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A CLOSED FORM FOR THE DENSITY FUNCTIONS OF RANDOM WALKS IN ODD DIMENSIONS

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Abstract

We derive an explicit piecewise-polynomial closed form for the probability density function of the distance travelled by a uniform random walk in an odd-dimensional space.

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1. Introduction

In [1], the authors explore the distance travelled by a uniform n-step random walk in $\mathbb{R}^d$ with unit step length. Following their lead, we denote the probability density function of this distance by $p_n(m - 1/2; x)$, where $m = (d - 1)/2$.

We recall that the density can be expressed in terms of an integral engaging the *normalised Bessel function of the first kind* of order ν, defined by

$$j_\nu(x) = \nu!\Big(\frac{2}{x}\Big)^\nu J_\nu(x) = \nu! \sum_{k\geq 0} \frac{(-x^2/4)^k}{k!(k+\nu)!}.$$

With this normalisation, we have $j_\nu(0) = 1$ and obtain the following theorem.

Theorem 1.1 (Bessel representation [1, 4]). *The probability density function of the distance to the origin in $d \geq 2$ dimensions after $n \geq 2$ steps is, for $x > 0$,*

$$p_n(m - 1/2; x) = \frac{2^{-m+1/2}}{\Gamma(m + 1/2)} \int_0^\infty (tx)^{m+1/2} J_{m-1/2}(tx)\, j^n_{m-1/2}(t)\, dt,$$

where $m = (d - 1)/2$.

The study of the density $p_n(\nu; x)$ is quite classical, originating in the early twentieth century [2, 4–7]. The most fundamental cases are those of two and three dimensions [2, 7]. The Bessel representation of the density is valuable for its generality and its analytically-pleasing structure, which form the basis for many related results [1, 4].

Additionally, when Theorem 1.1 is used for half-integer m, one can symbolically integrate any given small-order case, although the structure of the closed form is obscured in the process.

While some probabilistic results such as Theorem 1.1 hold in all dimensions, many arithmetic and analytic results are distinct between odd and even dimensions. Indeed, even dimensional results often involve elliptic integrals [1, 2], while odd dimensional results are typically resolvable in terms of elementary functions. For instance, noting that $j_{1/2}(x) = \operatorname{sinc}(x) = \sin(x)/x$ partly explains why analysis in three-dimensional space is relatively simple. More generally, $j_\nu(x)$ is elementary when ν is a proper half-integer [1, 4, 7]. In light of this discussion, it is striking that the next result is very recent.

Theorem 1.2 (Convolution formula for density in odd dimensions [3]). *Assume that the dimension $d = 2m+1$ is an odd number. Then for $x \geq 0$,*

$$p_n(m-1/2; x) = \frac{(2x)^{2m}\Gamma(m)}{\Gamma(2m)}\left(-\frac{1}{2x}\frac{d}{dx}\right)^m P_{m,n}(x),$$

where $P_{m,n}$ is the piecewise polynomial obtained from convolving

$$f_m(x) := \frac{\Gamma(m+1/2)}{\Gamma(1/2)\Gamma(m)}\begin{cases}(1-x^2)^{m-1} & \text{if } x \in [-1,1],\\ 0 & \text{otherwise,}\end{cases}$$

$n-1$ times with itself.

The expression in Theorem 1.2 above is both elegant and compact. It shows easily that in odd dimensions the density is a piecewise polynomial, but it can be difficult to manipulate or compute with or without a computer algebra system such as *Maple* or *Mathematica*. Note also that $p_n(m-1/2; x) = p_n(m-1/2; -x)$ in all cases.

2. Main result

We now use Theorem 1.2 to obtain an entirely explicit and tractable, convolution and differentiation free formula for $p_n(m-1/2; x)$, valid for all lengths and in all odd dimensions. We begin with a preliminary result which simplifies $P_{m,n}(x)$. We shall employ the *Heaviside* step function $H(x)$ which has $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$ and $H(0) = 1/2$. We also use the notation $[x^j]Q(x)$ to denote the coefficient of x^j in a polynomial Q.

Proposition 2.1. *Let $n \geq 1$ and $m \geq 1$. Then for $|x| \leq n$,*

$$\begin{aligned}P_{m,n}(x) = \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}H(n-2r+x)\\ \times \sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}[x^j]C_m(x)^r C_m(-x)^{n-r},\end{aligned} \tag{2.1}$$

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where

$$C_m(x) := \sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!} x^k.$$

Note that $C_m(x)$ satisfies the useful recurrence

$$C_m(x) = (2m-3)\, x\, C_{m-1}(x) + C_{m-2}(x).$$

Moreover, in terms of hypergeometric functions, $C_m(x) = {}_2F_0(m, 1-m; -x/2)$.

Proof. By the convolution theorem for the Fourier transform,

$$\mathcal{F}(P_{m,n}(x)) = \mathcal{F}(f_m(x))^n = \left(\frac{\Gamma(m+1/2)}{\Gamma(1/2)\Gamma(m)} \int_{-1}^{1} (1-x^2)^{m-1} e^{-iwx}\, dx\right)^n.$$

Observe that, for $m \geq 3$, $\mathcal{F}(f_m(x))$ satisfies the recurrence

$$T_m = \frac{(2m-1)(2m-3)}{w^2}(T_{m-1} - T_{m-2}),$$

which is also satisfied by

$$G_m(w) := \left(\frac{\Gamma(2m)}{2^m \Gamma(m)}\right) \sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!} (-1)^m \frac{2\cos(w + \frac{1}{2}\pi(m+k))}{w^{m+k}}.$$

This can be checked by hand. It can also easily be shown with the following *Maple 18* code.

```
with(inttrans ,fourier ):
f:=m -> piecewise(-1<=x and x<=1,
  GAMMA(m+1/2)/(GAMMA(m)*GAMMA(1/2)) * (1-x^2)^(m-1),0):
F:=m -> fourier(f(m),x,w):
simplify(F(m)-(2*m-1)*(2*m-3)/w^2*(F(m-1)-F(m-2)));
```

The above code returns 0 to indicate that $\mathcal{F}(f_m(x))$ satisfies the recurrence.

Correspondingly, we may execute the following *Maple 18* code.

```
G := m -> (GAMMA(2*m)/(2^m*GAMMA(m)))
    * sum( (m-1+k)!/(2^k*k!*(m-1-k)!)
    * (-1)^m * (2*cos(w+Pi/2*(m+k))/w^(m+k)), k=0..m-1 ):
simplify(G(m)-(2*m-1)*(2*m-3)/w^2*(G(m-1)-G(m-2)));
```

This returns 0 to show that $G_m(x)$ satisfies the same recurrence.

We can easily check that $\mathcal{F}(f_m(x))$ and $G_m(x)$ agree for $m = 1$ and $m = 2$, and so we may conclude that $\mathcal{F}(f_m(x)) = G_m(x)$ for all $m \geq 1$.

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Therefore,

$$\begin{aligned}
\mathcal{F}(P_{m,n}(x)) &= \Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n\Big(\sum_{k=0}^{m-1}\frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot(-1)^m\frac{2\cos(w+\frac{1}{2}\pi(m+k))}{w^{m+k}}\Big)^n \\
&= \Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n\Big(\sum_{k=0}^{m-1}\frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot(-1)^m\frac{(-1)^{m+k}e^{iw}+e^{-iw}}{(iw)^{m+k}}\Big)^n \\
&= \Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n\Big(\sum_{k=0}^{m-1}\frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot\frac{(-1)^m e^{iw}}{(-iw)^{m+k}} \\
&\qquad + \sum_{k=0}^{m-1}\frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot\frac{(-1)^m e^{-iw}}{(iw)^{m+k}}\Big)^n \\
&= \Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n\Big(e^{iw}\Big(\frac{1}{iw}\Big)^m C_m\Big(\frac{-1}{iw}\Big)+e^{-iw}\Big(\frac{-1}{iw}\Big)^m C_m\Big(\frac{1}{iw}\Big)\Big)^n \\
&= \Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n\sum_{r=0}^{n}\binom{n}{r}\frac{e^{iw(n-2r)}}{(iw)^{mn}}\Big((-1)^m C_m\Big(\frac{1}{iw}\Big)\Big)^r C_m\Big(\frac{-1}{iw}\Big)^{n-r} \\
&= \Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n\sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{(m-1)n}\frac{e^{iw(n-2r)}}{(iw)^{mn+j}}[x^j]C_m(x)^r C_m(-x)^{n-r}.
\end{aligned}$$

We can now reconstruct $P_{m,n}(x)$ from its Fourier transform, since

$$\mathcal{F}^{-1}\Big(\frac{e^{iw(n-2r)}}{(iw)^{mn+j}}\Big)=\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x)-\frac{1}{2}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}.$$

Thus, taking the inverse Fourier transform of $\mathcal{F}(P_{m,n}(x))$,

$$\begin{aligned}
P_{m,n}(x) = \Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n &\sum_{r=0}^{n}\binom{n}{r}(-1)^{mr} \\
&\times\sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x)[x^j]C_m(x)^r C_m(-x)^{n-r} \\
&+\frac{1}{2}\Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n\sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!} \\
&\times[x^j]C_m(x)^r C_m(-x)^{n-r}.
\end{aligned}\tag{2.2}$$

It remains only to show that the second double sum above is zero. Observe that when $x<-n$, $P_{m,n}(x)$ simplifies to

$$\frac{1}{2}\Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n\sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}[x^j]C_m(x)^r C_m(-x)^{n-r}.\tag{2.3}$$

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From the definition of convolution, we can easily deduce that $P_{m,n}(x)$ vanishes for $|x| > n$. It follows that (2.3) is zero for $x < -n$, but since it is a polynomial it must be zero everywhere. Thus, the latter term in (2.2) is zero, yielding (2.1). □

Next, we deal with the differential operator in Theorem 1.2.

LEMMA 2.2. *For all $F(x)$ and $m \geq 1$,*

$$\left(-\frac{1}{2x}\frac{d}{dx}\right)^m F(x) = \sum_{k=1}^{m} \frac{(-1)^k(2m-1-k)!}{2^{2m-k}(m-k)!(k-1)!}\frac{1}{x^{2m-k}}\left(\frac{d}{dx}\right)^k F(x). \tag{2.4}$$

PROOF. We proceed by induction. It is trivial to see that (2.4) is true for $m = 1$. Suppose it holds for some $m \geq 1$. Then

$$\begin{aligned}
&\left(-\frac{1}{2x}\frac{d}{dx}\right)^{m+1} F(x) \\
&\quad = \left(-\frac{1}{2x}\frac{d}{dx}\right)\sum_{k=1}^{m} \frac{(-1)^k(2m-1-k)!}{2^{2m-k}(m-k)!(k-1)!}\frac{1}{x^{2m-k}}\left(\frac{d}{dx}\right)^k F(x) \\
&\quad = \sum_{k=1}^{m} \frac{(-1)^{k+1}(2m-1-k)!}{2^{2m-k+1}(m-k)!(k-1)!}\left(\frac{1}{x^{2m-k+1}}\left(\frac{d}{dx}\right)^{k+1} F(x) - \frac{2m-k}{x^{2m-k+2}}\left(\frac{d}{dx}\right)^k F(x)\right) \\
&\quad = \sum_{k=2}^{m+1} \frac{(-1)^k(2m-k)!}{2^{2m-k+2}(m+1-k)!(k-2)!}\frac{1}{x^{2m-k+2}}\left(\frac{d}{dx}\right)^k F(x) \\
&\qquad + \sum_{k=1}^{m} \frac{(-1)^k(2m-k)!}{2^{2m-k+1}(m-k)!(k-1)!}\frac{1}{x^{2m-k+2}}\left(\frac{d}{dx}\right)^k F(x) \\
&\quad = \sum_{k=1}^{m+1} \frac{(-1)^k(2m+1-k)!}{2^{2m+2-k}(m+1-k)!(k-1)!}\frac{1}{x^{2m+2-k}}\left(\frac{d}{dx}\right)^k F(x).
\end{aligned}$$

Thus, (2.4) holds for all $m \geq 1$, proving the lemma. □

We are now ready to approach the probability density. Combining our previous results will allow us to fully expand $p_n(m - 1/2; x)$.

THEOREM 2.3 (Densities in odd dimensions). *Let $n \geq 2$ and $m \geq 1$. Then for $x \geq 0$,*

$$\begin{aligned}
p_n(m-1/2;x) = \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n &\sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}H(n-2r+x) \\
&\times \sum_{k=1}^{m}(-2)^k\binom{m-1}{k-1}\frac{(2m-1-k)!}{(2m-1)!}x^k \\
&\times \sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j-k}}{(mn-1+j-k)!}[x^j]C_m(x)^r C_m(-x)^{n-r},
\end{aligned} \tag{2.5}$$

where $H(x)$ is the Heaviside step function and

$$C_m(x) = \sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!} x^k.$$

Proof. By Theorem 1.2, Lemma 2.2 and Proposition 2.1,

$$\begin{aligned}
p_n(m-1/2;x) &= \frac{(2x)^{2m}\Gamma(m)}{\Gamma(2m)}\Big(-\frac{1}{2x}\frac{d}{dx}\Big)^m P_{m,n}(x) \\
&= \frac{(2x)^{2m}\Gamma(m)}{\Gamma(2m)} \sum_{k=1}^{m} \frac{(-1)^k(2m-1-k)!}{2^{2m-k}(m-k)!(k-1)!}\frac{1}{x^{2m-k}}\Big(\frac{d}{dx}\Big)^k P_{m,n}(x) \\
&= \Big(\frac{\Gamma(2m)}{2^m\Gamma(m)}\Big)^n \sum_{k=1}^{m} (-2)^k \binom{m-1}{k-1}\frac{(2m-1-k)!}{(2m-1)!}x^k \\
&\qquad \times \sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{mn-n}[x^j]C_m(x)^r C_m(-x)^{n-r}\Big(\frac{d}{dx}\Big)^k \\
&\qquad \times \frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x).
\end{aligned}$$

We can evaluate the above derivative directly, but we must be careful since there are jump discontinuities at $n-2r$ for $0 \le r \le n$. We shall see that these points are not an issue. Applying the general Leibniz rule, we obtain

$$\begin{aligned}
&\Big(\frac{d}{dx}\Big)^k \frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x) \\
&\quad = \sum_{a=0}^{k}\binom{k}{a}\Big(\Big(\frac{d}{dx}\Big)^a \frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}\Big)\Big(\Big(\frac{d}{dx}\Big)^{k-a} H(n-2r+x)\Big) \\
&\quad = \sum_{a=0}^{k}\binom{k}{a}\frac{(n-2r+x)^{mn-1+j-a}}{(mn-1+j-a)!}\Big(\Big(\frac{d}{dx}\Big)^{k-a} H(n-2r+x)\Big).
\end{aligned}$$

We shall see that the terms of this sum vanish except when $a = k$. Suppose $a < k$ and consider one such term. Clearly, $(d/dx))^{k-a}H(n-2r+x) = 0$ for $x \neq -n+2r$. Additionally, since $a < k \le m$ and $n \ge 2$ the exponent $mn-1+j-a$ is strictly positive, so $(n-2r+x)^{mn-1+j-a} = 0$ at $x = -n+2r$. Thus, the summand above vanishes for $a < k$, yielding

$$\Big(\frac{d}{dx}\Big)^k \frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x) = \frac{(n-2r+x)^{mn-1+j-k}}{(mn-1+j-k)!}H(n-2r+x).$$

We apply the above relation and the result follows from a simple rearrangement. $\square$

The formula we have presented is derived from the convolution form in Theorem 1.2 and produces an even function. However, $p_n(m-1/2;x)$ is the probability density function of a nonnegative random variable, so it must be 0 for negative values of x. We may use this fact to significantly reduce the number of terms in our formula, halving the time needed to compute $p_n(m-1/2;x)$ for given values of n and m.

Corollary 2.4. *Let $n \geq 2$ and $m \geq 1$. Then for $x \geq 0$,*

$$p_n(m-1/2;x) = \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{r}(-1)^{mr} H(n-2r-x)$$
$$\times \sum_{k=1}^{m} 2^k \binom{m-1}{k-1}\frac{(2m-1-k)!}{(2m-1)!} x^k$$
$$\times \sum_{j=0}^{(m-1)n} \frac{(n-2r-x)^{mn-1+j-k}}{(mn-1+j-k)!}[x^j]C_m(x)^r C_m(-x)^{n-r}.$$

Proof. Since our formula (2.5) is even (easily seen in Theorem 1.2), for $x \geq 0$

$$p_n(m-1/2;x) = p_n(m-1/2;-x)$$
$$= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n} \binom{n}{r}(-1)^{mr} H(n-2r-x)$$
$$\times \sum_{k=1}^{m} 2^k \binom{m-1}{k-1}\frac{(2m-1-k)!}{(2m-1)!} x^k$$
$$\times \sum_{j=0}^{(m-1)n} \frac{(n-2r-x)^{mn-1+j-k}}{(mn-1+j-k)!}[x^j]C_m(x)^r C_m(-x)^{n-r},$$

by Theorem 2.3. Observe that when $r > \lfloor (n-1)/2 \rfloor$, $H(n-2r-x)$ is zero on $(0,\infty)$. At $x=0$, every term is 0 for all values of r. Thus, when $x \geq 0$, we may simply omit the terms where $r > \lfloor (n-1)/2 \rfloor$. So we let r range from 0 to $\lfloor (n-1)/2 \rfloor$ in the sum, which yields our result directly. □

We finish with two examples echoing the direct analyses in [7].

Example 2.5 (Density in three dimensions). In $\mathbb{R}^3$, we have $C_1(x)=1$ so for $n \geq 2$ and $x \geq 0$, the density reduces to

$$p_n(1/2;x) = \frac{-x}{2^{n-1}} \sum_{r=0}^{n} \binom{n}{r}(-1)^r H(n-2r+x)\frac{(n-2r+x)^{n-2}}{(n-2)!}.$$

In particular, as shown in Figure 1,

$$p_2(1/2;x) = \begin{cases} 0 & \text{if } x<0, \\ x/2 & \text{if } x \in [0,2), \\ 0 & \text{if } x>2, \end{cases}$$

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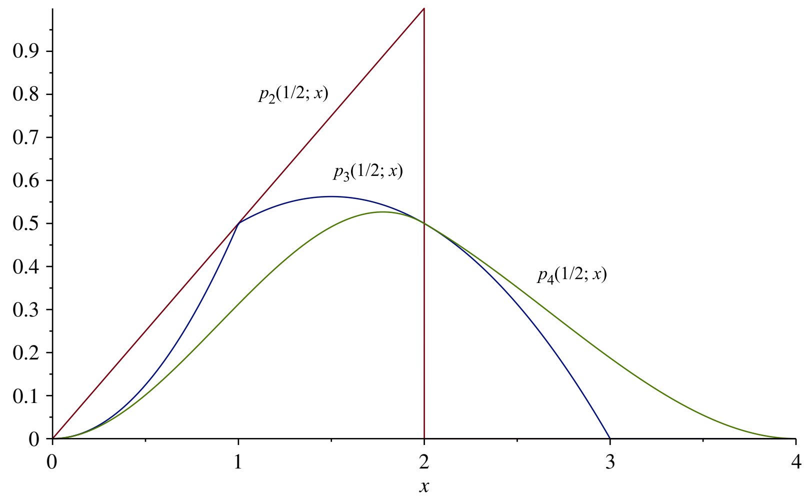

FIGURE 1. $p_n(1/2; x)$ for $n = 2, 3, 4$.

$$p_3(1/2; x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2}x^2 & \text{if } x \in [0, 1), \\ -\frac{1}{4}x^2 + \frac{3}{4}x & \text{if } x \in [1, 3), \\ 0 & \text{if } x > 3, \end{cases}$$

$$p_4(1/2; x) = \begin{cases} 0 & \text{if } x < 0, \\ -\frac{3}{16}x^3 + \frac{1}{2}x^2 & \text{if } x \in [0, 2), \\ \frac{1}{16}x^3 - \frac{1}{2}x^2 + x & \text{if } x \in [2, 4), \\ 0 & \text{if } x > 4. \end{cases}$$

EXAMPLE 2.6 (Density in five dimensions). In $\mathbb{R}^5$, we have $C_2(x) = 1 + x$, so for $n \geq 2$ and $x \geq 0$, the density reduces to

$$\begin{aligned} p_n(3/2; x) = {} & \left(\frac{3}{2}\right)^{n-1} \sum_{r=0}^{n} \binom{n}{r} H(n - 2r + x) \\ & \times \sum_{j=0}^{n} \frac{(n - 2r + x)^{2n-3+j}}{(2n - 3 + j)!} \left(x^2 - x \frac{(n - 2r + x)}{(2n - 2 + j)}\right) \sum_{l=0}^{j} (-1)^{j-l} \binom{r}{l} \binom{n - r}{j - l}. \end{aligned}$$

In particular, as shown in Figure 2,

$$p_2(3/2; x) = \begin{cases} 0 & \text{if } x < 0, \\ -\frac{3}{16}x^5 + \frac{3}{4}x^3 & \text{if } x \in [0, 2), \\ 0 & \text{if } x > 2, \end{cases}$$

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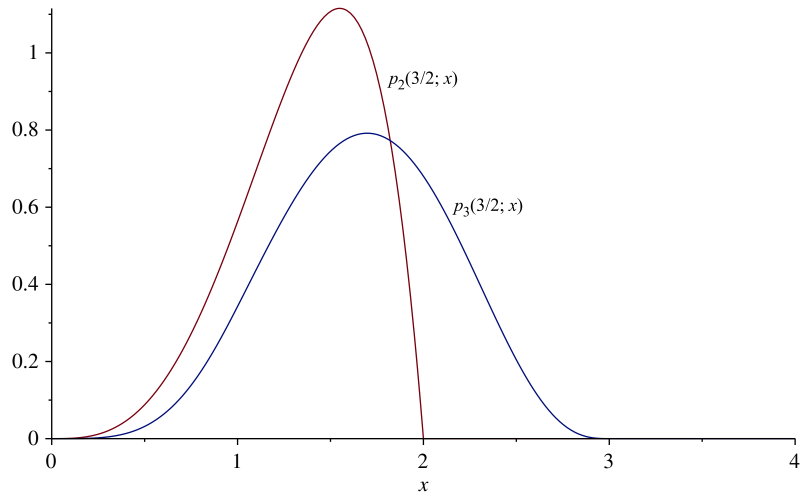

FIGURE 2. $p_n(3/2; x)$ for $n = 2, 3$.

$$p_3(3/2; x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{3}{560}x^8 - \frac{9}{40}x^6 + \frac{9}{16}x^4 & \text{if } x \in [0, 1), \\ -\frac{3}{1120}x^8 + \frac{9}{80}x^6 - \frac{9}{32}x^5 - \frac{9}{32}x^4 + \frac{81}{80}x^3 - \frac{243}{1120}x & \text{if } x \in [1, 3), \\ 0 & \text{if } x > 3. \end{cases}$$

As these examples demonstrate, Theorem 2.3 always provides an explicit, workable expression for $p_n(m - 1/2; x)$ with clearly indicated structure. We finish by observing that, if $W_n(m - 1/2, s) := \int_{x=0}^{n} x^s\, p_n(m - 1/2; x)\, dx$ denotes the *moment function*, we may also obtain an explicit formula for $W_n(m - 1/2, s)$.

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References

[1] J. M. Borwein, A. Straub and C. Vignat, 'Densities of short uniform random walks in higher dimensions', submitted for publication, 2015. Preprint, arXiv:1508.04729.

[2] J. M. Borwein, A. Straub, J. Wan and W. Zudilin, 'Densities of short uniform random walks', *Canad. J. Math.* **64**(5) (2012), 961–990; with an Appendix by Don Zagier.

[3] R. García-Pelayo, 'Exact solutions for isotropic random flights in odd dimensions', *J. Math. Phys.* **53**(10) 103504 (2012), 6–7.

[4] B. D. Hughes, *Random Walks and Random Environments,* Vol. 1, in: *Random Walks* (Clarendon Press, Oxford, 1995). See, in particular, pages 53–73 and 86–93.

[5] J. C. Kluyver, 'A local probability problem', *Nederl. Acad. Wetensch. Proc.* **8** (1906), 341–350.
[6] K. Pearson, 'A mathematical theory of random migration', in: *Drapers Company Research Memoirs, Biometric Series, 3* (Dulau and Company, London, 1906), 4–24.
[7] L. Rayleigh, 'On the problem of random vibrations, and of random flights in one, two, or three dimensions', *Phil. Mag. Ser. 6* **37**(220) (1919), 321–347.

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