PARETO OPTIMAL RISK EXCHANGES AND A SYSTEM OF DIFFERENTIAL EQUATIONS: A DUALITY THEOREM

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Abstract

This article, based on a result of Borch and an extension of Bühlmann, gives a complete characterization of Pareto optimal risk exchanges by a system of differential equations linking the derivate of agents contributions to their risk aversion coefficients.

Keywords

Pareto optimal risk exchange; Bernoulli utility function; absolute risk aversion; system of differential equations.

1. Introduction

This article extends a result of Bühlmann (1984). Starting from Borch's theorem (1960), Bühlmann found a system of differential equations with a Pareto optimal risk exchange as the solution. Here we are starting from these differential equations and prove existence and uniqueness of a solution without assuming any further condition. This solution depends on initial values which satisfy a certain clearing condition. It will turn out that it can be identified in a bijective way with the set of Pareto optimal risk exchanges.

2. Model

We consider a risk pool with n participants. Participant i $(1 \leq i \leq n)$ is characterized by

r_i : initial wealth
X_i: initial risk (random variable defined on a probability space $(\Omega, \mathfrak{A}, P)$; we assume that the expected values $E[X_i]$ exists)
u_i : Bernoulli utility function (defined on $\mathbb{R}$, increasing, strictly concave and twice differentiable: $u_i' < 0$, $u_i'' > 0$)
ρ_i : absolute risk aversion ($\rho_i := -u_i''/u_i'$. Notice $u_i' > 0$ and $\rho_i > 0$ i.e. the participants are risk averse; see Pratt (1964)).

By a risk pool we mean any formal mutual agreement among the n participants to redistribute their total initial risk $\sum_{i=1}^{n} X_i$.

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The intial risk vector $\boldsymbol{X}$,

$$\boldsymbol{X} := (X_1, \dots, X_n),$$

is called risk vector before exchange whereas a risk vector $\boldsymbol{Y}$,

$$\boldsymbol{Y} := (Y_1, \dots, Y_n),$$

defined on the same probability space $(\Omega, \mathfrak{A}, P)$ and satisfying the clearing condition

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_i ,$$

is denoted as risk vector after exchange or briefly as risk exchange.

Furthermore a risk exchange $\boldsymbol{Y}^* := (Y_1^*, \dots, Y_n^*)$ is called Pareto optimal if there does not exist another risk exchange $\boldsymbol{Y} := (Y_1, \dots, Y_n)$ with

$$E[u_i(r_i - Y_i^*)] \leq E[u_i(r_i - Y_i)] \qquad \text{for all } i$$

$$E[u_{i^o}(r_{i^o} - Y_{i^o}^*)] < E[u_{i^o}(r_{i^o} - Y_{i^o})] \qquad \text{for at least one } i^o.$$

In the sequel we are interested in Pareto optimal risk exchanges.

Remark. The motivation of a person for participating in a risk pool is to improve his initial expected utility $E[u(r - X)]$. Therefore a risk exchange Y has to satisfy the individual rationality condition

$$E[u_i(r_i - X_i)] \leq E[u_i(r_i - Y_i)] \qquad \text{for all } i$$

in addition to the pool condition of Pareto optimality. Unfortunately there are many Pareto optimal risk exchanges violating this condition. In order to preserve the beauty of the main result we drop the individual rationality condition and deal in this article with general Pareto optimal risk exchanges.

In order to simplify our notation we introduce the shifted disutility functions v_i

$$v_i(x) := u_i(r_i - x) \qquad i = 1, \dots, n. \qquad (v_i' < 0, v_i'' < 0)$$

With W_i we denote the range of the derivative of v_i

$$W_i := \{v_i'(x) \mid x \in \mathbb{R}\}.$$

3. MAIN RESULT

Now we show the existence of a bijective mapping between the set of Pareto optimal risk exchanges and the set of solutions of a system of differential equations satisfying a constrained boundary condition.

THEOREM

Let $\boldsymbol{w}$, $\boldsymbol{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n$, be a vector with $\sum_{i=1}^{n} w_i = 0$.

(i) Let (A) be the system of differential equations

$$\text{(A)} \qquad Y_i'(z) = \frac{\dfrac{1}{\rho_i(r_i - Y_i(z))}}{\displaystyle\sum_{j=1}^{n} \frac{1}{\rho_j(r_j - Y_j(z))}} \qquad i = 1, \ldots, n.$$

There exists a uniquely defined solution $\boldsymbol{Y}(z) = (Y_1(z), \ldots, Y_n(z))$ of (A) satisfying the boundary condition $Y_i(0) = w_i$, $i = 1, \ldots, n$.

(ii) If $\boldsymbol{Y}(z) = (Y_1(z), \ldots, Y_n(z))$ is the solution of (A) with boundary condition $Y_i(0) = w_i$, $i = 1, \ldots, n$, then

$$\boldsymbol{Y}\left(\sum_{i=1}^{n} X_i\right)$$

is a Pareto optimal risk exchange.

(iii) If $\boldsymbol{Y}^* := (Y_1^*, \ldots, Y_n^*)$ is a Pareto optimal risk exchange then there exists a solution $\boldsymbol{Y}(z) = (Y_1(z), \ldots, Y_n(z))$ of (A) satisfying a uniquely defined boundary condition $Y_i(0) = w_i$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} w_i = 0$, with

$$\boldsymbol{Y}^* = \boldsymbol{Y}\left(\sum_{i=1}^{n} X_i\right) \qquad \text{almost surely.}$$

PROOF

(i) Existence of a solution:

Let $f_{\boldsymbol{k}}$ be the function $f_{\boldsymbol{k}}(x) := \sum_{i=1}^{n} (v_i')^{-1}\left(\dfrac{x}{k_i}\right)$ with $\boldsymbol{k} := (k_1, \ldots, k_n)$,

$k_i := \dfrac{-1}{v_i'(w_i)} > 0$. $f_{\boldsymbol{k}}$ is a strictly decreasing and differentiable function defined on W

$$W := \bigcap_{i=1}^{n} \{x\, k_i \mid x \in W_i\}$$

with range $\mathbb{R}$. (see Lemma 1, Appendix). Furthermore $f_{\boldsymbol{k}}(-1) = 0$. (see Proof of Lemma 1, Appendix). We have

$\boldsymbol{Y}(z) := (Y_1(z), \ldots, Y_n(z))$ with $Y_i(z) = (v_i')^{-1}\left(\dfrac{1}{k_i}(f_{\boldsymbol{k}})^{-1}(z)\right)$, $i = 1, \ldots, n$, and $k_i := \dfrac{-1}{v_i'(w_i)}$, $i = 1, \ldots, n$, is a solution of (A).

Uniqueness of the solution:

Let $\tilde{Y}(z) = (\tilde{Y}_1(z), \ldots, \tilde{Y}_n(z))$ be another solution of (A) satisfying the same boundary condition. We define differentiable functions $g_i(z)$, $i = 2, \ldots, n$:

$$g_i(z) := k_1 v_1'(\tilde{Y}_1(z)) - k_i v_i'(\tilde{Y}_i(z)), \qquad z \in \mathbb{R}.$$

We have $g_i(0) = 0$ for all i and for the derivatives $g_i'(z)$, $i = 2, \ldots, n$, we get

$$\begin{aligned} g_i'(z) &= k_1 v_1''(\tilde{Y}_1(z))\, \tilde{Y}_1'(z) - k_i v_i''(\tilde{Y}_i(z))\, \tilde{Y}_i'(z) \\ &= \frac{\dfrac{k_1 v_1''(\tilde{Y}_1(z))}{\rho_1(r_1 - \tilde{Y}_1(z))} - \dfrac{k_i v_i''(\tilde{Y}_i(z))}{\rho_i(r_i - \tilde{Y}_i(z))}}{\sum_{j=1}^{n} \dfrac{1}{\rho_j(r_j - \tilde{Y}_j(z))}} \quad \text{(with (A))} \\ &= \frac{g_i(z)}{\sum_{j=1}^{n} \dfrac{1}{\rho_j(r_j - \tilde{Y}_j(z))}}, \quad z \in \mathbb{R}. \end{aligned}$$

Because the homogeneous linear differential equations

$$g_i'(z) = \frac{g_i(z)}{\sum_{j=1}^{n} \dfrac{1}{\rho_j(r_j - \tilde{Y}_j(z))}}, \quad z \in \mathbb{R}, \quad i = 2, \ldots, n$$

have only solutions of the form

$$g_i(z) = c_i \exp\left(\int_0^z \frac{1}{\sum_{j=1}^{n} \dfrac{1}{\rho_j(r_j - \tilde{Y}_j(t))}}\, dt \right), \quad c_i \in \mathbb{R}, \quad i = 2, \ldots, n$$

we get together with $g_i(0) = 0$: $c_i = 0$ and therefore $g_i(z) = 0$ for all $z \in \mathbb{R}$ and $i = 2, \ldots, n$.

This means

$$k_i v_i'(\tilde{Y}_i(z)) = k_1 v_1'(\tilde{Y}_1(z)) \quad \text{for all} \quad z \in \mathbb{R} \quad \text{and} \quad i = 2, \ldots, n.$$

Because $\sum_{i=1}^{n} \tilde{Y}_i'(z) = 1$ for all $z \in \mathbb{R}$ it follows together with the boundary condition that $\sum_{i=1}^{n} \tilde{Y}_i(z) = z$ for all $z \in \mathbb{R}$.

Because $\tilde{Y}(z)$ and $Y(z)$ satisfy both the equations (**) of Lemma 2 (see Appendix) we conclude by uniqueness of the solution that

$$\tilde{Y}(z) = Y(z) \qquad \text{for all} \qquad z \in \mathbb{R}.$$

(ii) $Y(z) := (Y_1(z), \ldots, Y_n(z))$ with $Y_i(z) = (v_i')^{-1}\left(\frac{1}{k_i}(f_k)^{-1}(z)\right)$, $i = 1, \ldots, n$,

and $k_i := \dfrac{-1}{v_i'(w_i)}$, $i = 1, \ldots, n$, is the unique solution of (A)

satisfying the boundary condition $Y_i(0) = w_i$, $i = 1, \ldots, n$. Because this solution satisfies (**) of Lemma 2, (see Appendix), it follows from BORCH'S theorem (see BORCH, (1960))that $Y(\Sigma X_i)$ is a Pareto optimal risk exchange.

(iii) It follows from BORCH'S theorem (see BORCH, (1960)) that there are strictly positive constants k_i, $i = 1, \ldots, n$, with

$$k_i v_i'(Y_i^*) = k_1 v_1'(Y_1^*) \quad \text{almost surely for} \quad i = 2, \ldots, n.$$

Let $\omega \in \Omega$ be an element of Ω for which the condition of BORCH is satisfied, $\boldsymbol{k}$ the vector $\boldsymbol{k} := (k_1, \ldots, k_n)$ and $f_{\boldsymbol{k}}$ the function as defined above. Because $f_{\boldsymbol{k}}(x)$ is defined for $x := k_1 v_1'(Y_1^*(\omega))$

$$f_{\boldsymbol{k}}(x) = \sum_{i=1}^{n} (v_i')^{-1}\left(\frac{x}{k_i}\right) = \sum_{i=1}^{n} (v_i')^{-1}\left(\frac{k_i v_i'(Y_i^*(\omega))}{k_i}\right) = \sum_{i=1}^{n} Y_i^*(\omega)$$

(condition of BORCH)

it follows analogously to Lemma 1, (see Appendix), that $f_{\boldsymbol{k}}$ is defined on some interval (a, b) with range $\mathbb{R}$. Therefore $(f_{\boldsymbol{k}})^{-1}(0)$ exists. We define the vector $\boldsymbol{w} = (w_1, \ldots, w_n)$ by

$$w_i := (v_i')^{-1}\left(\frac{1}{k_i}(f_{\boldsymbol{k}})^{-1}(0)\right), \qquad i = 1, \ldots, n.$$

The unique solution $Y(z) = (Y_1(z), \ldots, Y_n(z))$ of (A) with boundary condition $Y_i(0) = w_i$, $i = 1, \ldots, n$, satisfies the equations (**) of Lemma 2, i.e.

$$\sum_{i=1}^{n} Y_i(z) = z \qquad \text{for all} \qquad z \in \mathbb{R}$$

$$k_i v_i'(Y_i(z)) = k_1 v_1'(Y_1(z)) \quad \text{for} \quad i = 2, \ldots, n \quad \text{and all} \quad z \in \mathbb{R}.$$

We conclude by uniqueness of the solution that

$$Y^* = Y\left(\sum_{i=1}^{n} X_i\right) \qquad \text{almost surely}.$$

(REMARK: Because $-1/k_i$ is possibly not in the range of v_i' we cannot define w_i by $w_i := (v_i')^{-1}\left(\frac{-1}{k_i}\right)$). QED

4. EXAMPLE

We assume that the participants are using exponential utility functions, i.e. $\rho_i(x) = a_i$ for all $x \in \mathbb{R}$ and $i = 1, \ldots, n$, where ρ_i denotes the absolute risk aversion of participant i. In this case the system of differential equations (A) becomes very simple

$$\text{(A)} \qquad Y_i'(z) = \frac{\frac{1}{a_i}}{\sum_{j=1}^{n} \frac{1}{a_j}}, \qquad i = 1, \ldots, n.$$

We therefore have

$$Y_i(z) = \frac{\frac{1}{a_i}}{\sum_{j=1}^{n} \frac{1}{a_j}} z + \beta_i, \qquad i = 1, \ldots, n,$$

where the β_i's satisfy the clearing condition

$$\sum_{i=1}^{n} \beta_i = 0.$$

For further examples, e.g. for utility functions of the HARA-type, see LIENHARD, (1986).

APPENDIX

To conclude the two technical lemmas already used in the proof of the main Theorem are discussed.

LEMMA 1

Let $\quad \boldsymbol{w}, \boldsymbol{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n$, be a vector with $\sum_{i=1}^{n} w_i = 0$

and $\quad \boldsymbol{k}, \boldsymbol{k} := (k_1, \ldots, k_n) \in \mathbb{R}^n$ the vector with $k_i := \frac{-1}{v_i'(w_i)} > 0$.

Then $f_{\boldsymbol{k}}(x) := \sum_{i=1}^{n} (v_i')^{-1}\left(\frac{x}{k_i}\right)$ is a strictly decreasing and differentiable function defined on W

$$W := \bigcap_{i=1}^{n} \{x\,k_i \mid x \in W_i\}$$

with range $\mathbb{R}$.

PROOF

Obviously W is an open interval. W is not empty because it contains -1.

$$f_{\boldsymbol{k}}(-1) = \sum_{i=1}^{n} (v_i')^{-1}\left(\frac{-1}{k_i}\right) = \sum_{i=1}^{n} (v_i')^{-1}(v_i'(w_i)) = \sum_{i=1}^{n} w_i = 0\,.$$

We denote by (a_i, b_i) the open interval W_i and by (a, b) that of W. We have

$$a = a_i k_i \qquad \text{for at least one} \qquad i$$

and therefore

$$\lim_{x\to a} f_{\boldsymbol{k}}(x) = \lim_{x\to a} \sum_{i=1}^{n} (v_i')^{-1}\left(\frac{x}{k_i}\right) \geq \lim_{\substack{x\to a_i k_i\\ x>a}} (v_i')^{-1}\left(\frac{x}{k_i}\right) = \lim_{\substack{y\to a_i\\ y>a_i}} (v_i')^{-1}(y) = \infty\,.$$

Analoguously we get

$$\lim_{x\to b} f_{\boldsymbol{k}}(x) = -\infty\,.$$

It follows that the continuous function $f_{\boldsymbol{k}}$ has range $\mathbb{R}$. Obviously $f_{\boldsymbol{k}}$ is differentiable on W with derivative

$$f_{\boldsymbol{k}}'(x) = \sum_{i=1}^{n} \frac{1}{k_i} \frac{1}{v_i''\left((v_i')^{-1}\left(\frac{x}{k_i}\right)\right)} < 0\,. \qquad \text{QED}$$

LEMMA 2

Let $\boldsymbol{w}, \boldsymbol{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n$, be a vector with $\sum_{i=1}^{n} w_i = 0$

and $\boldsymbol{k}, \boldsymbol{k} := (k_1, \ldots, k_n) \in \mathbb{R}^n$ the vector with $k_i := \dfrac{-1}{v_i'(w_i)} > 0\,.$

Furthermore let (*) and (**) respectively denote the system of equations (in $Y_1(z), \ldots, Y_n(z)$)

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(*) $$z = f_k(k_i v_i'(Y_i(z))) \qquad i = 1, \ldots, n .$$

(**) $$\begin{cases} \sum_{i=1}^{n} Y_i(z) = z & \text{for all} \quad z \in \mathbb{R} \\ k_i v_i'(Y_i(z)) = k_1 v_1'(Y_1(z)) & \text{for} \quad i = 2, \ldots, n . \end{cases}$$

Then $Y(z) := (Y_1(z), \ldots, Y_n(z))$ with $Y_i(z) = (v_i')^{-1}\left(\frac{1}{k_i}(f_k)^{-1}(z)\right), i = 1, \ldots, n,$

is the unique solution of (*) resp. (**). The functions $Y_i(z)$, $i = 1, \ldots, n$, are strictly increasing and differentiable. They satisfy $Y_i(0) = w_i$ for $i = 1, \ldots, n$.

PROOF

From Lemma 1 it follows that $(f_k)^{-1}$ exists and is defined on $\mathbb{R}$. Therefore $Y(z)$ is well defined, strictly increasing and differentiable. By inverting equation (*) we see that $Y(z)$ is a solution and even the unique solution of (*). Obviously $Y(z)$ is also a solution of (**).

Note that

$$\sum_{i=1}^{n} Y_i(z) = f_k((f_k)^{-1}(z)) = z$$

$$k_i v_i'(Y_i(z)) = (f_k)^{-1}(z) = k_1 v_1'(Y_1(z)) \qquad \text{for} \qquad i = 2, \ldots, n .$$

Furthermore

$$Y_i(0) = (v_i')^{-1}\left(\frac{1}{k_i}(-1)\right) = w_i \qquad \text{(see proof of Lemma 1).}$$

Let $\tilde{Y}(z) := (\tilde{Y}_1(z), \ldots, \tilde{Y}_n(z))$ be another solution of (**). Then we have

$$z = \sum_{i=1}^{n} \tilde{Y}_i(z) = \sum_{i=1}^{n} (v_i')^{-1}\left(\frac{k_1}{k_i} v_1'(\tilde{Y}_1(z))\right)$$

$$= f_k(k_1 v_1'(\tilde{Y}_1(z))) = f_k(k_i v_i'(\tilde{Y}_i(z))) .$$

But the solution of (*) is unique, so we have $\tilde{Y}(z) = Y(z)$ for all $z \in \mathbb{R}$. This completes the proof.

QED

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