) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$S_{(2,g^{-1},1)} S_{(1,g,2)} = \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{im}(\phi) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

If we put

$$\epsilon_{(1,e,1)} = \epsilon_{(1,g,2)} = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{(2,e,2)} = \epsilon_{(2,g^{-1},1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix},$$

then, by Proposition 37 this yields an epsilon-strong $(I \times G \times I)$-grading on S.

7. Epsilon-strongly groupoid-graded modules. In this section, we define epsilon-strongly groupoid-graded modules (see Definition 46) and we provide a characterization of them (see Proposition 47) that generalizes a result [**15**, Theorem I.3.4] previously obtained for strongly group-graded modules. At the end of this section, we show (see Proposition 48) that the multiplication maps $m_{g,h} : S_g \otimes_R S_h \to \epsilon_g S_{gh} = S_{gh}\epsilon_{h^{-1}}$, for $(g, h) \in G_2$, are R-bimodule isomorphisms. In particular, this implies that for every $g \in G_1$, the sextuple

$$\left(\epsilon_g R, \epsilon_{g^{-1}} R, S_g, S_{g^{-1}}, m_{g,g^{-1}}, m_{g^{-1},g}\right)$$

is a set of equivalence data. Throughout this section, S denotes a ring which is graded by a small groupoid G, and we put $R = \oplus_{e\in G_0} S_e$. For the entirety of this section also let M be a graded left (right) S-module. Recall that this means that there to each $g \in G_1$ is an additive subgroup M_g of M such that $M = \oplus_{g\in G_1} M_g$, as additive groups, and for all $g, h \in G_1$, the inclusion $S_g M_h \subseteq M_{gh}$ (or $M_g S_h \subseteq M_{gh}$) holds, if $(g, h) \in G_2$, and $S_g M_h = \{0\}$ (or $M_g S_h = \{0\}$), otherwise. Recall that M is called *strongly graded* if for all $(g, h) \in G_2$ the equality $S_g M_h = M_{gh}$ (or $M_g S_h = M_{gh}$) holds.

DEFINITION 46. We say that M is *epsilon-strongly graded* if, for each $g \in G_1$, $S_g S_{g^{-1}}$ is a unital ideal of $S_{c(g)}$ such that for all $(g, h) \in G_2$ the equality $S_g M_h = S_g S_{g^{-1}} M_{gh}$ ($M_g S_h = M_{gh} S_{h^{-1}} S_h$) holds.

PROPOSITION 47. *The following assertions are equivalent:*

(a) The ring S is epsilon-strongly graded;
(b) Every graded left S-module is epsilon-strongly graded; and
(c) Every graded right S-module is epsilon-strongly graded.

Proof. Suppose that (a) holds. First we show that (b) holds. Let M be a G-graded left S-module and take $(g, h) \in G_2$. Then

$$S_g S_{g^{-1}} M_{gh} \subseteq S_g M_{g^{-1}gh} = S_g M_h = S_g S_{g^{-1}} S_g M_h \subseteq S_g S_{g^{-1}} M_{gh}.$$

Next, we show that (c) holds. Let M be a G-graded right S-module and take $(g, h) \in G_2$. Then

$$M_{gh} S_{h^{-1}} S_h \subseteq M_g S_h = M_g S_h S_{h^{-1}} S_h \subseteq M_{gh} S_{h^{-1}} S_h.$$

It is clear that (b) (or (c)) implies (a). □

PROPOSITION 48. *Suppose that S is epsilon-strongly graded, and let $\{\epsilon_g\}_{g\in G_1}$ be the family of central idempotents of R provided by Proposition 37. Then for all $(g, h) \in G_2$, the following assertions hold:*

(a) For every graded left S-module M, the multiplication map $m_{g,h} : S_g \otimes_{S_{d(g)}} M_h \to \epsilon_g M_{gh}$ is an isomorphism of R-bimodules.
(b) For every graded right S-module M, the multiplication map $m'_{g,h} : M_g \otimes_{S_{d(g)}} S_h \to M_{gh}\epsilon_{h^{-1}}$ is an isomorphism of R-bimodules.
(c) The multiplication map $m_{g,h} : S_g \otimes_R S_h \to \epsilon_g S_{gh} = S_{gh}\epsilon_{h^{-1}}$ is an isomorphism of R-bimodules.
(d) For every $g \in G_1$, the sextuple

$$(\epsilon_g R, \epsilon_{g^{-1}} R, S_g, S_{g^{-1}}, m_{g,g^{-1}}, m_{g^{-1},g})$$

is a set of equivalence data.

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Proof. Take $(g, h) \in G_2$.

(a) Let M be a G-graded left S-module. From Proposition 47(b), it follows that $m_{g,h}$ is surjective. Now we show that $m_{g,h}$ is injective. To this end, take a positive integer n and $s_g^{(i)} \in S_g$ and $l_h^{(i)} \in M_h$, for $i \in \{1, \ldots, n\}$, such that $m_{g,h}(x) = 0$, where $x = \sum_{i=1}^n s_g^{(i)} \otimes l_h^{(i)} \in S_g \otimes_{S_{d(g)}} M_h$. Take a positive integer m and $u_g^{(j)} \in S_g$ and $v_{g^{-1}}^{(j)} \in S_{g^{-1}}$, for $j \in \{1, \ldots, m\}$, such that $\epsilon_g = \sum_{j=1}^m u_g^{(j)} v_{g^{-1}}^{(j)}$. Then

$$x = \sum_{i=1}^n s_g^{(i)} \otimes l_h^{(i)} = \sum_{i=1}^n \epsilon_g s_g^{(i)} \otimes l_h^{(i)} = \sum_{i=1}^n \sum_{j=1}^m u_g^{(j)} v_{g^{-1}}^{(j)} s_g^{(i)} \otimes l_h^{(i)}$$
$$= \sum_{i=1}^n \sum_{j=1}^m u_g^{(j)} \otimes v_{g^{-1}}^{(j)} s_g^{(i)} l_h^{(i)} = \sum_{j=1}^m u_g^{(j)} \otimes v_{g^{-1}}^{(j)} m_{g,h}(x) = 0.$$

(b) Let M be a G-graded right S-module. From Proposition 47(c), it follows that $m'_{g,h}$ is surjective. Now we show that $m'_{g,h}$ is injective. To this end, take a positive integer n and $m_g^{(i)} \in M_g$ and $s_h^{(i)} \in S_h$, for $i \in \{1, \ldots, n\}$, such that $m'_{g,h}(x) = 0$, where $x = \sum_{i=1}^n l_g^{(i)} \otimes s_h^{(i)} \in M_g \otimes_{S_{d(g)}} S_h$. Take a positive integer m, and $u_{h^{-1}}^{(j)} \in S_{h^{-1}}$ and $v_h^{(j)} \in S_h$, for $j \in \{1, \ldots, m\}$, such that $\epsilon_{h^{-1}} = \sum_{j=1}^m u_{h^{-1}}^{(j)} v_h^{(j)}$. Then

$$x = \sum_{i=1}^n l_g^{(i)} \otimes s_h^{(i)} = \sum_{i=1}^n l_g^{(i)} \otimes s_h^{(i)} \epsilon_{h^{-1}} = \sum_{i=1}^n \sum_{j=1}^m l_g^{(i)} \otimes s_h^{(i)} u_{h^{-1}}^{(j)} v_h^{(j)}$$
$$= \sum_{i=1}^n \sum_{j=1}^m l_g^{(i)} s_h^{(i)} u_{h^{-1}}^{(j)} \otimes v_h^{(j)} = \sum_{j=1}^m m'_{g,h}(x) u_{h^{-1}}^{(j)} \otimes v_h^{(j)} = 0.$$

(c) and (d) follow immediately from (a) or (b). □

REMARK 49. Take $g \in G_1$. It is clear from the definition of epsilon-strongly groupoid-graded rings that the sextuple

$$(\epsilon_g R, \epsilon_{g^{-1}} R, S_g, S_{g^{-1}}, m_{g,g^{-1}}, m_{g^{-1},g})$$

is a set of pre-equivalence data with $m_{g,g^{-1}}$ and $m_{g^{-1},g}$ surjective. Thus, injectivity of the maps $m_{g,g^{-1}}$ and $m_{g^{-1},g}$ also follow from Proposition 20(a).

8. Generalized Epsilon-crossed products. In this section, we introduce generalized epsilon-crossed groupoid products (see Definition 53), and we show that they parameterize the family of epsilon-strongly groupoid-graded rings (see Propositions 55 and 56). Throughout this section, G denotes a small groupoid with G_0 finite.

DEFINITION 50. Suppose that $F : G \to \mathrm{PIC}_{cat}$ is a partial functor of inverse categories. For each $e \in G_0$, define the ring A_e by $F(e) = A_e$. For each $g \in G_1$, put $F(g) = \left[{}^{I_g}_{A_{c(g)}}(P_g)^{J_g}_{A_{d(g)}}\right]$ for some $A_{c(g)}$–$A_{d(g)}$-bimodule P_g, some unital ideal I_g of $A_{c(g)}$, and some unital ideal J_g of $A_{d(g)}$, making P_g an I_g–J_g-bimodule. For the time being, assume that the bimodules P_g, for $g \in G_1$, are fixed. From the equality $F(g^{-1}) = F(g)^*$, it follows by the proof of Theorem 28 that $J_g = I_{g^{-1}}$, so we may write $F(g) = \left[{}^{I_g}_{A_{c(g)}}(P_g)^{I_{g^{-1}}}_{A_{d(g)}}\right]$. For each $g \in G_1$, put $\epsilon_g = 1_{I_g}$.

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PROPOSITION 51. *Suppose that* $(g, h) \in G_2$. *Then* $\gamma_{P_{gh}}(\epsilon_{(gh)^{-1}}\epsilon_{h^{-1}}) = \epsilon_g\epsilon_{gh}$. *In particular,* $\epsilon_g P_{gh} = P_{gh}\epsilon_{h^{-1}}$.

Proof. From (15), it follows that

$$\begin{aligned} F(g)F(g^{-1})F(gh) &= \left[{}^{I_g}_{A_{c(g)}}(P_g)^{I_{g^{-1}}}_{A_{d(g)}}\right]\left[{}^{I_{g^{-1}}}_{A_{d(g)}}(P_{g^{-1}})^{I_g}_{A_{c(g)}}\right]\left[{}^{I_{gh}}_{A_{c(g)}}(P_{gh})^{I_{(gh)^{-1}}}_{A_{d(h)}}\right] \\ &= \left[{}^{I_g}_{A_{c(g)}}(I_g)^{I_g}_{A_{c(g)}}\right]\left[{}^{I_{gh}}_{A_{c(g)}}(P_{gh})^{I_{(gh)^{-1}}}_{A_{d(h)}}\right] \\ &= \left[{}^{\gamma_{I_g}(\epsilon_g\epsilon_{gh})A_{c(g)}}_{A_{c(g)}}(I_g \otimes_{A_{c(g)}} P_{gh})^{\gamma^{-1}_{P_{gh}}(\epsilon_g\epsilon_{gh})A_{d(h)}}_{A_{d(h)}}\right] \\ &= \left[{}^{\epsilon_g\epsilon_{gh}A_{c(g)}}_{A_{c(g)}}(I_g \otimes_{A_{c(g)}} P_{gh})^{\gamma^{-1}_{P_{gh}}(\epsilon_g\epsilon_{gh})A_{d(h)}}_{A_{d(h)}}\right] \end{aligned}$$

and

$$\begin{aligned} F(gh)F(h^{-1})F(h) &= \left[{}^{I_{gh}}_{A_{c(g)}}(P_{gh})^{I_{(gh)^{-1}}}_{A_{d(h)}}\right]\left[{}^{I_{h^{-1}}}_{A_{d(h)}}(P_{h^{-1}})^{I_h}_{A_{c(h)}}\right]\left[{}^{I_h}_{A_{c(h)}}(P_h)^{I_{h^{-1}}}_{A_{d(h)}}\right] \\ &= \left[{}^{I_{gh}}_{A_{c(g)}}(P_{gh})^{I_{(gh)^{-1}}}_{A_{d(h)}}\right]\left[{}^{I_{h^{-1}}}_{A_{d(h)}}(I_{h^{-1}})^{I_{h^{-1}}}_{A_{d(h)}}\right] \\ &= \left[{}^{\gamma_{P_{gh}}(\epsilon_{(gh)^{-1}}\epsilon_{h^{-1}})A_{c(g)}}_{A_{c(g)}}(P_{gh} \otimes_{A_{d(h)}} I_{h^{-1}})^{\epsilon_{(gh)^{-1}}\epsilon_{h^{-1}}A_{d(h)}}_{A_{d(h)}}\right]. \end{aligned}$$

Thus, from the equality $F(g)F(g^{-1})F(gh) = F(gh)F(h^{-1})F(h)$ and Proposition 18, we get that $\gamma_{P_{gh}}(\epsilon_{(gh)^{-1}}\epsilon_{h^{-1}}) = \epsilon_g\epsilon_{gh}$. Finally, $\epsilon_g P_{gh} = \epsilon_g\epsilon_{gh}P_{gh} = \gamma_{P_{gh}}(\epsilon_{(gh)^{-1}}\epsilon_{h^{-1}})P_{gh} = P_{gh}\epsilon_{(gh)^{-1}}\epsilon_{h^{-1}} = P_{gh}\epsilon_{h^{-1}}$. □

REMARK 52. Let $F : G \to \mathrm{PIC}_{cat}$ be a partial functor of inverse categories. Then for every $(g, h) \in G_2$, there are $A_{c(g)}$–$A_{d(h)}$-bimodule isomorphisms

$$P_g \otimes_{A_{d(g)}} P_h \cong P_g \otimes_{A_{d(g)}} P_{g^{-1}} \otimes_{A_{c(g)}} P_{gh} \cong I_g \otimes_{A_{c(g)}} P_{gh} \cong \epsilon_g P_{gh}.$$

DEFINITION 53. Let $F : G \to \mathrm{PIC}_{cat}$ be a partial functor of inverse categories. A *partial factor set associated with F* is a family $f = \{f_{g,h} \mid (g, h) \in G_2\}$, where each $f_{g,h} : P_g \otimes_{A_{d(g)}} P_h \to \epsilon_g P_{gh} = P_{gh}\epsilon_{h^{-1}}$ is an isomorphism of $A_{c(g)}$–$A_{d(h)}$-bimodules, making the following diagram commutative:

$$\begin{array}{ccc} P_g \otimes_{A_{d(g)}} P_h \otimes_{A_{d(h)}} P_r & \xrightarrow{\mathrm{id}_{P_g}\otimes f_{h,r}} & P_g \otimes_{A_{d(g)}} P_{hr}\epsilon_{r^{-1}} \\ {\scriptstyle f_{g,h}\otimes \mathrm{id}_{P_r}}\downarrow & & \downarrow{\scriptstyle f_{g,hr}} \\ \epsilon_g P_{gh} \otimes_{A_{d(h)}} P_r & \xrightarrow{f_{gh,r}} & \epsilon_g P_{ghr}\epsilon_{r^{-1}}, \end{array} \tag{24}$$

for all $(g, h, r) \in G_3$. If f is a partial factor set associated with F, then we define the partial generalized epsilon-crossed product (F, f) as the additive group $\oplus_{g\in G_1} P_g$ with multiplication defined by the biadditive extension of the relations $x \cdot y = f_{g,h}(x \otimes y)$, if $(g, h) \in G_2$, and $x \cdot y = 0$, otherwise, for all $x \in P_g$ and $y \in P_h$ and all $g, h \in G_1$. It is clear that if for each $g \in G_1$, we put $(F, f)_g = P_g$, then (F, f) is a groupoid-graded ring.

REMARK 54. We notice that the notion of partial factor sets already exists in the literature, in a close but different sense. Indeed, partial projective representations and their corresponding factor sets were introduced in [6]. Later, in [7], these factor sets were called *partial factor sets*. For a detailed account of these notions, we refer the reader to the survey [18].

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PROPOSITION 55. *If $F: G \to \mathrm{PIC}_{cat}$ is a partial functor of inverse categories, then the ring (F, f) is epsilon-strongly G-graded.*

Proof. Put $S = (F, f)$. By (24), the multiplication is associative. By the definition of the multiplication, for all $(g, h) \in G_2$, the equality $S_g S_h = \epsilon_g S_{gh} = S_{gh}\epsilon_{h^{-1}}$ holds. All that is left to show is that S has a multiplicative identity element. Take $e \in G_0$ and put $c_e = f_{e,e}(1_{A_e} \otimes 1_{A_e})$. Take $a_e \in A_e$. Then, since $f_{e,e}$ is an A_e-bimodule homomorphism, it follows that

$$\begin{aligned} a_e c_e &= a_e f_{e,e}(1_{A_e} \otimes 1_{A_e}) = f_{e,e}(a_e \otimes 1_{A_e}) = f_{e,e}(1_{A_e} \otimes a_e) \\ &= f_{e,e}(1_{A_e} \otimes 1_{A_e}) a_e = c_e a_e. \end{aligned}$$

Thus, $c_e \in Z(A_e)$. Since $f_{e,e}$ is surjective, there are $a, a' \in A_e$ such that $f_{e,e}(a \otimes a') = 1_{A_e}$. By A_e-bilinearity, it follows that $aa'c_e = c_e aa' = 1_{A_e}$. Hence $c_e \in U(Z(A_e))$. Now set $n_e = c_e^{-1}$. Then $f_{e,e}(n_e \otimes n_e) = n_e$. Hence $n := \sum_{e \in G_0} n_e$ is a multiplicative identity element of S. In fact, take $g \in G_1$ and $x \in P_g$. Then there is $y \in P_g$ such that $x = f_{c(g),g}(n_{c(g)} \otimes y)$. Thus, by (24), we get that

$$\begin{aligned} n \cdot x = n_{c(g)} \cdot x &= f_{c(g),g}(n_{c(g)} \otimes x) = f_{c(g),g}(n_{c(g)} \otimes f_{c(g),g}(n_{c(g)} \otimes y)) \\ &= f_{c(g),g}(f_{c(g),c(g)}(n_{c(g)} \otimes n_{c(g)}) \otimes y) = f_{c(g),g}(n_{c(g)} \otimes y) = x. \end{aligned}$$

Analogously, $x \cdot n = x$. □

PROPOSITION 56. *If S is an epsilon-strongly graded ring, then there is a partial functor of inverse categories $F: G \to \mathrm{PIC}_{\mathrm{cat}}$ and a partial factor set f associated with F such that $S = (F, f)$.*

Proof. Define $F: G \to \mathrm{PIC}_{\mathrm{cat}}$ by $F(g) = \left[{}^{\epsilon_g S_{c(g)}}_{S_{c(g)}} S_g {}^{\epsilon_{g^{-1}} S_{d(g)}}_{S_{d(g)}}\right]$, $g \in G_1$, and a partial factor set f associated with F by the multiplication maps $f_{g,h}: S_g \otimes_{S_{d(g)}} S_h \to \epsilon_g S_{gh} = S_{gh}\epsilon_{h^{-1}}$ for $(g, h) \in G_2$. The claim now follows immediately from Proposition 48(c). □

DEFINITION 57. Let F and F' be partial functors of inverse categories from G to $\mathrm{PIC}_{\mathrm{cat}}$ that coincide on G_0. Take partial factor sets f and f' associated with F and F', respectively and put $F(g) = \left[{}^{I_g}_{A_{c(g)}} (P_g) {}^{J_g}_{A_{d(g)}}\right]$, $F'(g) = \left[{}^{I'_g}_{A_{c(g)}} (P'_g) {}^{J'_g}_{A_{d(g)}}\right]$, $g \in G_1$. A *morphism from f to f'* is a family $\alpha = (\alpha_g)_{g \in G_1}$, where each $\alpha_g: P_g \to P'_g$ is an $A_{c(g)}$–$A_{d(g)}$-bimodule homomorphism, such that the diagram

$$\begin{array}{ccc} P_g \otimes_{A_{d(g)}} P_h & \xrightarrow{f_{g,h}} & \epsilon_g P_{gh} \\ {\scriptstyle \alpha_g \otimes \alpha_h}\downarrow & & \downarrow{\scriptstyle \alpha_{gh}} \\ P'_g \otimes_{A_{d(g)}} P'_h & \xrightarrow{f'_{g,h}} & \epsilon_g P'_{gh} \end{array} \tag{25}$$

is commutative for all $(g, h) \in G_2$.

LEMMA 58. *With the above notation, a morphism α from F to F' induces a homomorphism of graded rings α from (F, f) to (F', f'). Moreover, if each α_e, $e \in G_0$, is surjective, then $\alpha(1) = 1$. The map α is an isomorphism if and only if each α_e, $e \in G_0$, is bijective.*

Proof. Similar to the proof of [**13**, Lemma 4.2]. □

PROPOSITION 59. *The isomorphism class of (F, f) does not depend on the choice of the bimodules P_g.*

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Proof. Put $F(g) = \left[{}^{I_g}_{A_{c(g)}}(P_g)^{J_g}_{A_{d(g)}}\right] = \left[{}^{I_g}_{A_{c(g)}}(P'_g)^{J_g}_{A_{d(g)}}\right]$, for $g \in G_1$. Then there exists an $A_{c(g)}$–$A_{d(g)}$-bimodule homomorphism $\alpha_g : P_g \to P'_g$. If we now put $f'_{g,h} = \alpha_{gh}^{-1} \circ f_{g,h} \circ (\alpha_g \otimes \alpha_h)$, for $(g, h) \in G_2$, then f' is a factor set associated with F and (25) commutes. □

9. Partial cohomology of groupoids. In this section, we extend the construction of a partial cohomology theory for partial actions of groups on commutative monoids, from [**5**], to partial actions of groupoids. We follow closely the presentation and the proofs in [**5**]. Partial actions of groupoids on rings were first studied in [**2**]. Partial actions of categories on sets and topological spaces have been introduced in [**16**]. For the rest of this section, let G be a groupoid and suppose that B is the product of a collection of commutative semigroups $\{B_e\}_{e\in G_0}$.

REMARK 60. Let $\{\theta_g\}_{g\in G_1}$ be a unital partial groupoid action of G on B[1]. Note that if $e \in G_0$ and C and D are unital ideals of B_e, then $C \cap D = CD$, so it follows from [**2**, Lemma 1.1] that the properties (G2) and (G3) can be replaced by

(G2′) if $(g, h) \in G_2$, then $\theta_g(B_{g^{-1}}B_h) = B_g B_{gh}$, and
(G3′) if $(g, h) \in G_2$ and $x \in B_{h^{-1}}B_{h^{-1}g^{-1}}$, then $\theta_g(\theta_h(x)) = \theta_{gh}(x)$,

respectively.

A unital partial G-module is a pair (B, θ), where B is a commutative monoid and θ is a unital partial action of G on B.

DEFINITION 61. A morphism $\psi : (B, \theta) \to (B', \theta')$ of partial G-modules is a set of monoid homomorphisms $\psi = \{\psi_{c(g)} : B_{c(g)} \to B'_{c(g)}\}_{g\in G}$ such that

- $\psi_{c(g)}(B_g) \subseteq B'_g$,
- $\theta'_g \circ \psi_{c(g)} = \psi_{c(g)} \circ \theta_g$ on $B_{g^{-1}}$,

for all $g \in G_1$.

Recall that, if $n \geq 2$, then we let G_n denote the set of all $(g_1, \ldots, g_n) \in \times_{i=1}^n G_1$ that are composable, that is, such that for every $i \in \{1, \ldots, n-1\}$ the relation $d(g_i) = c(g_{i+1})$ holds.

We denote by $pMod(G)$ the category of (unital) partial G-modules. Sometimes, for convenience, (B, θ) will be replaced by B. Suppose that $B \in pMod(G)$. An *n-cochain of G with values in B* is a function $f : G_n \to B$ such that for every $(g_1, \ldots, g_n) \in G_n$, $f(g_1, \ldots, g_n)$ is an invertible element of $B_{(g_1,\ldots,g_n)} = B_{g_1}B_{g_1g_2}\cdots B_{g_1\cdots g_n}$. Denote the set of n-cochains by $C^n(G, B)$. We let $C^0(G, B)$ denote $U(B)$, the group of units in B.

PROPOSITION 62. *Let $B \in pMod(G)$. Then $C^n(G, B)$ is an abelian group under pointwise multiplication.*

Proof. This is clear if $n = 0$. Now suppose that $n \geq 1$. Define $e_n \in C^n(G, B)$ in the following way. Given $(g_1, \ldots, g_n) \in G_n$, put $e_n(g_1, \ldots, g_n) = 1_{g_1}1_{g_1g_2}\cdots 1_{g_1\cdots g_n}$. It is clear that e_n is an identity element of $C^n(G, B)$. Given $f \in C^n(G, B)$ and $(g_1, \ldots, g_n) \in G_n$, put $f^{-1}(g_1, \ldots, g_n) = f(g_1, \ldots, g_n)^{-1}$, where the inverse is taken in $B_{(g_1,\ldots,g_n)}$. It is clear that $ff^{-1} = f^{-1}f = e_n$. □

DEFINITION 63. Let $B \in pMod(G)$ and let n be a non-negative integer. Define a map $\delta_n : C^n(G, B) \to C^{n+1}(G, B)$ in the following way. Take $b = (b_e)_{e\in \mathrm{ob}(G)} \in U(B)$ and $g \in G_1$. Put

[1] In the sense of Definition 39, but in this case, $\{\theta_g\}_{g\in G_1}$ is a family of semigroup isomorphisms.

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$$\delta^0(b)(g) = \theta_g(1_{g^{-1}} b_{d(g)}) b_{c(g)}^{-1}.$$

Now suppose that n is a positive integer. Take $f \in C^n(G, B)$ and $(g_1, \dots, g_{n+1})$ in G_{n+1}. Put

$$\delta^n(f)(g_1, \dots, g_{n+1}) =$$
$$\theta_{g_1}(1_{g_1^{-1}} f(g_2, \dots, g_{n+1})) \left(\prod_{i=1}^{n} f(g_1, \dots, g_i g_{i+1}, \dots g_{n+1})^{(-1)^i} \right) f(g_1, \dots, g_n)^{(-1)^{n+1}}.$$

Adapting the proofs of [6, Proposition 1.5] and [6, Proposition 1.7] to our situation, we get the following.

PROPOSITION 64. *Suppose that $B \in pMod(G)$, and that n is a non-negative integer. Then the following assertions hold:*

(a) The map δ^n is a well-defined homomorphism of groups $C^n(G, B) \to C^{n+1}(G, B)$ satisfying $\delta^{n+1}\delta^n = e_{n+2}$.
(b) The map sending B to the sequence $\{\delta^n : C^n(G, B) \to C^{n+1}(G, B)\}_{n \in \mathbb{N}}$ is a functor from $pMod(G)$ to the category of complexes of abelian groups.

DEFINITION 65. Let $B \in pMod(G)$ and let n be a positive integer. The map δ^n is called a *coboundary homomorphism*. We define the abelian groups $Z^n(G, B) = \ker(\delta^n)$, $B^n(G, B) = \mathrm{im}(\delta^{n-1})$. The quotient group $H^n(G, B) = Z^n(G, B)/B^n(G, B)$ is called *the nth cohomology group of G with values in B*. We put $H^0(G, B) = Z^0(G, B) = \ker(\delta^0)$.

Let G be a groupoid. Denote by $I(X)_{cat}$ the subcategory of $\mathrm{BIJ}_{\mathrm{cat}}$ having as objects the collection $(X_e)_{e \in G_0}$ of abelian semigroups and morphisms $\left[{}^{X_g}_{X_{c(g)}} f_g {}^{X_{g^{-1}}}_{X_{d(g)}}\right]$, where X_g is a unital ideal of $X_{c(g)}$ and $f_g : X_{g^{-1}} \to X_g$, is a monoid isomorphism for all $g \in G_1$. The composition in $I(X)_{\mathrm{cat}}$ is defined in the same way as in ISO_{cat}, the map $* : (I(X)_{cat})_1 \to (I(X)_{cat})_1$ is also defined by restriction of the map $*$ defined in $(\mathrm{BIJ}_{cat})_1$. It follows from Proposition 13 that $I(X)_{cat}$ is an inverse category.

PROPOSITION 66. *If G is a groupoid and $X = \prod_{e \in G_0} X_e$, then $X \in pMod(G)$, if and only if, there is a partial functor of inverse categories $F : G \to I(X)_{cat}$.*

Proof. First we show the "only if" statement. Let $\{\theta_g : X_{g^{-1}} \to X_g\}_{g \in G_1}$ be a partial action of G on X and define $F : G \to I(X)_{cat}$, by $F(g) = \left[{}^{X_g}_{X_{c(g)}} \theta_g {}^{X_{g^{-1}}}_{X_{d(g)}}\right]$, for $g \in G_1$, and $F_e = X_e$, for any $e \in G_0$. Then F is a partial functor of inverse categories.

Now we show the "if" statement. Let $F : G \to I(X)_{\mathrm{cat}}$ be a partial functor of inverse categories. Put $F(g) = \left[{}^{X_g}_{X_{c(g)}} \theta_g {}^{X_{g^{-1}}}_{X_{d(g)}}\right]$, for $g \in G_1$. We shall show that the family $\{\theta_g\}_{g \in G_1}$ gives a partial action of G on $X = \prod_{e \in G_0} X_e$. It is clear that for each $g \in G_1$, B_g is an ideal of $B_{c(g)}$, $B_{c(g)}$ is an ideal of B, and $\theta_g : B_{g^{-1}} \to B_g$ is a monoid isomorphism. Now we check (G1), (G2), and (G3) from Definition 38.

(G1): Let $e \in G_0$. Then $F(e) = \left[{}^{X_e}_{X_e} (\theta_e) {}^{X_e}_{X_e}\right]$, and θ_e is the identity map on X_e.

(G2)–(G3): Let $(g, h) \in G_2$. Then $F(g)F(h) = F(gh)F(h)^* F(h)$, which by the definition of F implies $\theta_g \circ \theta_h = \theta_{gh} \circ \theta_h^{-1} \circ \theta_h = \theta_{gh} \circ \mathrm{id}_{X_{h^{-1}}}$, which in turn implies that θ_{gh} is an extension of $\theta_g \circ \theta_h$. □

Let $F : G \to \mathrm{PIC}_{cat}$ be a partial functor of inverse categories. For each $e \in G_0$, define the ring A_e by $F(e) = A_e$. Take $g \in G_1$. Put $F(g) = \left[{}^{I_g}_{A_{c(g)}} (P_g) {}^{I_{g^{-1}}}_{A_{d(g)}}\right]$. Define $B_g = Z(I_g)$ and $B = \prod_{e \in \mathrm{ob}(G)} Z(A_e)$.

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PROPOSITION 67. *With the above notation, we have* $B \in pMod(G)$.

Proof. By Proposition 31, there is a partial functor of inverse categories $L: \mathrm{PIC}_{cat} \to \mathrm{ISOC}_{cat}$. From Proposition 25, it follows that $l = L \circ F: G \to \mathrm{ISOC}_{cat}$ is a partial functor of inverse categories. But $l(g) = \left[{}_{B_{c(g)}}^{B_g}(\gamma_{P_g})_{B_{d(g)}}^{B_{g^{-1}}}\right]$, so $l: G \to I(X)_{cat}$, and hence we get that $B \in pMod(G)$. □

10. Proof of the main result. In this section, we prove Theorem 3 which was stated in Section 1. For the convenience of the reader, we shall now recall its exact wording.

THEOREM If G is a groupoid, $F: G \to \mathrm{PIC}_{cat}$ is a partial functor of inverse categories and f is a partial factor set associated with F, then the map $H^2(G, U(Z(A))) \to C(A, F)$, defined by $[q] \mapsto qf$, is bijective.

In order to prove the above theorem, we need the following result.

PROPOSITION 68. *Let f and f' be factor sets associated with F.*

(a) *If $q \in Z^2(G, B)$, then fq is a factor set associated with F.*
(b) *There is $q \in Z^2(G, B)$ such that $f' = qf$.*
(c) *A cocycle $q \in Z^2(G, B)$ belongs to $B^2(G, B)$ if and only if there is a graded ring isomorphism α from (F, f) to (F, qf) such that for all $g \in G_1$, the graded restriction α_g to P_g is an $A_{c(g)}$–$A_{d(g)}$-bimodule isomorphism.*
(d) *The map from $Z^2(G, B)$ to the collection of factor sets associated with F, defined by $q \mapsto qf$, is bijective.*

Proof. (a) Put $f'' = fq$. We need to verify that (24) commutes for f''. Take $(g, h, p) \in G_3$, $x \in P_g$, $y \in P_h$, and $z \in P_p$. Then

$$\begin{aligned}
\big(f''_{gh,p} \circ (f''_{g,h} \otimes \mathrm{id}_{P_p})\big)(x \otimes y \otimes z) &= q_{gh,p}q_{g,h}\big(f_{gh,p} \circ (f_{g,h} \otimes \mathrm{id}_{P_p})\big)(x \otimes y \otimes z)\\
&= \big(q_{g,hp}\gamma_{P_g}(q_{h,p})\big)\big(f_{g,hp} \circ (\mathrm{id}_{P_g} \otimes f_{h,p})\big)(x \otimes y \otimes z)\\
&= \big(q_{g,hp}\gamma_{P_g}(q_{h,p})\big)f_{g,hp}\big(x \otimes f_{h,p}(y \otimes z)\big)\\
&= f''_{g,hp}\big(\gamma_{P_g}(q_{h,p})x \otimes f_{h,p}(y \otimes z)\big)\big)\\
&= f''_{g,hp}\big(xq_{h,p} \otimes f_{h,p}(y \otimes z)\big)\\
&= f''_{g,hp}\big(x \otimes f''_{h,p}(y \otimes z)\big)\\
&= \big(f''_{g,hp} \circ (\mathrm{id}_{P_\sigma} \otimes f''_{h,p})\big)(x \otimes y \otimes x).
\end{aligned}$$

(b) Take $(g, h) \in G_2$. Then $f'_{g,h} \circ f^{-1}_{g,h}$ is an $A_{c(\sigma)}$–$A_{d(\sigma)}$-bimodule automorphism of P_{gh}. Hence, by Proposition 21, there is $q_{g,h} \in U(Z(I_{c(g)}))$ such that $(f'_{g,h} \circ f^{-1}_{g,h})(x) = q_{\sigma,\tau}x$, for $x \in P_{gh}$. By (24), it follows that $q \in Z^2(G, B)$.

(c) Suppose now that $q \in B^2(G, A)$. Then there is $c \in C^1(G, A)$ such that for all $(g, h) \in G_2$, it follows that $q_{g,h} = \gamma_{P_g}(c_h)c_g c^{-1}_{gh}$. Define a map α from (F, qf) to (F, f), by $\alpha(x) = c_g x$, for $x \in P_g$. If $x \in P_g$, $y \in P_h$, and $(g, h) \in G_2$, then $\alpha(xy) = c_{gh}q_{g,h}f_{g,h}(x \otimes y) = q^{-1}_{g,h}\gamma_{P_g}(c_h)c_g q_{g,h}f_{g,h}(x \otimes y) = f_{g,h}(\gamma_{P_g}x \otimes c_h y) = \alpha(x)\alpha(y)$. Clearly, for all $g \in G_1$, the map α_g is an $A_{c(g)}$–$A_{d(g)}$-bimodule isomorphism.

On the other hand, suppose that there is an isomorphism of graded rings β from (F, qf) to (F, f) such that for all $g \in G_1$ the map β_g is an $A_{c(g)}$–$A_{d(g)}$-bimodule isomorphism. From Proposition 21, it follows that for each $g \in G_1$, there is $d_g \in U(Z(I_{c(g)}))$ such that for all $x \in P_g$ the equation $\beta_g(x) = d_g x$ holds. Therefore, for all $x \in P_g$, $y \in P_h$, and all $(g, h) \in G_2$, we get that $\beta(xy) = \beta(x)\beta(y) \Leftrightarrow d_{gh} q_{g,h} f_{g,g}(x \otimes y) = f_{g,h}(d_g x \otimes d_h y) \Leftrightarrow d_{gh} q_{g,h} f_{g,h}(x \otimes y) = d_g \gamma_{P_g}(d_h) f_{g,h}(x \otimes y)$. Thus, $q \in B^2(G, A)$.

(d) This follows from (a), (b), and (c). □

Definition 69. If f and f' are factor sets associated with F, then we write $(F, f) \approx (F, f')$ if there is an isomorphism of graded rings from (F, f) to (F, f') such that each graded restriction to P_g, $g \in G_1$, is an $A_{c(g)}$–$A_{d(g)}$-bimodule isomorphism. Let $C(A, F)$ denote the collection of equivalence classes of generalized epsilon-crossed groupoid products (F, f) modulo $\approx$, where f runs over all factor sets associated with F.

Proof of Theorem 3. This follows immediately from Proposition 68. □

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