ON MR. WOOLHOUSE'S IMPROVED THEORY OF ANNUITIES AND ASSURANCES.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—In Mr. Woolhouse's very able paper on the continuous method of obtaining the values of annuities, assurances, etc., a general formula for payments by instalments is demonstrated, of which considerable use is afterwards made. Let V_x denote a function depending for its value upon that of the variable quantity x, and let x denote an interval of time measured from some given epoch; then Mr. Woolhouse shows that

$$\frac{1}{m}(V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} + \ldots + V_\omega)$$

$$= \int_0^\omega V dx + \frac{1}{2m}(V_0 + V_\omega) - \frac{1}{12m^2}\left\{\left(\frac{dV}{dx}\right)_0 - \left(\frac{dV}{dx}\right)_\omega\right\}$$

$$+ \frac{1}{720m^4}\left\{\left(\frac{d^3V}{dx^3}\right)_0 - \left(\frac{d^3V}{dx^3}\right)_\omega\right\} - \ldots\ldots$$

Now let $\frac{1}{m}(V_0 + V_{\frac{1}{m}} + V_{\frac{2}{m}} + \ldots + V_\omega)$ be represented by $\Sigma^{(m)}V$; it is then shown that

$$\Sigma^{(m)}V = \Sigma^{(1)}V - \frac{m-1}{2m}V + \frac{m^2-1}{12m^2}\frac{dV}{dx} - \frac{m^4-1}{720m^4}\frac{d^3V}{dx^3} + \&c. \quad . \quad (1)$$

where V, $\frac{dV}{dx}$, $\frac{d^3V}{dx^3}$, etc. are all initial values, since in all matters connected with annuities and assurances V_ω, $\left(\frac{dV}{dx}\right)_\omega$, $\left(\frac{d^3V}{dx^3}\right)_\omega$ etc. vanish.

The formula (1) for payments by instalments is obtained by Mr. Woolhouse in a very simple and elementary manner; but, as it promises to become of great importance in the theory of life assurance, I have ventured to send you another demonstration of it, mainly due to Sir John Lubbock.

That gentleman, in a paper on annuities reprinted in the fifth volume of the *Journal of the Institute*, shows (p. 277) that if y be any variable, and y_0, y_i, y_{2i}, $\ldots, y_{ni}$, $y_{(n+1)i}$ $\ldots y_{mni-i}(ni=1)$ its successive values, then

$$y_0 + y_i + y_{2i} + \ldots + y_{ni} + y_{(n+1)i} + \ldots + y_{(mni-i)}$$

$$= n(y_0 + y_1 + \&c. \ldots + y_{m-1}) + \frac{n-1}{2}(y_m - y_0) - \frac{n^2-1}{12n}(\Delta y_m - \Delta y_0)$$

$$+ \frac{n^2-1}{24n}(\Delta^2 y_m - \Delta^2 y_0) - \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the coefficient of $\Delta^q y_m - \Delta^q y_0$ is equal to the coefficient of x^{q-1} in the development of $\frac{(1+x)^{ni} - (1+x)}{1-(1+x)^i}$. This result is obtained, like Mr. Woolhouse's, almost from first principles. If, now, we go another step, and calculate the coefficient of $\Delta^3 y_m - \Delta^3 y_0$, we find it to be $-\frac{19n^4 - 20n^2 + 1}{720m^3}$; and since in life assurance calculations terminal values vanish, *i.e.* $y_m = 0$, $\Delta y_m = 0$, $\Delta^2 y_m = 0$, etc., the right-hand side of (2) becomes

$$= n(y_0 + y_1 + \dots + y_{m-1}) - \frac{n-1}{2} y_0 + \frac{n^2-1}{12n} \Delta y_0$$
$$- \frac{n^2-1}{24n} \Delta^2 y_0 + \frac{19n^4 - 20n^2 + 1}{720n^3} \Delta^3 y_0 - \&c.$$

In this expression write $(\varepsilon^{\overline{dx}} - 1)y_0$ for Δy_0, $(\varepsilon^{\overline{dx}} - 1)^2 y_0$ for $\Delta^2 y_0$, &c., the symbols of operation and quantity being separated. On doing this, expanding $(\varepsilon^{\overline{dx}} - 1)y_0$, $(\varepsilon^{\overline{dx}} - 1)^2 y_0$, $(\varepsilon^{\overline{dx}} - 1)^3 y_0$, &c., and collecting coefficients, it will be found that the coefficient of $\dfrac{d^2 y_0}{dx^2}$ vanishes, and that (3) now becomes

$$= n(y_0 + y_1 + \dots + y_{m-1}) - \frac{n-1}{2} y_0 + \frac{n^2-1}{12n} \frac{dy_0}{dx}$$
$$+ \frac{10n^4 - 10n^2 - 30n^4 + 30n^2 + 19n^4 - 20n^2 + 1}{720n^3} \frac{d^3 y_0}{dx^3} + \&c.$$

that is, $y_0 + y_i + y_{2i} + \&c. \dots + y_{ni} + y_{(n+1)i} + \dots + y_{(mni-i)} (ni=1)$

$$= n(y_0 + y_1 + \&c. \dots + y_{m-1}) - \frac{n-1}{2} y_0 + \frac{n^2-1}{12n} \frac{dy_0}{dx}$$
$$- \frac{n^4-1}{720n^3} \frac{d^3 y_0}{dx^3} + \&c.$$

or, dividing both sides by n,

$$\frac{1}{n} \{ y_0 + y_i + y_{2i} + \&c. \dots + y_{ni} + y_{(n+i)i} + \dots + y_{(mni-i)} \}$$
$$= (y_0 + y_1 + \&c. \dots + y_{m-i}) - \frac{n-1}{2n} y_0 + \frac{n^2-1}{12n^2} \frac{dy_0}{dx}$$
$$- \frac{n^4-1}{720n^4} \frac{d^3 y_0}{dx^3} + \&c.$$

or $\Sigma^{(n)} y = \Sigma^{(1)} y - \dfrac{n-1}{2n} y_0 + \dfrac{n^2-1}{12n^2} \dfrac{dy_0}{dx} - \dfrac{n^4-1}{720n^4} \dfrac{d^3 y_0}{dx^3} + \&c.$

which is Mr. Woolhouse's formula.

On my showing the above demonstration to Mr. Sprague, he remarked that it had occurred to him, that the method of separation of symbols might be made of more direct use in proving Mr. Woolhouse's formula. Acting upon that suggestion, I have obtained the following more direct proof, which will doubtless be of interest to some readers of the *Journal*.

Let u_x denote a function of x, and let D be put for $1 + \Delta$; then, separating the symbols of operation and quantity,

$$D^n u_x = (1 + \Delta)^n u_x = u_{x+n} :$$

Thus we have

$$u_0 + u_{\frac{1}{m}} + u_{\frac{2}{m}} + \dots + u_{\omega - \frac{1}{m}} = (1 + D^{\frac{1}{m}} + D^{\frac{2}{m}} + \dots + D^{\omega - \frac{1}{m}}) u_0$$
$$= \cdot \frac{1 - D^\omega}{1 - D^{\frac{1}{m}}} \cdot u_0$$
$$= \frac{1}{1 - D^{\frac{1}{m}}} \cdot u_0$$

(since in life assurance calculations $u_\omega = 0$)

$$= - \frac{1}{(D^{\frac{1}{m}} - 1)} \cdot u_0$$

$$= (D^{\frac{1}{m}} - 1)^{-1} u_0$$

$$= (\epsilon^{\frac{d}{mdx}} - 1)^{-1} u_0 \quad (\text{since } D = \epsilon^{\frac{d}{dx}})$$

$$= m \int_0^\omega u_x dx + \frac{1}{2} u_0 - \frac{1}{12m} \left(\frac{du_x}{dx}\right)_0 + \frac{1}{720m^3} \left(\frac{d^3 u_x}{dx^3}\right)_0 - \&c.$$

$$\therefore \quad \Sigma^{(m)} u = \frac{1}{m} (u_0 + u_{\frac{1}{m}} + u_{\frac{2}{m}} + \&c. \ldots + u_{\omega - \frac{1}{m}})$$

$$= \int_0^\omega u_x d_x + \frac{1}{2m} u_0 - \frac{1}{12m^2} \left(\frac{du_x}{dx}\right)_0 + \frac{1}{720m^4} \left(\frac{d^3 u_x}{dx^3}\right)_0 - \&c.$$

But $\Sigma^{(1)} u = \int_0^\omega u_x dx + \frac{1}{2} u_0 - \frac{1}{12} \left(\frac{du_x}{dx}\right)_0 + \frac{1}{720} \left(\frac{d^3 u_x}{dx^3}\right)_0 - \&c.$

$\therefore$ subtracting,

$$\Sigma^{(m)} u - \Sigma^{(1)} u = - \frac{m-1}{2m} u_0 + \frac{m^2 - 1}{12m^2} \left(\frac{du_x}{dx}\right)_0 - \frac{m^4 - 1}{720m^4} \left(\frac{d^3 u_x}{dx^3}\right)_0 + \&c.$$

or,

$$\Sigma^{(m)} u = \Sigma^{(1)} u - \frac{m-1}{2m} u_0 + \frac{m^2 - 1}{12m^2} \left(\frac{du_x}{dx}\right)_0 - \frac{m^4 - 1}{720m^4} \left(\frac{d^3 u_x}{dx^3}\right)_0 + \&c.$$

This result will appear pretty obvious from the consideration that the xth payment in the one case will coincide with the mxth in the other ; and so by putting mdx for dx, and dividing by m, in the expression for $u_0 + u_1 + \ldots + u_{\omega-1}$, we shall obtain the value of

$$\frac{1}{m} (u_0 + u_{\frac{1}{m}} + u_{\frac{2}{m}} + \ldots + u_{\omega - \frac{1}{m}}).$$

Mr. Woolhouse has, in his paper, deduced the values of annuities and assurances, when continuous. It is interesting in some respects to see how these results can be obtained from the corresponding values for yearly payments. Take the expression for A_x ; and we have $A_x = \dfrac{1 - i a_x}{1 + i}$, where i is the rate of interest, payable yearly, and a_x the present value of an annuity for the life of a person aged x years. This may be written

$$A_x = \frac{1}{1 + i} (1 - i a_x)$$

$$= \begin{Bmatrix} \text{Present value of £1 due} \\ \text{a year hence} \end{Bmatrix} \times \begin{Bmatrix} \text{£1} - (\text{interest on £1 for} \\ \text{a year}) \times a_x \end{Bmatrix}$$

Now change years into moments : then i becomes δ, A_x becomes $\overline{A}_x$, and a_x becomes $\overline{a}_x$: and $\overline{A}_x = 1 - \delta \overline{a}_x$, since value of £1 due instantly is £1.

(See Mr. Woolhouse's paper, page 115, in the July number of the *Journal*).

Again, the ordinary value of an assurance on the joint lives of x and y is given by the equation

$$
\begin{aligned}
\mathrm{A}_{xy} &= \frac{l_{xy} - l_{x+1,\,y+1}}{l_{xy}}\, v + \frac{l_{x+1,\,y+1} - l_{x+2,\,y+2}}{l_{xy}}\, v^2 + \&\mathrm{c}. \\
&= \frac{l_{xy}}{l_{xy}}\, v + \frac{l_{x+1,\,y+1}}{l_{xy}}\, v^2 + \&\mathrm{c}. - \left\{ \frac{l_{x+1,\,y+1}}{l_{xy}}\, v + \frac{l_{x+2,\,y+2}}{l_{xy}}\, v^2 + \&\mathrm{c} \right\} \\
&= \frac{l_{x-1,\,y-1}}{l_{xy}} \left\{ \frac{l_{xy}}{l_{x-1,\,y-1}}\, v + \frac{l_{x+1,\,y+1}}{l_{x-1,\,y-1}}\, v^2 + \&\mathrm{c}. \right\} \\
&\qquad - \left\{ \frac{l_{x+1,\,y+1}}{l_{xy}}\, v + \frac{l_{x+2,\,y+2}}{l_{xy}}\, v^2 + \&\mathrm{c}. \right\} \\
&= \frac{l_{x-1,\,y-1}}{l_{xy}}\, a_{x-1,\,y-1} - a_{xy}
\end{aligned}
$$

Now change the interval of time from a year to τ, where $m\tau =$ one year: then, instead of an assurance for £1, the above expression will give us the value of an assurance of £τ, payable at the end of the interval τ in which the life fails, so that we get, using the common notation,

$$
\begin{aligned}
\tau \mathrm{A}_{xy}^{(m)} &= \frac{l_{x-\tau,\,y-\tau}}{l_{xy}}\, a_{x-\tau,\,y-\tau}^{(m)} - a_{xy}^{(m)} . \\
&= \frac{l_{xy} - \Delta l_{x-\tau,\,y-\tau}}{l_{xy}}\, a_{x-\tau,\,y-\tau}^{(m)} - a_{xy}^{(m)} \\
&= a_{x-\tau,\,y-\tau}^{(m)} - a_{xy}^{(m)} - \frac{\Delta l_{x-\tau,\,y-\tau}}{l_{xy}}\, a_{x-\tau,\,y-\tau}^{(m)}
\end{aligned}
$$

$$
\therefore \qquad \mathrm{A}_{xy}^{(m)} = \frac{a_{x-\tau,\,y-\tau}^{(m)} - a_{xy}^{(m)}}{\tau} - \frac{1}{l_{xy}} \cdot \frac{\Delta l_{x-\tau,\,y-\tau}}{\tau}\, a_{x-\tau,\,y-\tau}^{(m)} .
$$

Now diminish τ indefinitely, and we have

$$
\begin{aligned}
\overline{\mathrm{A}}_{xy} &= -\frac{d\overline{a}_{xy}}{dt} + \mu_{xy}\overline{a}_{xy} \\
&= \mu_{xy}\overline{a}_{xy} - \frac{d\overline{a}_{xy}}{dx} - \frac{d\overline{a}_{xy}}{dy} .
\end{aligned}
$$

From the value just obtained for $\overline{\mathrm{A}}_{xy}$, we can obtain the value of $\overline{\mathrm{A}}_{\underset{xy}{1}}$ in a similar way.

By the ordinary method,

$$
\mathrm{A}_{\underset{xy}{1}} = \tfrac{1}{2} \left\{ \mathrm{A}_{xy} + \frac{a_{x-1,\,y}}{p_{x-1,\,1}} - \frac{a_{x,\,y-1}}{p_{y-1,\,1}} \right\}.
$$

The way in which this is obtained is well known, and will be seen to apply when the interval of time is τ, instead of a year. We shall thus have

$$\tau \mathop{\mathrm{A}}_{xy}^{(m)}=\tfrac{1}{2}\left\{\tau \mathrm{A}_{xy}^{(m)}+\frac{a_{x-\tau,\,y}^{(m)}}{p_{x-\tau,\,\tau}}-\frac{a_{x,\,y-\tau}^{(m)}}{p_{y-\tau,\,\tau}}\right\}$$

$$=\tfrac{1}{2}\left\{\tau \mathrm{A}_{xy}^{(m)}+\frac{l_x-\Delta l_{x-\tau}}{l_x}\,a_{x-\tau,\,y}^{(m)}-\frac{l_y-\Delta l_{y-\tau}}{l_y}\,a_{x,\,y-\tau}^{(m)}\right\}$$

$$=\tfrac{1}{2}\left\{\tau \mathrm{A}_{xy}^{(m)}+a_{x-\tau,\,y}^{(m)}-a_{xy}^{(m)}-\frac{\Delta l_{x-\tau}}{l_x}\,a_{x-\tau,\,y}^{(m)}-\left(a_{x\,y-\tau}^{(m)}-a_{xy}^{(m)}\right)\right.$$
$$\left.+\frac{\Delta l_{y-\tau}}{l_y}\cdot a_{x,\,y-\tau}^{(m)}\right\}$$

$$\therefore\quad \mathop{\mathrm{A}}_{xy}^{(m)}=\tfrac{1}{2}\left\{\mathrm{A}_{xy}^{(m)}+\frac{a_{x-\tau,\,y}^{(m)}-a_{xy}^{(m)}}{\tau}-\frac{1}{l_x}\cdot\frac{\Delta l_{x-\tau}}{\tau}\cdot a_{x-\tau,\,y}^{(m)}-\frac{a_{x,\,y-\tau}^{(m)}-a_{xy}^{(m)}}{\tau}\right.$$
$$\left.+\frac{1}{l_y}\cdot\frac{\Delta l_{y-\tau}}{\tau}\,a_{x,\,y-\tau}^{(m)}\right\}$$

Now, as before, diminish τ indefinitely, and we get

$$\overline{\mathrm{A}}_1=\tfrac{1}{2}\left\{\overline{\mathrm{A}}_{xy}-\frac{d\bar{a}_{xy}}{dx}+\mu_x.\bar{a}_{xy}+\frac{d\bar{a}_{xy}}{dy}-\mu_y.\bar{a}_{xy}\right\},$$

which, after substituting for $\overline{\mathrm{A}}_{xy}$ from the equation

$$\overline{\mathrm{A}}_{xy}=\mu_{xy}\bar{a}_{xy}-\frac{d\bar{a}_{xy}}{dx}-\frac{d\bar{a}_{xy}}{dy},$$

reduces to

$$\overline{\mathrm{A}}_1=\tfrac{1}{2}\left\{2\mu_x\bar{a}_{xy}-2\frac{d\bar{a}_{xy}}{dx}\right\},$$
$$=\mu_x\bar{a}_{xy}-\frac{d\bar{a}_{xy}}{dx}.$$

which is Mr. Woolhouse's result.

Now, approximately, $\bar{a}_{xy}=a_{xy}+\tfrac{1}{2}$, and $-\dfrac{d\bar{a}_{xy}}{dx}=\tfrac{1}{2}(a_{x-1,\,y}-a_{x+1,\,y})$. Making these substitutions, we get

$$\overline{\mathrm{A}}_1=\mu_x(a_{xy}+\tfrac{1}{2})+\tfrac{1}{2}(a_{x-1,\,y}-a_{x+1,\,y}),$$

the very convenient working formula proposed by Mr. Woolhouse.

These demonstrations, although of no intrinsic value, are not, I venture to think, without interest, as showing the connection between the ordinary and continuous methods of obtaining annuities and assurances.

I am, Sir,

Your most obedient servant,

December 20*th*, 1869. WILLIAM SUTTON.

P.S.—There is a misprint in Mr. Woolhouse's paper: the expressions on the right-hand side of equations (35) are the approximate values of $-\dfrac{d\bar{a}_{xy}}{dx},\ -\dfrac{d\bar{a}_{xy}}{dy},\ -\dfrac{d\bar{a}_{xy}}{dt}$, respectively, and not, as printed, of $\dfrac{d\bar{a}_{xy}}{dx},\ \dfrac{d\bar{a}_{xy}}{dy},\ \dfrac{d\bar{a}_{xy}}{dt}$.