ON THE HYPERPLANE SECTIONS THROUGH TWO GIVEN POINTS OF AN ALGEBRAIC VARIETY

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1. Let k be an infinite field and let V/k be an irreducible variety of dimension $\geqq 2$ in a projective n-space P^n over k. Let P and Q be two k-rational points on V In this paper, we describe ideal-theoretically the generic hyperplane section of V through P and Q (Theorem 1) and prove that the section is almost always an absolutely irreducible variety over k^{1/p^e} if V/k is absolutely irreducible (Theorem 3). As an application (Theorem 4), we give a new simple proof of an important special case of the existence of a curve connecting two rational points of an absolutely irreducible variety [**4**], namely any two k-rational points on V/k can be connected by an irreducible curve.

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2. Without loss of generality, we assume that V/k is affine in an affine space A^n and P and Q are in A^n. We choose a coordinate system for A^n, so that $P = (0)$, the origin of A^n, and the X_n-axis passes through P and Q. Consider a generic hyperplane $H_u: u_1X_1 + \ldots + u_{n-1}X_{n-1} = 0$ through the X_n-axis, where $u_1, \ldots, u_{n-1}$ are algebraically independent over k. Let $\mathfrak{p}$ be the prime ideal of V in the polynomial ring $k[X_1, \ldots, X_n]$. In the following, H_u will be used also as the polynomial $u_1X_1 + \ldots + u_{n-1}X_{n-1}$ in $k(u_1, \ldots, u_{n-1})[X_1, \ldots, X_n]$. Let $(\mathfrak{p}, H_u) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_\mu$ be an irredundant primary decomposition. Let $\mathfrak{q}_1$ be an isolated component with $\mathfrak{p}_u$ as its radical and let W_u be its variety, dim $W_u \geqq 1$, since it is well known that all the components of the intersection of V with a hyperplane have dimension $\geqq \dim V - 1$.

LEMMA 1. *Let* $(\xi) = (\xi_1, \ldots, \xi_n)$ *be a generic point of* W_u *over* $k(u_1, \ldots, u_{n-1})$. *If* dim $V \geqq 3$, *or, if* dim $V = 2$ *and* V *does not contain the line* PQ, *then* (ξ) *is a generic point of* V *over* k.

Proof. Let dim $V = r$. Denoting

$$k(\xi) = k(\xi_1, \ldots, \xi_n), \qquad k(u;\xi) = k(u_1, \ldots, u_{n-1}; \xi_1, \ldots, \xi_n),$$

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we have

$$\begin{aligned}\operatorname{tr}\deg_{k(\xi)}k(u;\xi)+\operatorname{tr}\deg_k k(\xi) &= \operatorname{tr}\deg_k k(u;\xi)\\ &= \operatorname{tr}\deg_k k(u)+\operatorname{tr}\deg_{k(u)}k(u;\xi) = (n-1)+(r-1),\end{aligned}$$

and $\operatorname{tr}\deg_{k(\xi)}k(u;\xi) \leqq n-2$. Thus $\operatorname{tr}\deg_k k(\xi) \geqq r$. But $(\xi) \in V$, therefore $\operatorname{tr}\deg_k k(\xi) = r$. Hence (ξ) is a generic point of V over k.

LEMMA 2. *Let (ξ) and V be the same as those in Lemma* 1. *If $\xi_i \neq 0$ for some i with $1 \leqq i \leqq n-1$, then $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{n-1}$ are algebraically independent over $k(\xi)$.*

Proof. Say $i = 1$. $\operatorname{tr}\deg_{k(\xi)}k(u;\xi)+\operatorname{tr}\deg_k k(\xi) = (n-2)+r$. Since

$$u_1 = -\frac{u_2\xi_2+\ldots+u_{n-1}\xi_{n-1}}{\xi_1} \in k(u_2, \ldots, u_{n-1}, \xi),$$

we have $k(u;\xi) = k(u_2, \ldots, u_{n-1}, \xi)$ and

$$\operatorname{tr}\deg_{k(\xi)}k(u_2, \ldots, u_{n-1};\xi)+r = r+n-2.$$

Therefore $\operatorname{tr}\deg_{k(\xi)}k(u_2, \ldots, u_{n-1};\xi) = n-2$, i.e. $u_2, \ldots, u_{n-1}$ are algebraically independent over $k(\xi)$.

PROPOSITION 1. *Let $\mathfrak{p}, H_u, \mathfrak{p}_u, (\xi)$, and W_u be as previously defined. If* $\dim V \geqq 3$, *or, if* $\dim V = 2$ *and V does not contain the line PQ, then*

$$(\mathfrak{p}, H_u):(X_1, \ldots, X_{n-1})^\gamma = \mathfrak{p}_u$$

for all sufficiently large integers γ, where

$$(X_1, \ldots, X_{n-1}) = (X_1, \ldots, X_{n-1})\cdot k(u_1, \ldots, u_{n-1})[X_1, \ldots, X_n].$$

Proof. In the proof, we write $k[X]$ for $k[X_1, \ldots, X_n]$. Let $F(u_1, \ldots, u_{n-1}; X)$ be a polynomial in $\mathfrak{p}_u$, we may assume that

$$F(u_1, \ldots, u_{n-1}; X) \in k[u_1, \ldots, u_{n-1}][X].$$

If $\xi_1 \neq 0$, then $F(u_1, \ldots, u_{n-1}; \xi) = 0$ implies that

$$F\left(-\frac{u_2\xi_2+\ldots+u_{n-1}\xi_{n-1}}{\xi_1}, u_2, \ldots, u_n; \xi\right) = 0.$$

Hence there exists a non-negative integer σ such that

$$X_1{}^\sigma F\left(-\frac{u_2X_2+\ldots+u_{n-1}X_{n-1}}{X_1}, u_2, \ldots, u_{n-1}; X\right) \in k(u_2, \ldots, u_{n-1})[X]$$

vanishes at (ξ). By Lemma 2, the prime ideal determined by (ξ) in $k(u_2, \ldots, u_{n-1})[X]$ is $\mathfrak{p}k(u_2, \ldots, u_{n-1})[X]$. Thus

$$X_1{}^\sigma F\left(-\frac{u_2X_2+\ldots+u_{n-1}X_{n-1}}{X_1}, u_2, \ldots, u_{n-1}; X\right) \in \mathfrak{p}k(u_2, \ldots, u_{n-1})[X].$$

But

$$X_1^{\sigma}F\left(-\frac{u_2X_2+\ldots+u_{n-1}X_{n-1}}{X_1}, u_2, \ldots, u_{n-1}; X\right) - X_1^{\sigma}F(u_1, \ldots, u_{n-1}; X)$$
$$\equiv 0 \mod (u_1X_1+\ldots+u_{n-1}X_{n-1})\cdot k(u_1, \ldots, u_{n-1})[X]$$

for large σ. We have

$$X_1^{\sigma}F(u_1, \ldots, u_{n-1}; X) \in (\mathfrak{p}, H_u)\cdot k(u_1, \ldots, u_{n-1})[X]$$

for large σ. The above discussion is symmetric with respect to those $\xi_i \neq 0$ $(i = 1, \ldots, n-1)$. Therefore for any $\xi_i \neq 0$ $(i = 1, 2, \ldots, n-1)$, we have $X_i^{\sigma}F(u_1, \ldots, u_{n-1}; X) \in (\mathfrak{p}, H_u)$ for large σ. For any j such that $\xi_j = 0$, $X_j \in \mathfrak{p}$, thus $X_j^{\sigma}F(u_1, \ldots, u_{n-1}; X) \in (\mathfrak{p}, H_u)$ for any $F \in \mathfrak{p}_u$ and any non-negative integer σ. Thus $(\mathfrak{p}, H_u):(X_1, \ldots, X_{n-1})^{\gamma} \supset \mathfrak{p}_u$ for all sufficiently large integers γ. We now show the other inclusion: Let $g(u_1, \ldots, u_{n-1}; X)$ be an element in $(\mathfrak{p}, H_u):(X_1, \ldots, X_{n-1})^{\gamma}$. Then for any $h(u_1, \ldots, u_{n-1}; X) \in (X_1, \ldots, X_{n-1})^{\gamma}$, $h(u; X)\cdot g(u; X) \in (\mathfrak{p}, H_u)$. Therefore there exists $m_i(u; X)$, $n(u; X)$ in $k(u_1, \ldots, u_{n-1})[X]$ such that

$$h(u;X)\cdot g(u;X) = \sum_{i=1}^{s} m_i(u;X)F_i(X) + n(u;X)H_u,$$

where $(F_1, \ldots, F_s)\cdot k[X] = \mathfrak{p}$. Thus $h(u;\xi)g(u;\xi) = 0$. If $g(u;\xi) \neq 0$, then $h(u;X) = 0$ at (ξ) for all $h(u;X) \in (X_1, \ldots, X_{n-1})^{\gamma}$, which implies that $\dim V \leqq 1$, a contradiction. Therefore $g(u;\xi) = 0$ and $g(u;X) \in \mathfrak{p}_u$, i.e. $(\mathfrak{p}, H_u):(X_1, \ldots, X_{n-1})^{\gamma} \subset \mathfrak{p}_u$. Hence $(\mathfrak{p}, H_u):(X_1, \ldots, X_{n-1})^{\gamma} = \mathfrak{p}_u$ for all sufficiently large integers γ.

COROLLARY 1. *Let V be the same as in the proposition; then $(\mathfrak{p}, H_u)$ has only one isolated component.*

Proof. Suppose that $\mathfrak{p}_2$, say, is another isolated component; then, by the proposition, we have $(\mathfrak{p}, H_u):(X_1, \ldots, X_{n-1})^{\gamma'} = \mathfrak{p}_2$ for all sufficiently large γ'. It follows that $\mathfrak{p}_u = \mathfrak{p}_2$.

THEOREM 1. *If* $\dim V \geqq 3$, *or, if* $\dim V = 2$ *and V does not contain the line PQ, then $(\mathfrak{p}, H_u)$ is either a prime or has an irredundant primary decomposition in which there is only one isolated component which is a prime ideal and the rest are the embedded components of dimension* 0 *and at most one embedded component of dimension* 1 *with the prime ideal of the line PQ, $(X_1, \ldots, X_{n-1})$, as its radical.*

Proof. Let $(\mathfrak{p}, H_u) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{\mu}$ be an irredundant primary decomposition and assume that $\mathfrak{q}_1$ is the only isolated component, according to Corollary 1, with $\mathfrak{p}_u$ as its radical.

$$(\mathfrak{p}, H_u):(X_1, \ldots, X_{n-1})^{\gamma} = \bigcap_{i=1}^{\mu} (\mathfrak{q}_i : (X_1, \ldots, X_n)^{\gamma})$$

for sufficiently large γ. First, if no radical of the $\mathfrak{q}_i$s contains any power of $(X_1, \ldots, X_{n-1})$, then $\mathfrak{q}_i : (X_1, \ldots, X_{n-1})^\gamma = \mathfrak{q}_i$, and

$$\mathfrak{p}_u = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_\mu = (\mathfrak{p}, H_u).$$

Thus the assertion is proved in this case. Secondly, if some of the radicals of $\mathfrak{q}_i$s contain a power of $(X_1, \ldots, X_{n-1})$, say, $\mathfrak{q}_t, \ldots, \mathfrak{q}_\mu$, then they contain $(X_1, \ldots, X_{n-1})$ and it follows that $0 \leqq \dim \mathfrak{q}_i \leqq 1$ for $t \leqq i \leqq \mu$. Hence for $i \leqq t-1$, $\mathfrak{q}_i : (X_1, \ldots, X_{n-1})^\gamma = \mathfrak{q}_i$ and $\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{t-1} = \mathfrak{p}_u$. Therefore $\mathfrak{p}_u \cap \mathfrak{q}_t \cap \ldots \cap \mathfrak{q}_\mu = \bigcap_{i=1}^{\mu} \mathfrak{q}_i = (\mathfrak{p}, H_u)$. Finally, if $\mathfrak{q}_i$, $t \leqq i \leqq \mu$, is of dimension 1, then $\sqrt{\mathfrak{q}_i} = (X_1, \ldots, X_{n-1})$. Hence $\mathfrak{p} \subset (X_1, \ldots, X_{n-1})$. The irredundancy of the decomposition implies that there is only one embedded component of dimension 1 if such exists. Hence, if $\dim V \geqq 3$, there is at most one embedded primary component of dimension 1.

COROLLARY 2. *Let V be the same as in the proposition. If V/k is normal, then $(\mathfrak{p}, H_u)$ is a prime ideal.*

Proof. This follows from the fact that the principal ideals in the coordinate ring of V over k are unmixed.

LEMMA 3. *Let K be a regular finitely generated extension of an infinite field k with* $\operatorname{tr} \deg_k K \geqq 3$. *Let x, y, and z be three elements of K algebraically independent over k, and $z/x \notin K^p k$, where p is the characteristic of k. Then for all but a finite number of constants $c \in k$, K is a regular extension of $k((y + cz)/x)$. Moreover, if τ is an indeterminate, then $K(\tau)$ is regular over $k(\tau)((y + \tau z)/x)$.*

Proof. By [**3**, p. 185, Proposition 1 and p. 186, Corollary to Proposition 2], the hypothesis $z/x \notin K^p k$ yields $D(z/x) \neq 0$ for some derivation D of K/k and separability for K over $k((y + cz)/x)$ except for only one c for which $D((y + cz)/x) = 0$. The rest of the lemma follows from [**5**, p. 369, Theorem 8 and p. 369, Corollary to Theorem 8] by taking y/x for ξ_1 and z/x for ξ_2.

THEOREM 2. *If V/k is an absolutely irreducible variety of dimension $r \geqq 3$ and if V is not a cylinder in the direction of line PQ, then W_u is an absolutely irreducible variety over $k(u)^{1/p^e}$.*

Proof. $W_u/k(u)$ is irreducible; let (ξ) be a generic point of W_u over $k(u)$. By Lemma 1, (ξ) is a generic point of V over k, hence $\operatorname{tr} \deg_k k(\xi) \geqq 3$ and $k(\xi)$ is a regular extension over k [**6**, p. 69, Proposition 1]. Let ξ_1, ξ_2, and ξ_{n-1} be algebraically independent over k. If ξ_n is separably algebraic over $k(\xi_1, \ldots, \xi_{n-1})$ and if we assume that $\{\xi_1, \xi_2, \xi_{n-1}\}$ is a subset of a separable transcendental base of $k(\xi)$, let $K = k(u_2, \ldots, u_{n-2})(\xi)$; u_{n-1} is then algebraically independent over K. Viewing $k(u_2, \ldots, u_{n-2})$ as the field k and u_{n-1} as τ in Lemma 3, we have $K(u_{n-1}) = k(u_2, \ldots, u_{n-2})(u_{n-1})(\xi) = k(u)(\xi)$. Let $y = -(u_2\xi_2 + \ldots + u_{n-2}\xi_{n-2})$, $z = -\xi_{n-1}$, and $x = \xi_1$, one sees that x, y, and z are algebraically independent over $k(u_2, \ldots, u_{n-2})$ and

$$\frac{z}{x} = -\frac{\xi_{n-1}}{\xi_1} \notin K^p \cdot k(u_2, \ldots, u_{n-2}).$$

By Lemma 3, we see that $K(u_{n-1})$ is a regular extension over

$$k(u_2, \ldots, u_{n-2})(u_{n-1})\left(\frac{y + u_{n-1}\xi_{n-1}}{x}\right) = k(u).$$

Therefore W_u is absolutely irreducible over $k(u)$. If ξ_n is not separable over $k(\xi_1, \ldots, \xi_{n-1})$, we consider the map τ of A^n to A^n such that $\tau(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1}, a_n{}^p)$. τ maps A^n onto itself. If U is an absolutely irreducible variety with k as a field of definition and $(\xi_1, \ldots, \xi_n)$ as a generic point over k, then $k(\xi_1, \ldots, \xi_{n-1}\xi_n{}^p)$ is regular over k and τ maps U onto the absolutely irreducible variety $\bar{U}$ having $(\xi_1, \ldots, \xi_{n-1}, \xi_n{}^p)$ as a generic point over k. τ maps distinct Us into distinct $\bar{U}$s, and the hyperplane section $u_1X_1 + \ldots + u_{n-1}X_{n-1} = 0$ of V onto the hyperplane section

$$u_1X_1 + \ldots + u_{n-1}X_{n-1} = 0$$

of $\bar{V}$. Then ξ_n can be replaced by $\xi_n{}^p$, and repeating by $\xi_n{}^{p^e}$ so that one reduces to the case that $\xi_n{}^{p^e}$ is separable over $k(\xi_1, \ldots, \xi_{n-1})$. Thus $\{\xi_1, \ldots, \xi_{n-1}\}$ contains a separating transcendency basis of $k(\xi_1, \ldots, \xi_{n-1}, \xi_n{}^{p^e})$. Replacing ξ_n by $\xi_n{}^{p^e}$ in the argument of the separable case above, we conclude that $K(u_{n-1}) = k(u_2, \ldots, u_{n-2})(\xi_1, \ldots, \xi_{n-1}, \xi_n{}^{p^e})(u_{n-1})$ is regular over $k(u)$. This yields that $k(u_1, \ldots, u_{n-1})(\xi_1{}^{p^e}, \ldots, \xi_n{}^{p^e})$ is regular over $k(u)$, whence $k(u)^{1/p^e}(\xi)$ is regular over $k(u)^{1/p^e}$ and W_u is absolutely irreducible over $k(u)^{1/p^e}$. Therefore we conclude that W_u is absolutely irreducible over $k(u)^{1/p^e}$. $k(u_1, \ldots, u_{n-1})^{1/p^e} = k^{1/p^e}(u_1{}^{1/p^e}, \ldots, u_{n-1}{}^{1/p^e})$ and $u_1{}^{1/p^e}, \ldots, u_{n-1}{}^{1/p^e}$ remain as $n - 1$ indeterminates over k^{1/p^e}. The substitution

$$\begin{aligned}(u^{1/p^e}) = (u_1{}^{1/p^e}, \ldots, u_{n-1}{}^{1/p^e}) &\to (a^{1/p^e}) \\ &= (a_1{}^{1/p^e}, \ldots, a_{n-1}{}^{1/p^e}), \qquad (a) = (a_1, \ldots a_{n-1}) \in k^{n-1}\end{aligned}$$

is the same as the substitution $(u) = (u_1, \ldots, u_{n-1}) \to (a) = (a_1, \ldots, a_{n-1})$. Therefore to specialize an ideal $\mathfrak{a}$ in $k(u)^{1/p^e}[X]$ to an ideal $\bar{\mathfrak{a}}$ in $k^{1/p^e}[X]$ by the substitution $(u) \to (a)$ is the same as to specialize $\mathfrak{a}$ in $k^{1/p^e}(u^{1/p^e})[X]$ to $\bar{\mathfrak{a}}$ in $k^{1/p^e}[X]$ by the substitution $(u^{1/p^e}) \to (a^{1/p^e})$. Therefore [**1**; **2**; **5**, Appendix] are applicable to the case of specializing ideals in $k(u)^{1/p^e}[X]$ by the substitution $(u) \to (a)$.

THEOREM 3. *If V/k is an absolutely irreducible variety of dimension $r \geqq 3$ and if V is not a cylinder in the direction of line PQ, then for almost all hyperplanes H_a:$a_1X_1 + \ldots + a_{n-1}X_{n-1} = 0$, where $a_1, \ldots, a_{n-1} \in k$, $H_a \cap V$ is absolutely irreducible over k^{1/p^e}.*

Proof. Let $\sqrt{((\mathfrak{p}, H_u))}$ be the radical of $(\mathfrak{p}, H_u)$ in the polynomial ring $k(u)^{1/p^e}[X]$. $\sqrt{((\mathfrak{p}, H_u))}$ is the prime ideal of W_u in $k(u)^{1/p^e}[X]$ and is absolutely prime according to Theorem 2. By [**2**, p. 136, *Satz* 16], $\overline{\sqrt{((\mathfrak{p}, H_u))}}$ is almost always prime and absolute prime. By [**5**, p. 379, Theorem 1, Appendix], we obtain $\overline{\sqrt{((\mathfrak{p}, H_u))}} \subset \sqrt{(\overline{(\mathfrak{p}, H_u)})}$ almost always. But $\sqrt{((\mathfrak{p}, H_u))}$ is prime almost always. Therefore $\overline{\sqrt{((\mathfrak{p}, H_u))}} = \sqrt{(\overline{(\mathfrak{p}, H_u)})}$ almost always. Therefore, for almost all $(a) \in k^{n-1}$, $\sqrt{((\mathfrak{p}, a_1X_1 + \ldots + a_{n-1}X_{n-1}))}$, which is the prime ideal of $H_a \cap V$, is absolutely prime. Thus $H_a \cap V$ is absolutely irreducible over k^{1/p^e}.

THEOREM 4. *If V/k is an absolutely irreducible variety of dimension $r > 1$, then any two k-rational points P and Q on V can be connected by an irreducible algebraic curve defined over $k^{1/p^e}(u)$.*

Proof. We may assume that $P = (0)$, and the X_n-axis passes through P and Q. The assertion is obvious if V is a cylinder in the direction of the line PQ or the line PQ lies on V. Hence we assume that the above are not the case. Repeating the hyperplane section and Theorem 3, we reduce the theorem to the case that $V/k^{1/p^e}$ is of dimension $r = 2$. One more application of Theorem 1 yields the desired curve.

REFERENCES

1. W. Krull, *Parameter spezializierung in Polynomringen*, Arch. Math. *1* (1948), 56–64.
2. ——— *Parameterspezializierung in Polynomringen*. II: *Das Grund polynom*, Arch. Math. *1* (1948), 129–137.
3. S. Lang, *Introduction to algebraic geometry* (Interscience, New York, 1964).
4. H. Nishimura and Y. Nakai, *On the existence of a curve connecting given points on an abstract variety*, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. *28* (1954), 267–270.
5. A. Seidenberg, *The hyperplane section of normal varieties*, Trans. Amer. Math. Soc. *69* (1950), 357–386.
6. A. Weil, *Foundations of algebraic geometry*, Amer. Math. Soc. Colloq. Publ., Vol. 29 (Amer. Math. Soc., Providence, R.I., 1962).

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