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Multiple Solutions for Nonlinear Periodic Problems

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Abstract. We consider a nonlinear periodic problem driven by a nonlinear nonhomogeneous differential operator and a Carathéodory reaction term $f(t,x)$ that exhibits a $(p-1)$-superlinear growth in $x \in \mathbb{R}$ near $\pm\infty$ and near zero. A special case of the differential operator is the scalar p-Laplacian. Using a combination of variational methods based on the critical point theory with Morse theory (critical groups), we show that the problem has three nontrivial solutions, two of which have constant sign (one positive, the other negative).

1 Introduction

The aim of this paper is to study the existence of multiple nontrivial solutions for the following nonlinear periodic problem:

$$\begin{aligned} &-\big(\alpha\big(t,u'(t)\big)\big)' = f\big(t,u(t)\big) \text{ a.e. on } T=[0,b], \\ &u(0)=u(b), \quad u'(0)=u'(b). \end{aligned} \tag{1.1}$$

Here $\alpha\colon T\times\mathbb{R}\to\mathbb{R}$ is a continuous map such that for all $t\in T$, $\alpha(t,\cdot)$ is strictly monotone and C^1 on $\mathbb{R}\setminus\{0\}$. A special case of the differential operator in (1.1), is the scalar p-Laplacian. The reaction term $f(t,x)$ is a Carathéodory function (*i.e.*, $t\to f(t,x)$ is measurable and $x\to f(t,x)$ is continuous), and we assume that $f(t,\cdot)$ exhibits a $(p-1)$-superlinear growth near $\pm\infty$ and near 0, $1<p<\infty$. However, to express this $(p-1)$-superlinearity at $\pm\infty$, we do not employ the Ambrosetti–Rabinowitz condition (AR-condition), which is normally used in such cases, but instead we use a less restrictive hypothesis.

Multiplicity results for nonlinear Sturm–Liouville and periodic problems were proved by Aizicovici, Papageorgiou, and Staicu [1]; Ben Naoum and De Coster [2]; De Coster [5]; del Pino, Manásevich, and Murúa [6]; Gasiński and Papageorgiou [8]; Manásevich, Njoku, and Zanolin [10]; Njoku and Zanolin [12]; Papageorgiou and Papageorgiou [14]; Papageorgiou and Papalini [15], and Yang [16]. In all of these papers, the differential operator is the scalar p-Laplacian, and the reaction term is either $(p-1)$-linear or $(p-1)$-sublinear near $\pm\infty$. It appears that the question of the existence of multiple solutions for "$(p-1)$-superlinear" periodic problems has not previously been addressed. We also emphasize that in contrast to the scalar p-Laplacian, the differential operator here needs not be homogeneous.

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2 Mathematical Background and Hypotheses

Our approach combines variational methods based on the critical point theory with Morse theory (critical groups).

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition* (C-condition) if the following is true: "Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1+\|x_n\|)\varphi'(x_n) \to 0$ in X^* as $n \to \infty$ admits a strongly convergent subsequence." Using this notion, we can have the following minimax characterization of certain critical values of φ, known in the literature as the *mountain pass theorem*.

Theorem 2.1 *If X is a Banach space and $\varphi \in C^1(X)$ satisfies the C-condition, and for $x_0, x_1 \in X$, $r > 0$, $\|x_0 - x_1\| > r$, we have*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\big[\varphi(x) : \|x - x_0\| = r\big] = \eta_r$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi\big(\gamma(t)\big)$, where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\},$$

then $c \geq \eta_r$ and c is a critical value of φ.

For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$ we set $\varphi^c = \{x \in X : \varphi(x) \leq c\}$ and $K_\varphi = \{x \in X : \varphi'(x) = 0\}$.

Let $Y_2 \subseteq Y_1 \subseteq X$. Then for every integer $k \geq 0$, let $H_k(Y_1, Y_2)$ denote the kth-singular relative homology group for the pair (Y_1, Y_2) with the coefficients in $\mathbb{Z}$. The critical groups of φ at an isolated critical point $x_0 \in X$ with $c = \varphi(x_0)$ are defined by $C_k(\varphi, x_0) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}\big)$ for all $k \geq 0$, where U is a neighborhood of x_0 such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology implies that this definition is independent of the particular choice of the neighborhood U. Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity, are defined by $C_k(\varphi, \infty) = H_k(X, \varphi^c)$ for all $k \geq 0$. The deformation theorem implies that this definition is independent of the choice of c. Suppose K_φ is finite. We set $M(t, x) = \sum_{k\geq 0} \operatorname{rank}\, C_k(\varphi, x) t^k$, $P(t, \infty) = \sum_{k\geq 0} \operatorname{rank}\, C_k(\varphi, \infty) t^k$. The *Morse relation* says that

$$\sum_{x\in K_\varphi} M(t, x) = P(t, \infty) + (1 + t)Q(t), \tag{2.1}$$

where $Q(t) = \sum_{k\geq 0} \beta_k t^k$ is a formal series with nonnegative integer coefficients β_k (see Chang [3]).

In the study of problem (1.1), we shall use the following two spaces:

$$W^{1,p}_{\text{per}}(0, b) = \{u \in W^{1,p}(0, b) : u(0) = u(b)\},$$
$$\widehat{C}(T) = C^1(T) \cap W^{1,p}_{\text{per}}(0, b) = \{u \in C^1(T) : u(0) = u(b)\}.$$

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Note that $\widehat{C}(T)$ is an ordered Banach space with positive cone $\widehat{C}_+ = \{u \in \widehat{C}(T) : u(t) \geq 0 \text{ for all } t \in T\}$. This cone has a nonempty interior $\operatorname{int} \widehat{C}_+ = \{u \in \widehat{C}_+ : u(t) > 0 \text{ for all } t \in T\}$.

The hypotheses on $\alpha(t,x)$ are the following:

H(α) : $\alpha(t,x) = h(t,|x|)x$ for all $(t,x) \in T \times \mathbb{R}$, where $h(t,x) > 0$ for all $t \in T$, all $x > 0$, and

- (i) $\alpha \in C(T \times \mathbb{R}) \cap C^1\big(T \times (\mathbb{R} \setminus \{0\})\big)$;
- (ii) there exist $0 < c_0 < c_1$ such that $c_0|x|^{p-2} \leq \alpha_x'(t,x) \leq c_1|x|^{p-2}$, $1 < p < \infty$, for all $(t,x) \in T \times (\mathbb{R} \setminus \{0\})$;
- (iii) if $G(t,x) = \int_0^x \alpha(t,s)ds$, then there exists $\eta \in L^1(T)$ such that $pG(t,x) - \alpha(t,x)x \geq \eta(t)$ for a.a. $t \in T$, all $x \in \mathbb{R}$.

Remark 2.2 Evidently, for a.a. $t \in T$, $\alpha(t,\cdot)$ is strictly monotone, $G(t,\cdot)$ is strictly convex, and $G(t,x) \leq \alpha(t,x)x$. Moreover,

$$|\alpha(t,x)| \leq \frac{c_1}{p-1}|x|^{p-1}, \quad \alpha(t,x)x \geq \frac{c_0}{p-1}|x|^p,$$

$$\text{and} \quad \frac{c_0}{p(p-1)}|x|^p \leq G(t,x) \leq \frac{c_1}{p(p-1)}|x|^p$$

for all $(t,x) \in T \times \mathbb{R}$.

Example 2.3 The following functions $\alpha(t,x)$ satisfy hypotheses H(α). Here $\vartheta \in C^1(T)$ with $\vartheta(t) > 0$ for all $t \in T$:

- $\alpha(t,x) = \vartheta(t)|x|^{p-2}x$, $1 < p < \infty$ (corresponds to the weighted scalar p-Laplacian);
- $\alpha(t,x) = \vartheta(t)\big[|x|^{p-2}x + |x|^{q-2}x\big]$ for $2 \leq p$;
- $\alpha(t,x) = \begin{cases} \vartheta(t)\big[|x|^{p-2}x + |x|^{q-2}x\big], & \text{if } |x| \leq 1, \\ \vartheta(t)\big[|x|^{p-2}x + c|x|^{r-2}x - (c-1)x\big] & \text{if } |x| > 1. \end{cases}$
  with $1 < r < p \leq q$ and $r < 2 \leq p$ or $2 < r \leq p < q$;
- $\alpha(t,x) = \vartheta(t)\Big[|x|^{p-2}x + \dfrac{|x|^{p-2}x}{1+|x|^p}\Big]$ for $1 < p \leq 2$.

H(f) : $f \colon T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(t,0) = 0$ for a.a. $t \in T$ and

- (i) $|f(t,x)| \leq \widehat{\alpha}(t)(1 + |x|^{r-1})$ for a.a $t \in T$, all $x \in \mathbb{R}$ with $\widehat{\alpha} \in L^1(T)_+$, $p < r < \infty$;
- (ii) if $F(t,x) = \int_0^x f(t,s)ds$, then $\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^p} = +\infty$ uniformly for a.a. $t \in T$ and there exist $\mu > r - p$ and $\beta_0 > 0$ such that

$$\beta_0 \leq \liminf_{|x|\to\infty} \frac{f(t,x) - pF(t,x)}{|x|^\mu} \text{ uniformly for a.a. } t \in T;$$

- (iii) $\lim_{x\to 0} \frac{f(t,x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $t \in T$;

(iv) there exist $\widehat{c} > 0$ and $\delta_0 > 0$ such that $f(t,x)x \geq -\widehat{c}|x|^p$ for a.a. $t \in T$, all $x \in \mathbb{R}$ and $F(t,x) \leq 0$ for a.a. $t \in T$, all $|x| \leq \delta_0$.

Remark 2.4 Hypotheses H(f)(ii) and (iii) imply the $(p-1)$-superlinear growth near $\pm\infty$ and near 0. However, we do not employ the AR-condition, which says that there exist $\tau > p$ and $M > 0$ such that $0 < \tau F(t,x) \leq f(t,x)x$ for a.a. $t \in T$ and all $|x| \geq M$. Integrating this inequality, we obtain $\widetilde{c}\,|x|^\tau \leq F(t,x)$ for a.a. $t \in T$ and all $|x| \geq M$. Therefore the AR-condition dictates at least a τ-growth near $\pm\infty$ for $F(t,\cdot)$. In contrast, H(f)(ii) is much weaker and permits slower growth near $\pm\infty$. Similar conditions were also used by Costa and Magalhaes [4] and Fei [7].

Example 2.5 The following functions satisfy H(f) (for the sake of simplicity we drop the t-dependence):

$$f_1(x) = |x|^{r-2}x - |x|^{p-2}x \text{ with } p < r \text{ and}$$
$$f_2(x) = |x|^{p-2}x(\ln|x|^p + 1).$$

Note that f_2 does not satisfy the AR-condition.

In what follows, for the sake of notational simplicity, we set $W = W^{1,p}_{\text{per}}(0,b)$. Let $A \colon W \to W^*$ be the nonlinear map defined by $\langle A(u), y\rangle = \int_0^b \alpha(t,u')y'\mathrm{d}t$ for all $u, y \in W$. From Papageorgiou and Kyritsi [13], we have the following proposition.

Proposition 2.6 *The map $A\colon W \to W^*$ defined above is maximal monotone, strictly monotone and of type $(S)_+$* i.e., *if $u_n \xrightarrow{w} u$ in W and* $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$, *then $u_n \to u$ in W.*

3 Solutions of Constant Sign

Let $\varphi\colon W \to \mathbb{R}$ be the Euler functional for problem (1.1) defined by

$$\varphi(u) = \int_0^b G(t,u')\mathrm{d}t - \int_0^b F(t,x)\mathrm{d}t \quad \text{for all } u \in W.$$

Evidently $\varphi \in C^1(W)$. Also for $\lambda > 0$, let

$$f_+^\lambda(t,x) = \begin{cases} 0 & \text{if } x \leq 0, \\ f(t,x) + \lambda x^{p-1} & \text{if } x > 0, \end{cases} \quad \text{and}$$

$$f_-^\lambda(t,x) = \begin{cases} f(t,x) + \lambda x^{p-1} & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Set $F_\pm^\lambda(t,x) = \int_0^b f_\pm^\lambda(t,s)\mathrm{d}s$ and consider the C^1-functionals $\varphi_\pm^\lambda \colon W \to \mathbb{R}$ defined by $\varphi_\pm^\lambda(u) = \int_0^b G(t,u')\mathrm{d}t + \frac{\lambda}{p}\|u\|_p^p - \int_0^b F_\pm^\lambda(t,u)\mathrm{d}t$.

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Proposition 3.1 *If hypotheses $H(\alpha)$ and $H(f)$ hold, then φ and $\varphi_{\pm}^{\lambda}$ satisfy the C-condition.*

Proof First we prove this for φ. So, let $\{u_n\}_{n\geq 1} \subseteq W$ be a sequence such that

$$|\varphi(u_n)| \leq M_1 \text{ for some } M_1 > 0, \quad \text{and } (1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^* \text{ as } n \to \infty. \tag{3.1}$$

From the convergence in (3.1) we have

$$\Big| \langle A(u_n), h\rangle - \int_0^b f(t, u_n) h \mathrm{d}t \Big| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{3.2}$$

for all $h \in W$, with $\varepsilon_n \to 0^+$,

$$\Rightarrow - \int_0^b \alpha(t, u_n') u_n' \mathrm{d}t + \int_0^b f(t, u_n) u_n \mathrm{d}t \leq \varepsilon_n \text{ for all } n \geq 1 \tag{3.3}$$

(choosing $h = u_n \in W$). From (3.1), we also have

$$\int_0^b pG(t, u_n')\mathrm{d}t - \int_0^b pF(t, u_n)\mathrm{d}t \leq pM_1 \text{ for all } n \geq 1, \tag{3.4}$$
$$\Rightarrow \int_0^b \Big[f(t, u_n)u_n - pF(t, u_n) \Big] \mathrm{d}t \leq M_2 \ \text{ for some } M_2 > 0, \ \text{all } n \geq 1$$

(adding (3.3) and using $H(\alpha)$(iii)).

Hypotheses $H(f)$(i) and (ii) imply that we can find $\beta_1 \in (0, \beta_0)$ and $\widehat{a}_1 \in L^1(T)_+$ such that

$$\beta_1 |x|^{\mu} - \widehat{a}_1(t) \leq f(t, x)x - pF(t, x) \text{ for all } t \in T, \ \text{all } x \in \mathbb{R}. \tag{3.5}$$

Using (3.5) in (3.4) we infer that $\{u_n\}_{n\geq 1} \subseteq L^{\mu}(T)$ is bounded. It is clear that we can always assume that $\mu < r$. Let $t \in (0, 1)$ be such that $\frac{1}{r} = \frac{1-t}{\mu}$. Invoking the interpolation inequality, we can find $M_3 > 0$ such that

$$\|u_n\|_r^r \leq M_3 \|u_n\|^{tr} \text{ for all } n \geq 1. \tag{3.6}$$

In (3.2) we choose $h = u_n \in W$, and using the properties of $\alpha(t, x)$, $H(f)$(i) and (3.6), we have

$$\frac{c_0}{p-1} \|u_n'\|_p^p \leq c_2 (1 + \|u_n\| + \|u_n\|^{tr}) \ \text{ for some } \ c_2 > 0, \ \text{all } n \geq 1. \tag{3.7}$$

Recall that $u \to \|u'\|_p + \|u\|_{\mu}$ is equivalent to the Sobolev norm. Since $\{u_n\}_{n\geq 1} \subseteq L^{\mu}(T)$ is bounded and using (3.7) (note $tr = r - \mu < p$), we infer that $\{u_n\}_{n\geq 1} \subseteq W$ is bounded. So, we may assume that $u_n \xrightarrow{w} u$ in W and $u_n \to u$ in $C(T)$. In (3.2) we

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set $h = u_n - u$ and pass to the limit as $n \to \infty$. We obtain $\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0$, and so $u_n \to u$ in W (see Proposition 2.6). Therefore, φ satisfies the C-condition.

Next, with a slight variation of the above proof, we show that $\varphi_\pm^\lambda$ satisfy the C-condition. So, as before, let $\{u_n\}_{n\geq 1} \subseteq W$ be a sequence such that

$$\text{(3.8)} \quad \left|\varphi_+^\lambda(u_n)\right| \leq M_3 \quad \text{for some } M_3 > 0, \quad \text{all } n \geq 1 \\ \text{and} \quad (1 + \|u_n\|)(\varphi_+^\lambda)'(u_n) \to 0 \quad \text{in } W^* \quad \text{as } n \to \infty.$$

From the convergence in (3.8), we have

$$\text{(3.9)} \quad \left| \langle A(u_n), h\rangle + \lambda \int_0^b |u_n|^{p-2} u_n h \mathrm{d}t - \int_0^b f(t, u_n^+) h \mathrm{d}t - \lambda \int_0^b (u_n^+)^{p-1} \right| \\ \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \text{ for all } n \geq 1.$$

In (3.9) we choose $h = -u_n^- \in W$ and obtain $\frac{c_0}{p-1}\|(u_n^-)'\|_p^p + \lambda\|u_n^-\|_p^p \leq \varepsilon_n$ for all $n \geq 1$, hence

$$\text{(3.10)} \qquad u_n^- \to 0 \quad \text{in} \quad W \quad \text{as } n \to \infty.$$

Next, if in (3.9) we choose $h = u_n^+ \in W$ and as before, we use (3.8) and (3.10), we obtain $\int_0^b \big[f(t, u_n^+)u_n^+ - pF(t, u_n^+)\big] \mathrm{d}t \leq M_4$ for some $M_4 > 0$, all $n \geq 1$. From this, as in the first part of the proof, using H(f)(ii) and the interpolation inequality, we show that $\{u_n^+\}_{n\geq 1} \subseteq W$ is bounded. This fact and (3.10), imply that $\{u_n\}_{n\geq 1} \subseteq W$ is bounded, from which via Proposition 2.6, we conclude that φ_+^λ satisfies the C-condition. We proceed similarly for φ_-^λ. ∎

Proposition 3.2 *If hypotheses H(α) and H(f) hold, then $u = 0$ is a local minimizer of $\varphi_\pm^\lambda$ and φ.*

Proof We prove this for φ_+^λ, the proofs for φ_-^λ and φ being similar. Let $\delta_0 > 0$ be as postulated by hypothesis H(f)(iv) and let $u \in \widehat{C}(T)$ such that $\|u\|_{C^1(T)} \leq \delta_0$. Then $\varphi_+^\lambda(u) \geq \frac{c_0}{p(p-1)}\|u'\|_p^p + \frac{\lambda}{p}\|u^-\|_p^p \geq 0 = \varphi_+^\lambda(0)$. Hence $u = 0$ is a local $\widehat{C}(T)$-minimizer of φ_+^λ. From Papageorgiou–Papalini [15, Proposition 5], we infer that $u = 0$ is also a local W-minimizer of φ_+^λ. ∎

Clearly hypothesis H(f)(ii) implies the following proposition.

Proposition 3.3 *If hypotheses H(α) and H(f) hold, then $\varphi_\pm^\lambda(\xi) \to \pm\infty$ as $\xi \to \pm\infty$, $\xi \in \mathbb{R}$.*

Now we are ready to produce two nontrivial solutions of constant sign.

Proposition 3.4 *If hypotheses H(α) and H(f) hold, then problem (1.1) has two nontrivial solutions of constant sign $u_0 \in \mathrm{int}\widehat{C}_+$, $v_0 \in -\mathrm{int}\widehat{C}_+$.*

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Proof From Proposition 3.2, we know that $u = 0$ is a local minimizer of φ_+^λ. We may assume that $u = 0$ is isolated, or, otherwise, we can easily see that we have a whole sequence of distinct positive solutions. Hence, reasoning as in Papageorgiou and Papalini [15, Proposition 6], we can find $\varrho \in (0,1)$ small such that

$$\varphi_+^\lambda(0) = 0 < \inf\big[\varphi_+^\lambda(u) : \|u\| = \varrho\big] = m_\varrho. \tag{3.11}$$

Then (3.11) together with Propositions 3.1 and 3.4 permits the use of Theorem 2.1 (the mountain pass theorem), and we obtain $u_0 \in W$ such that

$$\varphi_+^\lambda(0) = 0 < m_\varrho \le \varphi_+^\lambda(u_0) \quad \text{and} \quad (\varphi_+^\lambda)'(u_0) = 0. \tag{3.12}$$

From the inequality in (3.12), we infer that $u_0 \neq 0$. From the equality in (3.12), we have

$$A(u_0) + \lambda |u_0|^{p-2} u_0 = N_+^\lambda(u_0), \tag{3.13}$$

where $N_+^\lambda(u)(\cdot) = f_+^\lambda\big(\cdot, u(\cdot)\big)$ for $u \in W$. Acting on (3.13) with $-u_0^- \in W$, we obtain $u_0 \ge 0$, $u_0 \neq 0$. Then (3.13) becomes $A(u_0) = N(u_0)$, where $N(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in W$. Hence $u_0 \in \widehat{C}(T)$ and solves (1.1) (see [8]). Also, from $H(f)$(iv) we have $(\alpha(t, u'(t)))' \le \widehat{c} u_0(t)^{p-1}$ a.a. on T, hence $u_0 \in \operatorname{int} \widehat{C}_+$ (see Montenegro [11]). Similarly, working with φ_-^λ, we obtain another constant sign solution $v_0 \in -\operatorname{int} \widehat{C}_+$. ■

4 Critical Groups at Infinity

In this section we compute the critical groups at infinity for φ and $\varphi_\pm^\lambda$.

Proposition 4.1 *If hypotheses $H(\alpha)$ and $H(f)$ hold, then $C_k(\varphi, \infty) = 0$ for all $k \ge 0$.*

Proof Hypotheses $H(f)$(i) and (ii) imply that given any $\xi > 0$, we can find $\widehat{\alpha}_2 \in L^1(T)_+$ such that

$$F(t,x) \ge \frac{\xi}{p}|x|^p - \widehat{\alpha}_2(t) \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R}. \tag{4.1}$$

Let $u \in \partial B_1 = \{u \in W : \|u\| = 1\}$ and $s > 0$. Then

$$\begin{aligned} \varphi(su) &\le \frac{c_1 s^p}{p(p-1)} \|u'\|_p^p - \int_0^b F(t, su) dt \\ &\le \frac{s^p}{p}\Big(\frac{1}{p-1} - \xi \|u\|_p^p\Big) + \|\widehat{\alpha}_2\|_1 \quad \text{(see (4.1))}. \end{aligned} \tag{4.2}$$

Choosing $\xi > \frac{1}{(p-1)\|u\|_p^p}$, from (4.2) we see that

$$\varphi(su) \le \frac{c_1 s^p}{p(p-1)} \|u'\|_p^p - \int_0^b F(t, su) dt \to -\infty \quad \text{as } s \to +\infty. \tag{4.3}$$

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Using (3.5), for every $u \in W$ we have

$$\int_0^b \big[\, pF(t,su) - f(t,u)u\big]\,\mathrm{d}t \le -\beta_1 \int_0^b |u|^\mu \mathrm{d}t + \|\widehat{\alpha}_2\|_1. \tag{4.4}$$

Choose $\eta < -(\|\widehat{\alpha}_2\|_1 + 1) < 0$. From (4.3) we see that for $s > 0$ large and $u \in \partial B_1$, we have

$$p\,\varphi(su) \le \frac{c_1 s^p}{p-1} - \int_0^b pF(t,su)\mathrm{d}t \le \eta < 0 = \varphi(0) \quad (\text{since } \|u\| = 1). \tag{4.5}$$

Clearly then, we can find $s^* > 0$ such that $\varphi(s^* u) = \eta$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\,\varphi(su) = \langle \varphi'(su), u\rangle &= \int_0^b \alpha(t,su')u'\mathrm{d}t - \int_0^b f(t,su)u\mathrm{d}t \\
&\le \frac{1}{s}\Big[s^p \frac{c_1}{p-1} - \int_0^b pF(t,su)u\mathrm{d}t + \|\widehat{\alpha}_2\|_1\Big] \quad (\text{see } (4.4)) \\
&\le \frac{1}{s}\big[\eta + \|\widehat{\alpha}_2\|_1\big] < 0 \quad (\text{see } (4.5) \text{ and recall that } \eta < -(\|\widehat{\alpha}_2\|_1 + 1)).
\end{aligned}$$

It follows that the above $s^* > 0$ is unique, and we denote it by $\gamma(u) > 0$. We have $\varphi\big(\gamma(u)u\big) = \eta$, $u \in \partial B_1$, and moreover, the implicit function theorem implies that $\gamma \in C(\partial B_1)$. We set $\widehat{\gamma}(u) = \frac{1}{\|u\|}\gamma\big(\frac{u}{\|u\|}\big)$ for all $u \in W \setminus \{0\}$. Then $\widehat{\gamma} \in C(W \setminus \{0\})$ and $\varphi\big(\widehat{\gamma}(u)u\big) = \eta$ for all $u \in W \setminus \{0\}$. Also, $\varphi(u) = \eta$ implies $\widehat{\gamma}(u) = 1$. Hence, if for $u \neq 0$ we set

$$\widehat{\gamma}_0(u) = \begin{cases} 1 & \text{if } u \in \varphi^\eta, \\ \widehat{\gamma}(u) & \text{if } u \notin \varphi^\eta, \end{cases}$$

then $\widehat{\gamma}_0 \in C(W \setminus \{0\})$. Consider the homotopy $h(\tau,u) = (1-\tau)u + \tau\widehat{\gamma}_0(u)u$. We have $h(0,u) = u$, $h(1,u) \in \varphi^\eta$ and $h(\tau,\cdot)|_{\varphi^\eta} = \mathrm{id}|_{\varphi^\eta}$ for all $\tau \in [0,1]$, hence φ^η is a strong deformation retract of $W \setminus \{0\}$. Also, by considering the homotopy $h(\tau,u) = (1-\tau)u + \tau\frac{u}{\|u\|}$, we see that ∂B_1 is a strong deformation retract of $W \setminus \{0\}$. Thus φ^η and ∂B_1 are homotopy equivalent and ∂B_1 is contractible in itself. Therefore choosing $\eta < \inf \varphi(K_\varphi)$, we conclude that

$$C_k(\varphi,\infty) = H_k(W,\varphi^\eta) = H_k(W,\partial B_1) = 0 \quad \text{for all } k \ge 0$$

(see Granas–Dugundji [9]). ■

In a similar way, we show the triviality of the critical groups at infinity of $\varphi_\pm^\lambda$.

Proposition 4.2 *If hypotheses $H(\alpha)$ and $H(f)$ hold, then $C_k(\varphi_\pm^\lambda,\infty) = 0$ for all $k \ge 0$.*

Proof We prove this for φ_+^λ, the proof for φ_-^λ being similar. Again, from H(f)(i) and (ii) we see that for any $\xi > 0$, we can find $\widehat{\alpha}_3 \in L^1(T)_+$ such that

$$F(t, x^+) \geq \frac{\xi}{p}(x^+)^p - \widehat{\alpha}_3(t) \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R}. \tag{4.6}$$

Let $\partial B_+ = \{u \in \partial B_1 : u^+ \neq 0\}$. Then for $u \in \partial B_+$ and $s > 0$ we have

$$\varphi_+^\lambda(su) \leq s^p\Big[\frac{c_1}{p-1} + \lambda\|u^-\|_p^p - \xi\|u^+\|_p^p\Big] + \|\widehat{\alpha}_3\|_1 \to -\infty \quad \text{as } s \to +\infty \tag{4.7}$$

(see (4.6) and recall that $\xi > 0$ is arbitrary).

From H(f)(i) and (ii), we can find $\beta_1 \in (0, \beta_0)$ and $\widehat{\alpha}_4 \in L^1(T)_+$ such that

$$f(t, x^+)x^+ - pF(t, x^+) \geq \beta_1(x^+)^\mu - \widehat{\alpha}_4(t) \quad \text{a.a. } t \in T, \text{ all } x \in \mathbb{R}. \tag{4.8}$$

For $u \in W$, we have

$$\int_0^b \Big[pF(t, u^+) - f(t, u^+)u^+\Big]\,\mathrm{d}t \leq -\beta_1\int_0^b (u^+)^\mu \mathrm{d}t + \|\widehat{\alpha}_4\|_1 \quad \text{(see (4.8))}. \tag{4.9}$$

Choose $\eta < -(\|\widehat{\alpha}_4\|_1 + 1) < 0$. From (4.6) and (4.7), we see that for $s > 0$ large

$$s^p\Big[\frac{c_1}{p-1} + \lambda\|u^-\|_p^p - \int_0^b pF(t, su^+)\mathrm{d}t\Big] \leq \eta < 0 = \varphi_+^\lambda(0). \tag{4.10}$$

Hence we can find $\widehat{s} > 0$ such that $\varphi_+^\lambda(\widehat{s}u) = \eta$. Moreover, since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\varphi_+^\lambda(su) &= \big\langle(\varphi_+^\lambda)'(su), u\big\rangle \\
&\leq \frac{1}{s}\Big[\frac{c_1 s^p}{p-1} + s^p\lambda\|u^-\|_p^p - \int_0^b pF(t, su^+)\mathrm{d}t + \|\widehat{\alpha}_4\|_1\Big] \quad \text{(see (4.9))} \\
&\leq \frac{1}{s}\big[\eta + \|\widehat{\alpha}_4\|_1\big] < 0 \quad \text{(see (4.10) and recall that } \eta < -(\|\widehat{\alpha}_4\|_1 + 1)),
\end{aligned}$$

the above $\widehat{s} > 0$ is unique and we denote it by $\gamma^+(u) > 0$. Then $\varphi_+^\lambda\big(\gamma^+(u)u\big) = \eta$ for all $u \in \partial B_+$, and by the implicit function theorem, $\gamma^+ \in C(\partial B_+)$. Let $E_+ = \{u \in W : u^+ \neq 0\}$ and set $\widehat{\gamma}^+(u) = \frac{1}{\|u\|}\gamma^+\big(\frac{u}{\|u\|}\big)$ for all $u \in E_+$. Then $\widehat{\gamma}^+ \in C(E_+)$ and $\varphi_+^\lambda\big(\widehat{\gamma}^+(u)u\big) = \eta$ for all $u \in E_+$. In addition, $\varphi_+^\lambda(u) = \eta$ implies $\widehat{\gamma}^+(u) = 1$. So, if for $u \in E_+$ we set

$$\widehat{\gamma}_0^+(u) = \begin{cases} 1 & \text{if } u \in (\varphi_+^\lambda)^\eta, \\ \widehat{\gamma}(u) & \text{if } u \notin (\varphi_+^\lambda)^\eta, \end{cases}$$

then $\widehat{\gamma}_0^+ \in C(E_+)$. Let $h(t, u) = (1-t)u + t\widehat{\gamma}_0^+(u)u$. We have

$$(\varphi_+^\lambda)^\eta \ \text{ is a strong deformation retract of } E_+. \tag{4.11}$$

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Also, using $\widehat{h}(t,u) = \frac{(1-t)u+t\widehat{u}_0}{\|(1-t)u+t\widehat{u}_0\|}$, we see that

$$E_+ \text{ is contractible in itself.} \tag{4.12}$$

From (4.11) and (4.12), it follows that $H_k\big(W,(\varphi_+^\lambda)^\eta\big) = H_k(W,E_+) = 0$ for all $k \geq 0$ (see Granas and Dugundji [9]), hence choosing $\eta < \inf \varphi_+^\lambda(K_{\varphi_+^\lambda})$, we infer $C_k(\varphi_+^\lambda,\infty) = 0$ for all $k \geq 0$. Similarly we show that $C_k(\varphi_-^\lambda,\infty) = 0$ for all $k \geq 0$, this time using $\partial B_- = \{u \in \partial B_1 : u^- \neq 0\}$. ∎

5 Three Solutions Theorem

In this section, we prove the full multiplicity theorem for problem (1.1), producing three nontrivial solutions.

Theorem 5.1 *If hypotheses $H(\alpha)$ and $H(f)$ hold, then problem (1.1) has at least three nontrivial solutions: $u_0 \in \mathrm{int}\widehat{C}_+$, $v_0 \in -\mathrm{int}\widehat{C}_+$, and $y_0 \in C^1(T)$.*

Proof From Proposition 3.3, we already have two constant sign solutions $u_0 \in \mathrm{int}\,\widehat{C}_+$ and $v_0 \in -\mathrm{int}\,\widehat{C}_+$. Suppose that $\{0,u_0,v_0\}$ are the only critical points of φ. Then we can easily see that $\{0,u_0\}$ are the only critical points of φ_+^λ, and $\{0,v_0\}$ are the only critical points of φ_-^λ.

Claim 1: $C_k(\varphi_+^\lambda,u_0) = C_k(\varphi_-^\lambda,v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$.
Let $\widehat{\eta} < 0 = \varphi_+^\lambda(0) < \eta < \varphi_+^\lambda(u_0)$ (see (3.12)). We have $(\varphi_+^\lambda)^{\widehat{\eta}} \subseteq (\varphi_+^\lambda)^\eta \subseteq W$. We consider the long exact sequence of homological groups corresponding to this triple of sets:

$$\cdots \to H_k\big(W,(\varphi_+^\lambda)^{\widehat{\eta}}\big) \xrightarrow{i_*} H_k\big(W,(\varphi_+^\lambda)^\eta\big) \xrightarrow{\partial_*} H_{k-1}\big((\varphi_+^\lambda)^\eta,(\varphi_+^\lambda)^{\widehat{\eta}}\big) \to \cdots . \tag{5.1}$$

Here i is the embedding of $(\varphi_+^\lambda)^{\widehat{\eta}}$ into $(\varphi_+^\lambda)^\eta$, and ∂ is the boundary map. We have

$$H_k\big(W,(\varphi_+^\lambda)^{\widehat{\eta}}\big) = C_k(\varphi_+^\lambda,\infty) = 0 \quad \text{for all } k \geq 0 \text{ (see Proposition 4.2)}, \tag{5.2}$$

$$H_k\big(W,(\varphi_+^\lambda)^\eta\big) = C_k(\varphi_+^\lambda,0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0 \text{ (see Proposition 3.2)}. \tag{5.3}$$

From (5.2) and (5.3), we see that in (5.1) only the tail $k = 1$ is nontrivial. Moreover, from the exactness of (5.1) and the rank theorem, we see that $\operatorname{rank} C_1(\varphi_+^\lambda,u_0) = \operatorname{rank} H_1\big(W,(\varphi_+^\lambda)^\eta\big) \leq 1$. On the other hand recall that u_0 is a critical point of mountain pass type for φ_+^λ (see the proof of Proposition 3.4). So, $\operatorname{rank} C_1(\varphi_+^\lambda,u_0) \geq 1$. Therefore, we conclude that $C_k(\varphi_+^\lambda,u_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$. Similarly we show that $C_k(\varphi_-^\lambda,v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$. This proves Claim 1.

Claim 2: $C_k(\varphi,u_0) = C_k(\varphi_+^\lambda,u_0)$ and $C_k(\varphi,v_0) = C_k(\varphi_-^\lambda,v_0)$ for all $k \geq 0$.
Let $h_+(s,u) = (1-s)\varphi(u) + s\varphi_+^\lambda(u)$, $(s,u) \in [0,1] \times W$. We will show that there exists $\varrho \in (0,1)$ small such that u_0 is the only critical point in $\overline{B}_\varrho(u_0) = \{u \in W : \|u-u_0\| \leq \varrho\}$ of $h_+(s,\cdot)$ for all $s \in [0,1]$. Indeed, if this is not the case, then we can find $t_n \to t \in [0,1]$, $u_n \to u_0$ in W and $(h_+)'_u(s_n,u_n) = 0$ for all $n \geq 1$. From Papageorgiou and Papalini [15, proof of Proposition 5], we know that

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$\{u_n\}_{n\geq 1} \subseteq \widehat{C}(T)$ is relatively compact. So, we have $u_n \to u_0$ in $\widehat{C}(T)$ and since $u_0 \in \operatorname{int}\widehat{C}_+$, we will have $u_n \in \operatorname{int}\widehat{C}_+$ for all $n \geq n_0$. Because $\varphi\big|_{\widehat{C}_+} = \varphi_+^\lambda\big|_{\widehat{C}_+}$, it follows that $\{u_n\}_{n\geq n_0}$ are all distinct critical points of φ, a contradiction to our assumption. Also, reasoning as in the proof of Proposition 3.1, we show that for all $s \in [0,1]$, $h_+(s,\cdot)$ satisfies the C-condition. Then by virtue of the homotopy invariance property of the critical groups (see Chang [3]), we have

$$C_k(\varphi, u_0) = C_k\big(h_+(0,\cdot), u_0\big) = C_k\big(h_+(1,\cdot), u_0\big) = C_k(\varphi_+^\lambda, u_0)$$

for all $k \geq 0$. Similarly for the triple $\{\varphi, \varphi_-^\lambda, v_0\}$. This proves Claim 2.

From Propositions 3.2 and 4.1, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \ \text{ and } \ C_k(\varphi,\infty) = 0 \text{ for all } k \geq 0. \tag{5.4}$$

From Claims 1, 2, (5.4), and the Morse relation (see (2.1)), with $t = -1$, we have $2(-1)^1 + (-1)^0 = 0$, a contradiction. So, φ has a critical point $y_0 \notin \{0, u_0, v_0\}$. Then $y_0 \in C^1(T)$ solves problem (1.1). ■

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