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ON GROWTH-COLLAPSE PROCESSES WITH STATIONARY STRUCTURE AND THEIR SHOT-NOISE COUNTERPARTS

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Abstract

In this paper we generalize existing results for the steady-state distribution of growth-collapse processes. We begin with a stationary setup with some relatively general growth process and observe that, under certain expected conditions, point- and time-stationary versions of the processes exist as well as a limiting distribution for these processes which is independent of initial conditions and necessarily has the marginal distribution of the stationary version. We then specialize to the cases where an independent and identically distributed (i.i.d.) structure holds and where the growth process is a nondecreasing Lévy process, and in particular linear, and the times between collapses form an i.i.d. sequence. Known results can be seen as special cases, for example, when the inter-collapse times form a Poisson process or when the collapse ratio is deterministic. Finally, we comment on the relation between these processes and shot-noise type processes, and observe that, under certain conditions, the steady-state distribution of one may be directly inferred from the other.

Keywords: Growth-collapse process; risk process; shot-noise process

2000 Mathematics Subject Classification: Primary 60K30
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1. Introduction

In this paper we focus on growth-collapse processes with some general stationary structure. A growth-collapse process is a process that increases according to some mechanism and from time to time it is reduced to some (possibly random fraction) of its pre-collapse value. We show that with relatively general assumptions, the process has a stationary version and also converges in distribution to the stationary marginal for every initial condition. A simple relationship is established between the time-stationary distribution and the point-stationary distribution of the process embedded immediately after collapse times. The processes immediately before and immediately after collapse epochs are autoregressive processes with random coefficients of the type considered in [4].

For the case where there is an independent and identically distributed (i.i.d.) structure, more explicit results are attained and we show how to compute the moments in terms of the various building blocks of the model. When the growth follows some nondecreasing Lévy process, the results are even more explicit, particularly when it is linear, as is assumed in most of the existing literature on growth-collapse models.

We also comment on the relationship between growth-collapse models in general and shot-noise type processes, which are processes that decay exponentially between *shot* epochs and

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at these shot epochs jump up by some random amount. These processes can also be viewed as dam processes with linear release rate, that is, with a release rate which is a constant multiple of the level.

For recent papers on growth-collapse models and their applications, see [1], [2], [3], [5], [6], [8], [9], and the references therein.

2. Model description and some preliminaries

Consider a risk-type process called a growth-collapse process where between the $(n-1)$th and nth claims (*collapse*) premium accumulates (*growth*) according to some nondecreasing right-continuous process $I_n = \{I_n(t) \mid t \geq 0\}$ with $I_n(0) = 0$. The times between claims are denoted by $\tau_1, \tau_2, \dots$. The remaining funds after a given claim are a random proportion of the fund before the claim. Denote these random proportions by $X_1, X_2, \dots$. Thus, if immediately before the nth claim the fund level is v then immediately after it, it is vX_n, where X_n assumes values in $[0, 1]$. Set $T_0 = 0$ and $T_n = \sum_{i=1}^{n} \tau_i$ for $n \geq 1$ so that $N(t) = \sup\{n \geq 0 \mid T_n \leq t\}$ is the number of claims by time t. Now, set $Y_n = I_n(\tau_n)$ and note that necessarily $Y_n \geq 0$. If V_0 is the initial wealth and V_n is the wealth level immediately after the nth claim, then

$$V_n = (V_{n-1} + Y_n)X_n,$$

and, thus,

$$V_n = V_0 \prod_{j=1}^{n} X_j + \sum_{i=1}^{n} Y_i \prod_{j=i}^{n} X_j.$$

The wealth level immediately before the nth claim is

$$U_n = V_{n-1} + Y_n,$$

and satisfies $U_{n+1} = X_n U_n + Y_{n+1}$. Thus, the sequence of wealth levels immediately before and immediately after claims are both of the autoregressive type with stochastic coefficients.

Now set $W(0) = V_0$ and let $W(t)$ be the fund level at time t. Then the continuous-time growth-collapse process is

$$W(t) = V_{N(t)} + I_{N(t)+1}(t - T_{N(t)}). \tag{1}$$

3. A stationary setup

In this section we will observe that, under some general stationarity assumptions, a stationary version of the process embedded immediately before and immediately after collapse is stable in the sense that it has a stationary version and that it converges in distribution to the one-dimensional stationary marginal for any initial level.

Theorem 1. *Assume that* $\{(X_n, Y_n) \mid n \geq 1\}$ *is a stationary sequence with* $\mathrm{E}\, Y_1 < \infty$*, that*

$$\prod_{i=1}^{\infty} X_i = 0 \quad \textit{almost surely (a.s.),} \tag{2}$$

and that its two-sided extension $\{(X_n, Y_n) \mid n \in \mathbb{Z}\}$ *satisfies*

$$\limsup_{n\to\infty} \left(\prod_{i=-n}^{-1} X_i \right)^{1/n} \leq \rho < 1 \quad \textit{a.s.} \tag{3}$$

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Then $\{V_n \mid n \geq 0\}$ *has a stationary version* $\{V_n^* \mid n \geq 1\}$ *with* $\mathrm{P}[V_n^* < \infty] = 1$ *and* $V_n - V_n^* \to 0$ *a.s. for any initial* V_0.

We note that the result follows directly from [4] when (2) and (3) are replaced by the more restrictive assumption that $\{(X_n, Y_n) \mid n \geq 1\}$ is ergodic with $\mathrm{E}\, X_1 < 1$. In this case, (2) is automatically satisfied and the limit superior in (3) is actually a limit and is a.s. equal to $\exp(\mathrm{E} \log X_1) \leq \mathrm{E}\, X_1 < 1$, where $\log 0 \equiv -\infty$ and $\mathrm{e}^{-\infty} \equiv 0$.

Proof of Theorem 1. If we show that a stationary version exists then, for any V_0, we have

$$V_n - V_n^* = (V_0 - V_0^*) \prod_{i=1}^{n} X_i,$$

and, thus, the result would follow from the fact that $\prod_{i=1}^{\infty} X_i = 0$ a.s.

To show that a stationary version exists, as in the classical Loynes' construction, we let $\{(X_n, Y_n) \mid n \in \mathbb{Z}\}$ be the double-sided extension of the original sequence and consider the process

$$V_n^* = \sum_{i=-\infty}^{n} Y_i \prod_{j=i}^{n} X_j = V_0^* \prod_{j=1}^{n} X_j + \sum_{i=1}^{n} Y_i \prod_{j=i}^{n} X_j.$$

Clearly, $V_n^* = (V_{n-1}^* + Y_n) X_n$ for all $n \in \mathbb{Z}$. If V_{-1}^* is a.s. finite then this would immediately imply that $\{V_n^* \mid n \geq 0\}$ is stationary and a.s. finite for every n. Let N be an a.s. finite random integer such that, for $n \geq N$,

$$\left(\prod_{i=-n}^{-1} X_i \right)^{1/n} \leq \frac{1+\rho}{2}.$$

Then,

$$\sum_{i=-\infty}^{-N} Y_i \prod_{j=i}^{-1} X_j \leq \sum_{i=-\infty}^{-N} Y_i \left(\frac{1+\rho}{2} \right)^i \leq \sum_{i=-\infty}^{-1} Y_i \left(\frac{1+\rho}{2} \right)^i. \tag{4}$$

Since $\mathrm{E}\, Y_1 < \infty$, $Y_1 \geq 0$, a.s., it follows that the right-hand side of (4) has a finite expected value and is therefore a.s. finite. Since N is a.s. finite, this implies that $\sum_{i=-N+1}^{-1} Y_i \prod_{j=i}^{-1} X_i$ and, hence, V_{-1}^* is a.s. finite as well. This completes the proof.

We note that in fact it follows that, under the conditions of Theorem 1, $\{(V_{n-1}^*, X_n, Y_n) \mid n \geq 1\}$ is a stationary sequence and that $\{(V_{n-1}, X_n, Y_n) \mid n \geq 1\}$ converges in distribution to (V_0^*, X_1, Y_1). Therefore, $U_n = V_{n-1} + Y_n$, the state of the process immediately before collapse, also has the stationary version $U_n^* = V_{n-1}^* + Y_n$ and converges in distribution to $U^* = V_0^* + Y_1$ for every initial V_0.

As for the process $\{W(t) \mid t \geq 0\}$, we immediately see that if $W(0) = V_0$ and $W'(0) = V_0'$ are two initial conditions, and $W(t)$ and $W'(t)$ are the resulting processes, then, from (1),

$$W(t) - W'(t) = (W(0) - W'(0)) \prod_{i=1}^{N(t)} X_i.$$

Since $T_n < \infty$ for all $n \geq 0$, then $N(t) \to \infty$ a.s. and, thus, under the conditions of Theorem 1, we have $W(t) - W'(t) \to 0$ a.s. as $t \to \infty$. Thus, if $W(t)$ has a limiting distribution, it does not depend on initial conditions. However, in order to establish time stationarity, we need to assume a bit more. This is given as follows.

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Theorem 2. *Assume that* $\{(T_n, K_n) \mid n \geq 1\}$ *is an ergodic event-stationary marked point process with marks*

$$K_n = (X_n, \{I_n(t) \mid t \geq 0\}),$$

as well as $\mathrm{E}\,\tau_1 < \infty$*,* $\mathrm{E}\,X_1 < 1$*, and* $\mathrm{E}\,Y_1 = \mathrm{E}\,I_1(\tau_1) < \infty$*. Then* $\{W(t) \mid t \geq 0\}$ *has a stationary version and, for every function* f *which is bounded and Lipschitz continuous on* $[0, \infty)$*, we have, a.s.,*

$$\frac{1}{t}\int_0^t f(W(s))\,\mathrm{d}s \to \frac{1}{\mathrm{E}\,\tau_1}\mathrm{E}\int_0^{\tau_1} f(V_0^* + I_1(s))\,\mathrm{d}s \quad \textit{a.s. as } t \to \infty. \tag{5}$$

Consequently, $W(t)$ *converges in distribution to the stationary marginal for any initial* $W(0)$*.*

Proof. Observe that $\Psi = \{(T_n, (V_{n-1}^*, I_n(\cdot), X_n)) \mid n \geq 1\}$ is also an ergodic event-stationary marked point process and, thus, it has a time-stationary version (e.g. Proposition 4.6 and the preceding paragraph of [10, p. 100]). As in (1), for this time-stationary version, let

$$W_\Psi^*(t) = V_{N(t)}^* + I_{N(t)+1}(t - T_{N(t)})$$
$$\text{and} \quad \theta_s\Psi = \{(T_{N(s)+n} - s, (V_{N(s)+n-1}^*, I_{N(s)+n}(\cdot), X_{N(s)+n})) \mid n \geq 1\}.$$

The existence of a stationary version of $\{W(t) \mid t \geq 0\}$ follows from $W_\Psi \sim W_{\theta_s\Psi}$ once we observe that $W_{\theta_s\Psi}^*(t) = W_\Psi^*(s+t)$. To see this, we first note that

$$\theta_s(V_{n-1}^* + I_n(t - T_n)) = V_{N(s)+n-1}^* + I_{N(s)+n}(t - (T_{N(s)+n} - s)), \tag{6}$$

and then that $\theta_s N(t) = N(t+s) - N(s)$, so that by replacing n on the right-hand side of (6) with $N(t+s) - N(s)$ we indeed obtain $W_{\theta_s\Psi}^*(t) = W_\Psi^*(s+t)$.

To proceed, as for (5), we note that

$$\int_0^t f(W(s))\,\mathrm{d}s = \sum_{n=1}^{N(t)}\int_0^{\tau_n} f(V_{n-1} + I_n(s))\,\mathrm{d}s + \int_0^{t-T_{N(t)}} f(V_{N(t)} + I_{N(t)+1}(s))\,\mathrm{d}s.$$

Ergodicity implies that $T_n/n \to \mathrm{E}\,\tau_1$ a.s. so that $N(t)/t \to 1/\mathrm{E}\,\tau_1$ a.s., and in particular $N(t) \to \infty$. Thus, with $\sup_{x\geq 0}|f(x)| = B < \infty$,

$$\left|\frac{1}{t}\int_0^{t-T_{N(t)}} f(V_{N(t)} + I_{N(t)+1}(s))\,\mathrm{d}s\right| \leq B\frac{T_{N(t)+1} - T_{N(t)}}{N(t)+1}\frac{N(t)+1}{t} \to 0 \quad \text{a.s. as } t \to \infty. \tag{7}$$

Next we observe that if $|f(x) - f(y)| \leq a|x - y|$ then

$$\left|\sum_{n=1}^{N(t)}\int_0^{\tau_n} f(V_{n-1} + I_n(s))\,\mathrm{d}s - \sum_{n=1}^{N(t)}\int_0^{\tau_n} f(V_{n-1}^* + I_n(s))\,\mathrm{d}s\right| \leq a\sum_{n=1}^{N(t)}|V_{n-1} - V_{n-1}^*|\tau_n,$$

and since $V_n - V_n^* \to 0$ a.s., then

$$\frac{1}{t}\sum_{n=1}^{N(t)}|V_{n-1} - V_{n-1}^*|\tau_n = \frac{\sum_{n=1}^{N(t)}|V_{n-1} - V_{n-1}^*|\tau_n}{\sum_{n=1}^{N(t)}\tau_n}\frac{T_{N(t)}}{t} \to 0 \quad \text{a.s. as } t \to \infty.$$

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Finally, by ergodicity,

$$\frac{1}{m}\sum_{n=1}^{m}\int_0^{\tau_n} f(V_{n-1}^* + I_n(s))\,\mathrm{d}s \to \mathrm{E}\int_0^{\tau_1} f(V_0^* + I_1(s))\,\mathrm{d}s \quad \text{a.s. as } m \to \infty,$$

and, thus,

$$\frac{N(t)}{t}\frac{1}{N(t)}\sum_{n=1}^{N(t)}\int_0^{\tau_n} f(V_{n-1}^* + I_n(s))\,\mathrm{d}s \to \frac{1}{\mathrm{E}\,\tau_1}\mathrm{E}\int_0^{\tau_1} f(V_0^* + I_1(s))\,\mathrm{d}s.$$

The convergence in distribution follows from the paragraph preceding the statement of the theorem.

4. The i.i.d. case

Under the conditions of Theorem 2, let $\{W^*(t) \mid t \geq 0\}$ be the stationary version of $\{W(t) \mid t \geq 0\}$. Then $f(x) = \mathrm{e}^{-\alpha x}$ is bounded and Lipschitz on $[0, \infty)$ for every $\alpha \geq 0$. Thus,

$$\begin{aligned}\frac{1}{t}\int_0^t \mathrm{e}^{-\alpha W(s)}\,\mathrm{d}s &\to \mathrm{E}\,\mathrm{e}^{-\alpha W^*(0)}\\ &= \mathrm{E}\bigg(\exp(-\alpha V_0^*)\frac{1}{\mathrm{E}\,\tau_1}\int_0^{\tau_1}\exp(-\alpha I_1(s))\,\mathrm{d}s\bigg) \quad \text{a.s. as } t \to \infty. \end{aligned} \tag{8}$$

Therefore, if we assume in addition to the conditions of Theorem 2 that $\{(\tau_n, k_n) \mid n \geq 1\}$ is an i.i.d. sequence (and, thus, stationary and ergodic), then the right-hand side of (8) becomes

$$\mathrm{E}\exp(-\alpha V_0^*)\frac{1}{\mathrm{E}\,\tau_1}\mathrm{E}\int_0^{\tau_1}\exp(-\alpha I_1(s))\,\mathrm{d}s,$$

and, thus, we have the following result.

Corollary 1. *Assume that in addition to the conditions of Theorem 2 the sequence*

$$\{(\tau_n, k_n) \mid n \geq 1\}$$

is i.i.d. Then

$$W \sim V + I_e,$$

where V and I_e are independent, $W \sim W^*(0)$, $V \sim V_0^*$, and I_e is a random variable having distribution

$$\mathrm{P}[I_e \in A] = \frac{1}{\mathrm{E}\,\tau_1}\mathrm{E}\int_0^{\tau_1}\mathbf{1}_A(I_1(s))\,\mathrm{d}s.$$

Moreover, if τ_1 and $\{I_1(t) \mid t \geq 0\}$ are independent, then letting τ_e denote an independent random variable having stationary excess lifetime distribution associated with τ_1 we have $I_e \sim I_1(\tau_e)$.

We note that the last part of Corollary 1 is easily obtained by first conditioning on $\{I_1(t) \mid t \geq 0\}$ and then applying the well-known property of the stationary excess lifetime distribution, according to which

$$\mathrm{E}\,g(\tau_e) = \int_0^\infty g(s)\frac{\mathrm{P}[\tau_1 > s]}{\mathrm{E}\,\tau_1}\,\mathrm{d}s = \frac{1}{\mathrm{E}\,\tau_1}\mathrm{E}\int_0^{\tau_1} g(s)\,\mathrm{d}s$$

for any (Borel) function g for which this expected value is well defined.

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We also observe that, for the well-studied case where τ_1 has an exponential distribution, $\tau_e \sim \tau_1$ and, thus, we see, as expected by PASTA (Poisson arrivals see time averages), that W has the same distribution as $V + I_1(\tau_1)$, which has the steady-state distribution of the level immediately before collapse, that is, immediately before the Poisson arrivals.

As $V \sim (V+Y)X$, we also have the following trivial consequence (similarly as in [2] and [5], but here with a more general growth process and/or inter-collapse times).

Corollary 2. *If in addition to Corollary 1 we assume that* X_1, $\{I_1(t) \mid t \geq 0\}$, *and* τ_1 *are independent, then letting* $\tilde{Z}(\alpha) = \mathrm{E}\,\mathrm{e}^{-\alpha Z}$ *and* $F_Z(x) = \mathrm{P}[Z \leq x]$ *for some nonnegative random variable* Z *we have*

$$\tilde{V}(\alpha) = \int_{[0,1]} \tilde{V}(\alpha x)\tilde{Y}(\alpha x) F_X(\mathrm{d}x).$$

In particular, if $\mathrm{P}[X = q] = 1$ *for some* $0 < q < 1$ *then*

$$\tilde{V}(\alpha) = \tilde{V}(q\alpha)\tilde{Y}(q\alpha) = \prod_{i=1}^{\infty} Y(q^i\alpha).$$

Assume now that the $\{\tau_n \mid n \geq 1\}$ are independent and that, independent from this sequence, $\{I(t) \mid t \geq 0\}$ is a subordinator, that is, a nondecreasing Lévy process with exponent

$$-\eta(\alpha) = -\log \mathrm{E}\,\mathrm{e}^{-\alpha I(1)} = c\alpha + \int_{(0,\infty)} (1 - \mathrm{e}^{-\alpha x})\nu(\mathrm{d}x),$$

where $c \geq 0$ and ν is a Lévy measure satisfying $\int_{(0,\infty)} \min(x,1)\nu(\mathrm{d}x) < \infty$. Then, with $I_i(t) = I(t - T_{i-1})$, all of the conditions of Corollary 1 are satisfied. Therefore,

$$\mathrm{E}\exp(-\alpha I_e) = \mathrm{E}\exp(-\alpha I_1(\tau_e)) = \mathrm{E}\exp(-\eta(\alpha)\tau_e) = \frac{1 - \mathrm{E}\exp(-\eta(\alpha)\tau_1)}{\eta(\alpha)\,\mathrm{E}\,\tau_1}.$$

In particular, since $\mathrm{E}\, I_1(t) = \eta'(0)t$ and $\operatorname{var}(I_1(t)) = -\eta''(0)t$, then

$$\mathrm{E}\, I_e = \eta'(0)\,\mathrm{E}\,\tau_e = \eta'(0)\frac{\mathrm{E}\,\tau_1^2}{2\,\mathrm{E}\,\tau_1}$$

when $\mathrm{E}\,\tau_1^2 < \infty$ and $\eta'(0) < \infty$. When $\mathrm{E}\,\tau_1^3 < \infty$ and $-\eta''(0) < \infty$,

$$\operatorname{var}(I_e) = \mathrm{E}(-\eta''(0)\tau_e) + \operatorname{var}(\eta'(0)\tau_e) = -\eta''(0)\,\mathrm{E}\,\tau_e + (\eta'(0))^2 \operatorname{var}(\tau_e),$$

so that

$$\mathrm{E}\, I_e^2 = -\eta''(0)\,\mathrm{E}\,\tau_e + (\eta'(0))^2\,\mathrm{E}\,\tau_e^2 = -\eta''(0)\frac{\mathrm{E}\,\tau_1^2}{2\,\mathrm{E}\,\tau_1} + (\eta'(0))^2\frac{\mathrm{E}\,\tau_1^3}{3\,\mathrm{E}\,\tau_1}.$$

It is noted that $\eta'(0) = c + \int_{(0,\infty)} x\nu(\mathrm{d}x)$ and, for $n \geq 2$,

$$\eta^{(n)}(0) = (-1)^{n-1}\int_{(0,\infty)} x^n \nu(\mathrm{d}x),$$

finite or infinite.

For the special case where $I(t) = ct$, we have $\eta(\alpha) = c\alpha$, so that $\eta'(0) = c$ and $\eta^{(n)}(0) = 0$ for $n \geq 2$. Here $I_e = c\tau_e$ and, thus,

$$\mathrm{E}\, I_e^n = c\frac{\mathrm{E}\,\tau_1^{n+1}}{(n+1)\,\mathrm{E}\,\tau_1}.$$

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5. Moments for the i.i.d. case

In this section we show how to compute all existing moments in the i.i.d. case. Even though it is straightforward, it is given for ease of reference. Since $W \sim V + I_e$, then, for every $n \geq 1$ such that $\mathrm{E}\, V^n < \infty$ and $\mathrm{E}\, I_e^n < \infty$,

$$\mathrm{E}\, W^n = \sum_{k=0}^{n} \binom{n}{k} \mathrm{E}\, V^k \, \mathrm{E}\, I_e^{n-k}, \tag{9}$$

and, thus, whenever it is possible to compute the moments of V and I_e, there is a simple formula for the computation of the moments of W.

When $\{I_1(t) \mid t \geq 0\}$ is a subordinator, then $\mathrm{E}\, I_e^n$ is an expression involving $\{\eta^{(k)}(0) \mid 1 \leq k \leq n\}$ and $\{\mathrm{E}\, \tau_1^k \mid 1 \leq k \leq n+1\}$. An example of the first two moments was given in the previous section for a general subordinator and for all moments when $I_1(t) = ct$.

In order to compute moments for V, we assume that X_1, τ_1, and $\{I_1(t) \mid t \geq 0\}$ are independent, and we recall that $V \sim (V+Y)X$, where V, Y, and X are independent, $Y \sim I_1(\tau_1)$, and $X \sim X_1$. Then it is clear that, when $\mathrm{E}\, Y^n < \infty$,

$$\mathrm{E}\, V^n = \mathrm{E}\, X^n \sum_{k=0}^{n} \binom{n}{k} \mathrm{E}\, V^k \, \mathrm{E}\, Y^{n-k},$$

and, thus,

$$\mathrm{E}\, V^n = \frac{\mathrm{E}\, X^n}{1 - \mathrm{E}\, X^n} \sum_{k=0}^{n-1} \binom{n}{k} \mathrm{E}\, V^k \, \mathrm{E}\, Y^{n-k},$$

so that moments can be computed recursively provided that the moments of Y can be computed. For example,

$$\mathrm{E}\, V = \frac{\mathrm{E}\, X}{1 - \mathrm{E}\, X} \mathrm{E}\, Y$$

and

$$\mathrm{E}\, V^2 = \frac{\mathrm{E}\, X^2}{1 - \mathrm{E}\, X^2} \bigg(\mathrm{E}\, Y^2 + 2 \frac{\mathrm{E}\, X}{1 - \mathrm{E}\, X} (\mathrm{E}\, Y)^2 \bigg).$$

For the case of a subordinator, $\mathrm{E}\, Y = \eta'(0)\, \mathrm{E}\, \tau_1$ and

$$\operatorname{var}(\tau) = -\eta''(0)\, \mathrm{E}\, \tau_1 + (\eta'(0))^2 \operatorname{var}(\tau_1),$$

so that

$$\mathrm{E}\, Y^2 = -\eta''(0)\, \mathrm{E}\, \tau_1 + (\eta'(0))^2\, \mathrm{E}\, \tau_1^2.$$

For the linear case, of course, $\mathrm{E}\, Y^n = c\, \mathrm{E}\, \tau_1^n$ for all $n \geq 1$.

Since the stationary distribution of the level immediately before a collapse is distributed like $V + Y$, then moments for $V + Y$ are given via (9), where I_e is replaced by Y. For the well-studied case where τ_1 is exponential, we have in fact $I_e \sim Y$ and, thus, in this case it follows that the moments of $V + Y$ coincide with those of W, as expected and discussed earlier.

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6. The relation with shot-noise type processes

Here we point out a connection between growth-collapse processes and shot-noise type processes for the case where the X_i do not have an atom at zero and $-\mathrm{E}\log X_i < \infty$. Without these assumptions, this relation is either not valid or useless. In particular, if we let $\xi_i = -r^{-1}\log X_i$ for some $r > 0$, $S_0 = 0$, $S_n = \sum_{i=1}^{n} \xi_i$ for $n \geq 1$, and $M(t) = \sup\{n \geq 0 \mid S_n \leq t\}$, then with Y_i as defined before, the process

$$Z(t) = Z(0)\mathrm{e}^{-rt} + \sum_{n=1}^{M(t)} Y_i \exp(-r(t - S_i))$$

is a shot-noise type process which is also the unique solution of

$$Z(t) = Z(0) + \sum_{i=1}^{M(t)} Y_i - r\int_0^t Z(s)\,\mathrm{d}s,$$

where an empty sum is defined to be 0. It is easy to check that at jump epochs, the level of this process immediately before or after a jump has the same dynamics as the process $\{V_n \mid n \geq 0\}$ or, respectively, $\{V_n + Y_n \mid n \geq 0\}$, and, thus, the same stationary version. Therefore, the stationary behavior of the continuous-time process can be inferred from a similar averaging principle as for the growth-collapse process. That is, the stationary Laplace–Stieltjes transform (LST) is given by

$$\frac{\mathrm{E}\int_0^{\xi_1} \exp(-\alpha(V_0^* + Y_1)\mathrm{e}^{-rt})\,\mathrm{d}t}{\mathrm{E}\,\xi_1} = \frac{\mathrm{E}\int_0^{\infty} \exp(-\alpha(V_0^* + Y_1)\mathrm{e}^{-rt})\,\mathbf{1}_{\{X_1 \leq \mathrm{e}^{-rt}\}}\,\mathrm{d}t}{\mathrm{E}\int_0^{\infty} \mathbf{1}_{\{X_1 \leq \mathrm{e}^{-rt}\}}\,\mathrm{d}t},$$

which in the i.i.d. case, via the change of variables $\mathrm{e}^{-rt} = x$, becomes

$$\int_0^1 \tilde{V}(\alpha x)\tilde{Y}(\alpha x)\frac{F_X(x)}{x}\,\mathrm{d}x \bigg/ \int_0^1 \frac{F_X(x)}{x}\,\mathrm{d}x.$$

In particular, when $I_i(t) = ct$, $\mathrm{E}\,\tau_1 < \infty$ and $\mathrm{E}\,X_i < 1$. Then the conditions and, therefore, the conclusions of Theorem 5.1 and Theorem 5.2 of [7] are met and it follows that the stationary densities f_{GC} and f_{SN} for the growth-collapse process and of the shot-noise process, respectively, exist and satisfy the relation

$$cf_{\mathrm{GC}}(x)\,\mathrm{E}\,\tau_1 = rxf_{\mathrm{SN}}(x)\,\mathrm{E}\,\xi_1.$$

Therefore, if, for $i = \mathrm{GC}, \mathrm{SN}$ and $\alpha \geq 0$, we denote by $\psi_i(\alpha) = \int_0^{\infty} \mathrm{e}^{-\alpha x} f_i(x)\,\mathrm{d}x$ the associated LSTs, then we have

$$c\psi_{\mathrm{GC}}(\alpha)\,\mathrm{E}\,\tau_1 = -r\psi_{\mathrm{SN}}'(\alpha)\,\mathrm{E}\,\xi_1.$$

From this relationship between the two distributions, it follows that if we let $\mu_{\mathrm{GC}}(n)$ and $\mu_{\mathrm{SN}}(n)$ be the nth moments of the stationary distribution of the growth-collapse and shot-noise models, respectively, then these moments satisfy

$$c\mu_{\mathrm{GC}}(n)\,\mathrm{E}\,\tau_1 = r\mu_{\mathrm{SN}}(n+1)\,\mathrm{E}\,\xi_1.$$

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In particular, the stationary expected value for the shot-noise process is given by

$$\mu_{\mathrm{SN}}(1) = \frac{c\,\mathrm{E}\,\tau_1}{r\,\mathrm{E}\,\xi_1},$$

without any further conditions. For higher moments, more assumptions are needed, of course, as discussed earlier in this paper.

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