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WREATH PRODUCTS OF NONOVERLAPPING LATTICE ORDERED GROUPS

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Introduction. One of the fundamental tools in the theory of totally ordered groups is Hahn's Theorem (a detailed discussion may be found in Fuchs [3]), which asserts, roughly, that every abelian totally ordered group can be embedded in a lexicographically ordered (unrestricted) direct sum of copies of the ordered group of real numbers. Almost any general question regarding the structure of abelian totally ordered groups can be answered by reference to Hahn's theorem. For the class of nonabelian totally ordered groups, a theorem which parallels Hahn's Theorem is given in [5], and states that each totally ordered group can be o-embedded in an ordered wreath product of subgroups of the real numbers. In order to extend this theorem to include an "if and only if" statement, one must consider lattice ordered groups, as an ordered wreath product of subgroups of the real numbers is, in general, not totally-ordered, but lattice ordered. Theorem 4.2 states that a lattice ordered group G can be 1-embedded in an ordered wreath product of subgroups of the reals if, and only if, G is nonoverlapping (Def. 2.5). This class of lattice ordered groups is also the collection of lattice ordered groups which are, as Wolfenstein [7] has named them, normal valued groups (see theorem 2.8).

The first section defines the notion of an interval of an order preserving permutation of a chain, and gives several properties of intervals. Section 2 defines nonoverlapping lattice ordered groups and proves certain results concerning these groups. Section 3 discusses ordered wreath products, (borrowing strongly from [5]), and certain transitive ordered permutation groups. The final section, concerned with nonoverlapping 1-groups and wreath products, ends with the important theorem of the paper, theorem 4.3.

1. Intervals of Order preserving Permutations of a Chain. Let G denote an 1-subgroup of the 1-group $A(S)$ of all order preserving permutations of the Chain S. [4] For $g \in G$ and $s \in S$, the set $\{t \in S \mid \text{there exist integers } n, m \text{ such that } sg^n \leq t \leq sg^m\}$, is the *interval of s by g*, and is denoted by $(g)_s$ or g_s. For $a \leq b$, elements of S, the subsets $\{t \in S \mid a \leq t \leq b\}$, $\{t \in S \mid a < t \leq b\}$, $\{t \in S \mid a \leq t < b\}$, and $\{t \in S \mid a < t < b\}$ will be denoted respectively by $[a, b]$, $(a, b]$, $[a, b)$ and (a, b).

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THEOREM 1.1. *Let G be an* 1*-subgroup of* $A(S)$ *for some chain* S.

(1) *For each* $s \in S$ *and* $g \in G$, g_s *is convex and* $g_s=(g^n)_s=|g|_s$ *for all integers* n *different from* 0. $(g_s^0=s)$.

(2) *Let* $s \in S$ *and* $g \in G$. *If* $sg>s$, *then* $g_s=\{t \in S \mid sg^{-n} \leq t \leq sg^n$ *for some positive integer* $n\}$. *If* $sg<s$, *then* $g_s=\{t \in S \mid sg^n \leq t \leq sg^{-n}$ *for some positive integer* $n\}$. *If* $sg=s$, $g_s=s$.

(3) *For* $\{s, t\} \subseteq S$ *and* $g \in G$, *either* $g_s=g_t$ *or* $g_s \cap g_t=\varnothing$. *Thus if* $t \in g_s$, $g_s=g_t$.

(4) *If* $e \leq h \leq g$ *for some* $\{h, g\} \subseteq G$, *then* $h_s \subseteq g_s$ *for each* $s \in S$.

(5) *If* G *is abelian and* $s\,|f| \leq s\,|g|$ *for some* $s \in S$ *and* $\{f, g\} \subseteq G$, *then* $f_s \subseteq g_s$.

(6) *If* $[s, s\,|g|] \subseteq f_s$ *for some* $s \in S$ *and* $\{g, h\} \subseteq G$, *then* $s\,|f|^n > s\,|g|$ *for some positive integer* n.

(7) *Suppose* S *is Dedekind complete and* $\{f, g\} \subseteq G$. *Then* $|f|^n \leq |g|$ *for all integers* n, *if and only if, for each* $s \in S$ *there exists an element* $t \in S$ *such that* $f_s \subseteq [t, t\,|g|]$.

(8) *If* $s \neq x$ *and* $sh \geq s$ *while* $xh \leq x$, *then* $x \notin h_s$. *Thus if* $sf \geq sg$ *and* $xf \leq xg$, $x \notin (fg^{-1})_s$.

(9) *Let* $f \in G$ *and* $t \in S$. *If for some* $x \in S$, $x \notin f_t$, *then* $x>s$ *for all* $s \in f_t$ *and* $xf^n>s$ *for all integers* n, *or* $x<s$ *for all* $s \in f_t$ *and* $xf^n<s$ *for all integers* n.

(10) *For* $\{g, h\} \subseteq G$ *and* $s \in S$, $(g^{-1}gh)_{sg}=(h_s)g$ *and* $(g^{-1}hg)_s=(h_{sg^{-1}})g$.

Proof. (1) That g_s is convex and $g_s=(g^n)_s$ for all non-zero integers n is clear from the definition. If $sg \geq s$, then $s\,|g|^n=sg^n$ for all integers n, and if $sg<s$, then $s\,|g|^n=sg^{-n}$ for all integers n. In each case $|g|_s=g_s$.

(2) and (3) are clear.

(4) This follows directly from the relation $xg^{-m} \leq xh^{-m} \leq x \leq xh^n \leq xg^n$.

(5) If G is abelian, $s\,|g|^{-m} \leq s\,|f|^{-m} \leq s\,|f|^n \leq s\,|g|^n$ for all positive integers n and m. Thus by part 4, $|f|_s \subseteq |g|_s$, which by part 1 implies $f_s \subseteq g_s$.

(6) This follows directly from the definition of f_s.

(7) Fix $s \in S$. If $f_s \subseteq [t, t\,|g|]$ for some $t \in S$, it follows from the definition of an interval that $s\,|f|^n \leq t\,|g| \leq s\,|g|$.

Conversely, suppose $|f|^n \leq |g|$ for all integers n. Fix $s \in S$. If $sf=s$, $f_s \subseteq [s, s\,|g|]$. If $sf \neq s$, let $t=\{z \in S \mid z \leq y$ for each $y \in f_s\}$. Since S is dedekind complete, t defines a point of S and since $f_s \neq s$, $t<x$ for all $x \in f_s$. If there exists $y \in f_s$ such that $t\,|g|<y$, then $t<y\,|g|^{-1} \leq y$ so that $y\,|g|^{-1} \in f_s$. Part 3 of this theorem implies that there exists an m such that $(y\,|g|^{-1})f^m \geq y$. That is, $(y\,|g|^{-1})\,|g|<(y\,|g|^{-1})f^{m+1}$ which contradicts the assumption. Thus $t\,|g| \geq y$ for all $y \in f_s$ and hence $f_s \subseteq [t, t\,|g|]$.

(8) If $s<x$, then $sh^{-n}<s<sh^n<xh^n \leq x$ for all positive integers n, and hence $x \notin h_s$ by part 2.

If $x<s$, then $xh^{-n}<sh^{-n}$ for all positive integers n, so that $x \leq xh^{-n}<sh^{-n}<s<xh^n$ and hence $x \notin h_s$ by part 2.

(9) If $x \notin f_t$, then $f_x \cap f_t=\varnothing$ and since f_t is convex, $x>s$ for all $s \in f_t$ or $x<s$

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for all $s \in f_t$. In the first case, if $xf^n \leq s$ for some integer n and some $s \in f_t$, then $s \in f_t \cap f_x$ which is absurd, so that $xf^n > s$. Similarly if $x < s$ for all $s \in f_t$ and $s \leq xf^n$ for some integer n and some $s \in f_t$, then $s \in f_t \cap f_x$, which is absurd, so that $xf^n < s$.

(10) The statement

$$h_{sg^{-1}} = \{t \in S \mid sg^{-1}h^n \leq t \leq sg^{-1}h^m \text{ for integers } n \text{ and } m\}$$

implies

$$h_{sg^{-1}}g = \{p \in S \mid sg^{-1}h^n g \leq p \leq sg^{-1}h^m g \text{ for integers } n, m\}$$

But this latter equation is the definition of $(g^{-1}hg)_s$. That is, $(h_{sg^{-1}})g = (g^{-1}hg)_s$. By replacing the point s by sg, one has $(h_s)g = (g^{-1}hg)_{sg}$.

2. **Nonoverlapping lattice-ordered groups.** Let G be an l-subgroup of $A(S)$ for some chain S. To emphasize the present dependency of the following definition on the chain S, we will write (G, S) in place of G. It will be seen in Theorem 2.4 that the concept of nonoverlapping which we now define really depends only on the group structure of G. Immediately following theorem 2.4 we give the more general definition of nonoverlapping.

DEFINTION 2.0. If for each $\{g, h\} \subseteq (G, S)$, $g_s \cap h_t \in \{\varnothing, g_s, h_t\}$ whenever s and t are elements of S, then the l-group (G, S) is called *nonoverlapping* (i.e. no two intervals overlap).

LEMMA 2.1. *Let G be an* l*-subgroup of $A(S)$ for some chain S. If (G, S) fails to be nonoverlapping, there exists*

$$\{x, y, z \mid x < y < z\} \subseteq S \quad \text{and} \quad \{h, g\} \subseteq G$$

such that $xg = x$, $zh = z$, $h_x = h_y$ and $g_y = g_z$ (i.e. a special type of overlap).

Proof. Since (G, S) is not overlapping, there exists

$$\{k, f\} \subseteq G \quad \text{and} \quad \{a, b\} \subseteq S$$

such that $k_a \cap f_b \notin \{\varnothing, k_a, f_b\}$. By part 1 of theorem 1.1, we may assume $k \geq e$ and $f \geq e$. Let $\{r, s, t\} \subseteq S$ be such that $r \in k_a \backslash f_b$, $s \in k_a \cap f_b = I$, and $t \in f_b \backslash k_a$. With no loss of generality we may assume $r < s < t$, so that $r < I < t$.

(The comments in this paragraph will be referred to later as part A.) If $w \in S$ and $w \nless I$, then either $w \in f_b$ or $w > f_b$. In the former case, $wf^n \in f_b$ for all integers n, and in the latter case $wf^n > f_b$ for all integers n by part 9 of theorem 1.1. That is, $w \nless I$ implies that $wf^n \nless I$ for all integers n. If $t < I$, then by part 9 of Theorem 1.1, $tf^n < I$ for all integers n. Similarly if $w \ngtr I$ then $wk^n \ngtr I$, and if $w > I$ then $wk^n > I$ for all integers n.

Since s is assumed to belong to $I = k_r \cap f_t$, there exists a positive integer n such that $sk^{-n} < r$ and $sf^n > t$. Let $g = f^n k^{-n} \vee e$, $h = f^{-n}k^n \vee e$, $x = sh^{-1}$, $y = s$, and $z = sg = s(f^n k^{-n} \vee e)$.

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The relation $sf^n > t$ implies $sf^n > I$, so that by part A, $sfk^{-n} > I$. Since k and f are assumed positive, one has $sf^n \geq s(f^n k^{-n} \vee e) = sf^n k^{-n} > I$. That is, $sf^n \geq z > I$ and hence $s \geq zf^{-n}$ so $zf^{-n} \ngtr I$. Thus by part A, $zf^{-n}k^n \ngtr I$, which together with $z > I$ implies that $z(f^{-n}k^n \vee e) = z$. That is, $zh = z$.

Using the relation $sk^{-n} < r < I$, part A, and the fact that $f \geq e$, one has $sk^{-n} \leq sk^{-n}f^n < I$, so that $sk^{-n} \leq s(k^{-n}f^n \vee e) < I$. Thus $sk^{-n} \leq x < I$ which implies $s \leq xk^n$, and hence $xk^n \nless I$. By part A, $xk^n f^{-n} \nless I$ also, and this together with $x < I$ implies that $x(k^n f^{-n} \wedge e) = x$. That is, $xg^{-1} = x$, and hence $xg = x$.

Since $xh = y$ and $zg^{-1} = y$, it is clear that $h_x = h_y$ and $g_y = g_z$.

LEMMA 2.2. *Let* (G, S) *be a nonoverlapping* 1-*group and suppose* $\{h, g\} \subseteq G^+$ (*i.e.* $h \geq e, g \geq e$). *If there is an* $s \in S$ *such that* $h_s \neq g_s$, *then either* $h_s \subseteq g_s$ *and* $xh^n < xg$ *for all* $x \in g_s \cup h_s = g_s$ *and all positive integers* n, *or* $g_s \subseteq h_s$ *and* $xg^n < xh$ *for all* $x \in g_s \cup h_s = h_s$ *and all positive integers* n.

Proof. Since $s \in g_s \cap h_s$, either $g_s \cap h_s = g_s$ or $g_s \cap h_s = h_s$. The two cases are similar, and in the following we assume $g_s \cap h_s = h_s$. In order to prove the lemma in the restricted case in which we select $x \in h_s$, assume by way of contradiction that there exists an $x \in h_s$ such that $xh^n > xg$ for some integer $n > 0$. Let $k = h^n$, and let t be an element of $g_s \backslash k_s$. (Such an element exists since $k_s = h_s$ by theorem 1.1, and h_s is properly contained in g_s.)

Case 1. Suppose $t < x$, and hence $t < k_s$. Since $g_t = g_s$, there is a positive integer m such that $tg^m < k_s$ but $tg^{m+1} \nless k_s$. The relation $tg^m < k_s$ implies $tg^m < x$. Thus $tg^{m+1} < xg < xk \in k_s$, so that $tg^{m+1} \in k_s$ by convexity. Hence $tg^{m+1}k^{-1} \in k_s$ and so $tg^{m+1}k^{-1} > tg^m$. That is $tg^m(gk^{-1} \vee e) > tg^m$. Thus if we denote tg^m by p, and we set F equal to the interval of p by $(gk^{-1} \vee e)$, we have $p \in F \backslash k_s$ and $p(gk^{-1} \vee e) \in F \cap k_s$. Since $xg < xk = xh^n$, we have $xgk^{-1} < x$, which together with $pgk^{-1} > p$ and part 8 of Theorem 1.1, implies that $x \in k_s \backslash F$. That is, for $w = p(gk^{-1} \vee e)$, $(gk^{-1} \vee e)_w \cap k_w$ is not in $\{\varnothing, (gk^{-1} \vee e)_w = F, k_w = k_s\}$, which contradicts (G, S) being nonoverlapping.

Case 2. Suppose $t > s$ and hence $t > k_s$. Since $g_t = g_x$, there exists a positive integer m such that $tg^{-m} > k_s$ but $tg^{-m-1} \ngtr k_s$. The relation $xg < xk$ implies $(xk)g^{-1} > (xk)k^{-1} = x$. However $tg^{-m} > k_s$ so that $tg^{-m} > xk$ and hence $tg^{-m}g^{-1} > xkg^{-1} > xkk^{-1} = x$.

This together with $tg^{-m-1} \ngtr k_s$ means that $tg^{-m-1} \in k_s$ by convexity.

Let $p = tg^{-m} > k_s$. Since $pg^{-1} \ngtr k_s$, then by part 9 of Theorem 1.1 $pg^{-1}k \ngtr k_s$ and hence $pg^{-1}k < p$. Thus $pk^{-1}g > p$ and hence $(k^{-1}g)_p = (k^{-1}g \vee e)_p$. The relation $(xk)g^{-1} > (xk)k^{-1}$ of the last paragraph implies that $(xk)g^{-1}k > xk$ and so by part 8 of Theorem 1.1 $xk \notin (k^{-1}g \vee e)_p$. Thus $xk \in k_s \backslash (k^{-1}g \vee e)_p$.

However $(k^{-1}g \vee e)^{-1} = g^{-1}k \wedge e$ and so by the last paragraph, $p \in (k^{-1}g \vee e)_p \backslash k_s$. Since $p(g^{-1}k \wedge e) \in (k^{-1}g \vee e)_p \cap k_s$, we have

$$(k^{-1}g \vee e)_p \cap k_s \notin \{\varnothing, (k^{-1}g \vee e)_p, k_s\}$$

which contradicts (G, S) being nonoverlapping.

Thus $xh^n \leq xg$ for all positive integers n and all $x \in h_s$. If $xh^n = xg$ for some x and some n, then since $xh > x$ implies $xh^{n+1} > xh^n = xg$, a contradiction, we must have $xh = x$. But then $xh^n = xg$ and $h_x = g_x$ which again is a contradiction. Thus $xh^n < xg$ for all $x \in h_s$ and all positive integers n.

Finally if $y \in g_s$, then h_y is proper in $g_y = g_s$ since h_s is proper in g_s. Thus by the preceding, $yh^n < yg$ for all positive integers n.

LEMMA 2.3. *Suppose* (G, S) *is a nonoverlapping* 1-*group, and suppose* $\{h, g\} \subseteq G^+$. *If for some* $s \in S$, $h_s = g_s = I$, *then one of the following is true.*

(1) $sg = sh = s$

(2) $xg < xh$ *for each* $x \in I$

(3) $xg > xh$ *for each* $x \in I$

(4) $xg < xh^2$ *and* $xg^2 > xh$ *for each* $x \in I$.

Proof. If 1, 2, and 3 fail, there exists $\{u, v\} \subseteq I$ such that $ug \geq uh$ and $vg \leq vh$. By part 8 of Theorem 1.1, each of $(gh^{-1})_u$ and $(hg^{-1})_v$ is proper in I. Hence by lemma 2.2, $xgh^{-1} < xh$ and $xhg^{-1} < xg$, for each $x \in I$. That is, $xg < xh^2$ and $xh < xg^2$ for each $x \in I$.

THEOREM 2.4. *For an* 1-*group* G, *the following are equivalent.*

(1) *For each pair* $\{h, g\} \subseteq G^+$, $hg = g^2h^2$

(2) *The* 1-*subgroup* H *of* $A(S)$ *is nonoverlapping whenever* H *is* 1-*isomorphic to* G.

(3) *There exists an* 1-*subgroup* H *of some* $A(S)$ *which is* 1-*isomorphic to* G *such that* (H, S) *is nonoverlapping.*

Proof. 1⇒2 By way of contradiction, suppose there is an (H, S) as in 2 which fails to be nonoverlapping. By Lemma 2.1 there exists $\{g, h\} \subseteq H^+$ and $\{x, y, z \mid x < y < z\} \subseteq S$ such that $h_x = h_y$, $g_y = g_z$, $zh = z$ and $xg = x$. Thus for some positive integer n, $xh^n > y$ and $yg^n > z$. Let $f = h^n$ and $k = g^n$. Then $xk = x$, $yk > z$, $xf > y$, $zf = z$, and by part 9 of Theorem 1.1, $xf^n < z$ for all integers n. Thus $xk^2f^2 = xf^2 < z$ and $xfk > yk > z$ which contradicts 1.

2⇒3 This is clear when one recalls Holland's Theorem [4] that each 1 group G is 1-isomorphic to an 1-subgroup H of some $A(S)$.

3⇒1 Suppose $\{h, g\} \subseteq H^+$, and fix $s \in S$. If $h_s \neq g_s$ we may assume $h_s \subseteq g_s$, as the case $g_s \subseteq h_s$ is similar, and so by lemma 2.2, $sh < sg$ which implies $shg < sg^2 \leq sg^2h^2$. If $h_s = g_s$, then by Lemma 2.3 we consider the following four cases.

(1) $sg = sh = s$ so that $shg = sg^2h^2$.

(2) $xg < xh$ for all $x \in h_s$ and hence $shg < shh \leq sg^2h^2$.

(3) $xh < xg$ for all $x \in h_s$ and hence $shg < sgg \leq sg^2h^2$.

(4) $xg < xh^2$ and $xh < xg^2$ for all $x \in h_x$ and hence $shg < sg^2h^2$.

Thus for each $s \in S$, $shg \leq sg^2h^2$. Since G is 1-isomorphic to H, this element-wise condition also holds in G.

W. C. Holland has shown [4] that each 1-group G is 1-isomorphic to an 1-subgroup H of some $A(S)$. (The 1-group (H, S) is called a Holland-representation of G.)

This together with the theorem above allows us to make the following definition which is a generalization of definition 2.0.

DEFINITION 2.5. An 1-group is *nonoverlapping* if, and only if, one (and hence all) of its Holland-representations is nonoverlapping in the sense of definition 2.0.

DEFINITION 2.6. A convex 1-subgroup of the 1-group G that is maximal with respect to not containing some g in G is called a *regular subgroup* of G. Let $\Gamma(G)$ be an index set for the collection of all regular subgroups G_γ of G. For each $\gamma \in \Gamma(G)$ there exists a unique convex 1-subgroup G^γ of G which is minimal with respect to properly containing G_γ, called the *cover* of G_γ. If g belongs to G^γ but not to G_γ, then γ (or G_γ) is said to be a value of g. A regular subgroup G_γ is called *special* if there exists an element g in G such that G_γ is the unique value of g. If this is the case, then g is also called special. For $\gamma, \lambda \in \Gamma(G)$, we define $\gamma \leq \lambda$ if $G_\gamma \subseteq G_\lambda$. A subset Δ of $\Gamma(G)$ is said to be *plenary* if

(i) each $e \neq g$ in G has at least one value in Δ,

(ii) if $g \notin G_\delta$ $(\delta \in \Delta)$, then there exists $\lambda \geq \delta$, $(\lambda \in \Delta)$ such that λ is a value for g.

DEFINITION 2.7 (Byrd [1]). Let $NP = \{G \mid G$ is an 1-group and there exists a plenary subset $\Delta(G) \subseteq \Gamma(G)$ such that each G_δ is normal in its cover, for each $\delta \in \Delta\}$.

THEOREM 2.8. *Let G be an* 1-*group. The following are equivalent.*

(1) *G is nonoverlapping*

(2) *for each pair $\{h, g\} \subseteq G^+$, $hg \leq g^2h^2$*

(3) *$G \in NP$*

(4) *each regular* 1-*subgroup of G is normal in its cover* (*i.e. G is normal valued*).

Proof. Wolfenstein proves the equivalence of 2, 3, and 4 in [7]. As a result he refers to such groups as normal valued groups. In [6], the author has independently proved the equivalence of 1, 3 and 4. Either reference together with theorem 2.4 completes the proof.

3. **Lattice ordered wreath products and transitive ordered permutation groups.** Familiarity with the material contained in [5] is assumed. The notation in this section is that of [5].

Let S be a chain and F a set of permutations of the chain S. A convex F-congruence on S is an equivalence relation Q on S such that each Q-cass is convex ($x \leq y \leq z$ and xQz implies xQy), and such that if xQz and $f \in F$, then $(xf)Q(zf)$.

Let (G, S) be a transitive o-permutation group, that is, G is a group of o-permutations of S such that given s and t in S, there exists a $g \in G$ such that $sg = t$. Let $\gamma = (Q_\gamma, Q^\gamma)$ be a pair of convex G-congruences of the transitive o-permutation group (G, S), with Q_γ properly contained in Q^γ. Let o be any point in S. Let S be the chain oQ^γ/Q_γ and let $G_{oQ_\gamma} = \{g \in G \mid (oQ^\gamma)g = oQ^\gamma\}$. Let G be the image in $A(S_\gamma)$ of G_{oQ_γ} under the natural o-homomorphism $g \to \bar{g}$, where $(xQ_\gamma)\bar{g} = (xg)Q_\gamma$.

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Then (G_γ, S_γ) is called the *γ-component* of (G, S). It is an easy consequence of transitivity that, to within an o-isomorphism, (G_γ, S_γ) is independent of the choice of $o \in S$.

The transitive o-permutation group (G, S) is said to be o-*primitive* if the only convex G-congruences on S are the two trivial ones. If C and K are convex G-congruences with $C \subseteq K$, it is easy to check that K covers C if and only if the (C, K)-component (G_γ, S_γ) of (G, S) is o-primitive. The o-*primitive components* (G_γ, S_γ) of (G, S) are those γ-components of (G, S) which are o-primitive. These are primarily those components arising from covering pairs $\gamma=(C_\gamma, C^\gamma)$ of convex G-congruences.

THEOREM 3.1. *Let (G, S) be a transitive lattice-ordered permutation group. If $(W, R)= * \prod_{\gamma\in\Gamma} (G_\gamma, S_\gamma)$ is the ordered wreath product of the* o-*primitive components (G_γ, S_γ) of (G, S), then (W, R) is a transitive lattice ordered permutation group, and there is an embedding $\phi=(G, S)\to(W, R)$ of the following sort. (Let (C_γ, C^γ) and (K_γ, K^γ) be the covering pairs of (G, S) and (W, R), respectively.) First $\phi:S\to R$ is an* o-*embedding and $(C_\gamma, C^\gamma)\phi=(K_\gamma, K^\gamma) \mid S\phi$. Next $\phi:G\to W$ is an* 1-*embedding of the lattice ordered group G in the lattice ordered group W. Finally $(s\phi)(g\phi)=(sg)\phi$ for all $s \in S$ and $g \in G$.*

Proof. Theorem 12, theorem 15, theorem 16, and theorem 17 of [5].

THEOREM 3.2. *If (G, S) is nonoverlapping and G is transitive on S, then G can be embedded as an* 1-*subgroup of an ordered wreath product of subgroups of the real numbers (permuting themselves in the right regular representation).*

Proof. It is enough by the preceding theorem to show that each o-primitive component of (G, S) is o-isomorphic with a subgroup of the real numbers, permuting itself in the right regular representation.

Fix $o \in S$ and suppose C is a convex G-congruence on S. Let $H_\gamma=\{g \mid (oC_\gamma)g\subseteq oC_\gamma\}$. Then H_γ is a convex 1-subgroup of G. Conversely each convex 1-subgroup H of G defines a convex G-congruence r on S by srt if and only if s belongs to the convexification of the orbit of t under H.

Thus each covering pair (C_γ, C^γ) of convex G-congruences on S determines a covering pair of convex 1-subgroups (H_γ, H^γ) of G. Let $s \in oC^\gamma\backslash oC_\gamma$. Since G is transitive on S, there exists $g \in G$ with $og=s$. Then $g \in H^\gamma\backslash H_\gamma$, and so H_γ is regular missing g, and hence normal in H^γ. (Theorem 2.8.) Since H^γ/H_γ is totally ordered and (H_γ, H^γ) is a covering pair, it follows that H^γ/H_γ is o-isomorphic to a subgroup of the real numbers. Hence the o-primitive component (G_γ, S_γ) constructed from (C_γ, C^γ) is 1-isomorphic with $(H^\gamma/H_\gamma, H^\gamma/H_\gamma)$. That is, each o-primitive component of (G, S) is o-isomorphic with a subgroup of the real numbers.

THEOREM 3.3. *If (W, B) is the ordered wreath product $* \prod_{\gamma\in r} (G_\gamma, S_\gamma)$ of right regular representations of subgroups of the real numbers, then (W, B) is nonoverlapping.*

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Proof. As in [5], we may view each point r of B as a function from Γ into the reals R, such that for a fixed point in B, say 0 where $0(\gamma)=0$, $\{\gamma \mid r(\gamma)\neq 0(\gamma)=0\}$ is inversely well ordered. We may also represent each $r \in B$ as a Γ-tuple $(\ldots, r(\gamma), \ldots)$. For $g \in W$ the Γ-tuple rg is then $(\ldots, r(\gamma)+\bar{g}_{\gamma,r}, \ldots)$ where $\bar{g}_{\gamma,r}$ is the real number by which the function $g_{\gamma,r}$ translates the points of G_γ.

Let $h \in W$ and suppose α is the largest element of the set $\{\gamma \mid r(\gamma)\neq(rh)(\gamma)\}$.

Claim. $\{s \mid s(\gamma)=r(\gamma)$ for $\gamma>\alpha\}=h_r$ (the interval of r by h).

Proof. Choose $s \in B$ so that $s(\gamma)=r(\gamma)$ for $\gamma>\alpha$. Let d denote the real number $|r(\gamma)-rh(\gamma)|$. Then for some $n \in Z^+$, $r(\alpha)-nd<s(\alpha)<r(\alpha+nd)$ so that s is between rh^{-n} and rh^n, and hence $s \in h_r$ which proves one inclusion.

To prove the other inclusion, suppose $t \in B$ and $\beta>\alpha$ where β is the largest element of $\{\gamma \mid t(\gamma)\neq r(\gamma)\}$. By the definition of α, $rh(\gamma)=r(\gamma)$ for $\gamma>\alpha$. By the action of elements of G on points of B, it follows that $rh^n(\gamma)=r(\gamma)$ for $\gamma>\alpha$. If $r(\beta)<t(\beta)$, then $rh^n(\beta)=r(\beta)<t(\beta)$, and $rh^n(\gamma)=r(\gamma)=t(\gamma)$ for $\gamma>\beta$. Thus $rh^n<t$ for all $n \in Z$ and hence $t \notin h_r$. In a similar fashion, if $r(\beta)>t(\beta)$, then $rh^n>t$ for all $n \in Z$, so that again, $t \notin h^r$. This proves the claim.

To see that (W, B) is nonoverlapping, suppose that $h_s \cap g_v \neq \varnothing$ for $\{s, v\}\subseteq B$ and $\{h, g\} \subseteq W^+$. Fix $t \in h_s \cap g_v$. If α is the largest element of $\{\gamma \mid t(\gamma)\neq th(\gamma)\}$ and β is the largest element of $\{\gamma \mid t(\gamma)\neq tg(\gamma)\}$, then by the claim proved above,

$$h_s = h_t = \{s \in B \mid s(\gamma) = t(\gamma) \text{ for } \gamma > \alpha\}$$

$$g_v = g_t = \{s \in B \mid s(\gamma) = t(\gamma) \text{ for } \gamma > \beta\}$$

Clearly $h_t \cap g_t \in \{\phi, h_t, g_t\}$, that is, $h_s \cap g_v \in \{\phi, h_s, g_v\}$, so (W, B) is nonoverlapping.

4. **Nonoverlapping lattice ordered groups and wreath products of reals.** A proof of the following theorem requires a straightforward vertification of certain conditions. As the process is long, however, the proof will be omitted. A proof for this theorem is given in section 5 of [6].

THEOREM 4.1. *Suppose that (G, S) is an 1-group (i.e. G is an 1-subgroup of some $A(S)$), A is a chain, and Γ_α is a chain for each $\alpha \in A$. Let $(G, S)_{\gamma_\alpha}=(G, S)$ and let $(W_\alpha, C_\alpha)= * \prod_{\gamma_\alpha\in\Gamma_\alpha} (G, S)_{\gamma_\alpha}$, the ordered wreath product. There exists a chain B, and an injective lattice homomorphism from $\sum_{\alpha\in A} \{* \prod_{\gamma_\alpha\in\Gamma_\alpha} (G, S)_{\gamma_\alpha}\}$ into $* \prod_{\beta\in B} (G, S)_\beta$. Here $(G, S)_\beta=(G, S)$.*

THEOREM 4.2. *The 1-group G is nonoverlapping if and only if G is 1-isomorphic with an 1-subgroup of a wreath product of the reals.*

Proof. By Theorem 3.3, a wreath product of the reals is nonoverlapping. Clearly an 1-subgroup of a nonoverlapping 1-group is nonoverlapping.

For the converse, suppose that G is nonoverlapping. The 1-group G can be embedded 1-isomorphically as a subcartesian sum of transitive 1-subgroups of o-preserving permutations of chains S_α, for α belonging to some index set A. (This is the embedding theorem of W. C. Holland [4].) Also since G restricted to each S_α is a homomorphic image of G, then $(G \mid S_\alpha, S_\alpha)$ is nonoverlapping. (Use the condition $hg \leq g^2h^2$ directly, or note that this equation defines a variety.) Thus by Theorem 3.2, G is 1-isomorphic with an 1-subgroup of

$$\sum_{\alpha \in A} \left\{ * \prod_{\gamma\alpha \in \Gamma_\alpha} (R_{\gamma_\alpha}, R_{\gamma_\alpha}) \right\}$$

where for each α in the chain A, the set Γ_α is a chain, and each $(R_{\gamma_\alpha}, R_{\gamma_\alpha})$ is a subgroup of the reals permutting itself in the right regular representation. By Theorem 4.1 it follows that G can be embedded by an injective lattice homomorphism in

$$* \prod_{\beta \in B} (R, R)_\beta \text{ for some chain } B.$$

In [7], Wolfenstein defines an *s-group* G as an 1-group for which the special values form a plenary subset of the set of all values of G. (See definition 2.6.) He has shown in [7] that each 1-subgroup of an s-group is nonoverlapping. Using the notation of [5], we show the converse.

THEOREM 4.3. *If the* 1*-group G is nonoverlapping, then G can be embedded as an* 1*-subgroup of an s-group.*

Proof. It is enough to show that $(W, S) = * \prod_{\beta \in B} (R, R)_\beta$, a wreath product of the reals, is an s-group.

Fix $s_0 \in S$ and $\beta_0 \in B$, and define $g \in W$ as

$$g_{\beta,s} = \begin{cases} 1 & \text{if } \beta = \beta_0, \quad s = s_0 \\ 0 & \text{otherwise} \end{cases}$$

Define

$$G_{\beta_0,s_0} = \{h \in W \mid h_{\beta_0,s_0} = 0 \text{ for } \beta \geq \beta_0\}$$

Then G_{β_0,s_0} is a special value of g, and by varying β_0 and s_0 it follows easily that the special values of (W, S) thus defined form a plenary subset of the set of all values of (W, S). We summarize the preceding results in the following.

THEOREM 4.4. *Let G be a lattice ordered group. The following are equivalent:*

(1) *G is nonoverlapping*

(2) *for each pair* $\{h, g\} \subseteq G^+$, $hg \leq g^2h^2$

(3) $G \in NP$

(4) *G is normal valued*

(5) *G can be embedded as an* 1*-subgroup of a wreath product of the reals.*

(6) *G can be embedded as an* 1*-subgroup of an s-group.*

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References

1. R. D. Byrd, *Archimidean Closures in Lattice-Ordered Groups*, Canadian J. Math. **21** (1969). 1004–1012.

2. —— and J. T. Lloyd, *Closed Subgroups and Complete Distributivity in Lattice-Ordered Groups*, Math. Zeitschr, **101** (1967), 123–130.

3. L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, New York, 1963.

4. W. Charles Holland, *The Lattice-Ordered Groups of Auotmorphisms of an Ordered Set*, Michigan Math. J. **10** (1963), 399–408.

5. —— and S. H. McCleary, *Wreath Products of Ordered Permutation Groups*, Pacific J. Math. **31.** (1969), 703–716.

6. J. A. Read, *Nonoverlapping Lattice-Ordered Groups*, Ph.D. thesis, The University of Wisconsin, Madison, Wisconsin, 1971.

7. S. Wolfenstein, *Valeurs normales dans un groupe réticulé*, Accademia Nazionale dei Lincei, **44,** (1968), 337–342.

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