A GENERALIZATION OF WYTHOFF'S GAME*

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1. *Wythoff's game.* W.A. Wythoff [1] in 1907 defined a modification of the game of Nim by the following rules:

(i) there are two players who play alternately;

(ii) initially there are two piles of matches, an arbitrary number in each pile;

(iii) a player may take an arbitrary number of matches from one pile or an equal number from both piles but he must take at least one match;

(iv) the player who takes the last match wins the game.

If, after his move, a player leaves one match in one pile and two in the other he can force a win; for if his opponent takes one match from the pile containing two he can take both remaining matches; and similarly for the other possibilities. Thus (1,2) is called a winning pair and in Table I each pair (u_n, v_n) is a winning pair in the following sense:

(a) in a single move a player can change a non-winning pair into a winning pair;

(b) any move will change a winning pair into a non-winning pair. (These statements are special cases of Theorem 2 below).

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If a player's move results in a winning pair (u_n, v_n) his opponent's move will result in a non-winning pair which he can transform into a winning pair (u_n', v_n') on his next move. Thus he can force a win, for ultimately his move will result in the pair (0, 0) which means he took the last match.

n	u_n	v_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26

Table I

The table was constructed by the rules:

(1) $u_o = 0$;

(2) $v_n = u_n + n$;

(3) u_n is the least positive integer distinct from the 2n integers $u_o, u_1, \ldots, u_{n-1}, v_o, v_1, \ldots, v_{n-1}$. Thus each positive integer occurs just once in the sequences

$$\{u_n\} \quad , \quad \{v_n\} \quad , n = 1, 2, 3, \ldots ,$$

and for this reason $\{u_n\}$ and $\{v_n\}$ are called complementary sequences.

Explicit formulas for u_n and v_n are

(1) $$u_n = \left[\tfrac{1}{2}n(1 + \sqrt{5})\right] ,$$

(2) $$v_n = \left[\tfrac{1}{2}n(3 + \sqrt{5})\right] ,$$

where square brackets denote the integral part function. (These formulas are special cases of Theorem 1.)

2. <u>The generalized game</u>. Define the k-Wythoff game by the rules:

(i) there are two players who play alternately;

(ii) initially there are two piles of matches, an arbitrary number in each pile;

(iii) a player may take an equal number of matches from both piles or a multiple of k matches from one pile but he must take at least one match;

(iv) the player who cannot make a legitimate move loses the game.

The original Wythoff game corresponds to $k = 1$.

The set of all winning pairs is characterized by the following three properties:

(a) an arbitrary pair is in the set or can be reduced to a pair in the set in just one move;

(b) a move on a pair in the set produces a pair not in the set;

(c) all pairs on which no legitimate move can be made are in the set.

As we saw in the case $k = 1$, if a player's move creates a winning pair his opponent's move will create a losing pair which he in turn can reduce to a winning pair. By repeated application of this strategy he will ultimately win.

Define the following k pairs of sequences:

(3) $$u_{i,n} = [(n + i/k)(1 + 1/A)] ,$$

(4) $$v_{i,n} = [(n + i/k)(1 + A)] ,$$

for $i = 0, 1, 2, \ldots, k-1$, and $n = 0, 1, 2, 3, \ldots,$ where

(5) $$A = \tfrac{1}{2}(k + \sqrt{k^2 + 4}) .$$

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For $k = 1, 2, 3, \ldots$ A is irrational since the continued fraction expansion is infinite:

$$A = k + \frac{1}{k+}\ \frac{1}{k+}\ \cdots .$$

However we can show this directly. For suppose $k^2 + 4 = a^2$. If k is odd then $k^2 \equiv 1 \bmod 8$ and $a^2 \equiv 5 \bmod 8$, which is impossible. If $k = 2k_1$ then $a = 2a_1$ and $k_1{}^2 + 1 = a_1{}^2$, which is also impossible. Hence $k^2 + 4$ cannot be a perfect square and A is irrational.

Now $u_{i,o} = 0$ since

$$u_{i,o} \leqslant [(1 + 1/A)(k-1)/k],$$

and we must show that

$$(1 + 1/A)(k-1)/k < 1,$$

i.e., $(k-1)(1 + A) < kA$, or $k-1 < A$, and squaring, this is $k^2 - 2k + 1 < k^2 + 4$, which is true for $k = 1, 2, 3, \ldots$.

Since

$$1 + A = 1 + A^{-1} + k,$$

(6) $$v_{i,n} = u_{i,n} + nk + i .$$

Thus $v_{i,o} = i$.

Defining $v_{k,n-1} = v_{o,n}$ we have,

THEOREM 1. The sequences $\{u_{i,n}\}$ and $\{v_{k-i,n-1}\}$ for $n = 1, 2, 3, \ldots$ and for each value of $i = 0, 1, 2, \ldots, k-1$, are complementary.

Proof. By (4)

(7) $$v_{k-i,n-1} = [(n - i/k)(1 + A)] .$$

Now $u_{i,n} \geqslant 1$ for $n \geqslant 1$ and $i = 0, 1, \ldots, k-1$. By (6) $v_{k-i,n-1} \geqslant 1$ for $n \geqslant 1$ and $i = 1, 2, \ldots, k-1$. Also $v_{k,n-1} = v_{o,n} \geqslant 1$ for $n \geqslant 1$. Thus for a given i the two sequences consist of positive integers.

Suppose the first N positive integers contain r members of $\{u_{i,n}\}$ and s members of $\{v_{k-i,n-1}\}$. Then by (3) and (7),

$$(r + i/k)(1 + 1/A) < N + 1 < (r + 1 + i/k)(1 + 1/A),$$

or $$(r + i/k)(1 + A) < (N + 1)A < (r + 1 + i/k)(1 + A),$$

and $$(s - i/k)(1 + A) < N + 1 < (s + 1 - i/k)(1 + A).$$

(Strict inequalities since A is irrational.) Adding the last two lines and dividing the result by $1 + A > 0$ we get

$$r + s < N + 1 < r + s + 2$$

or, since all these quantities are integers,

$$r + s = N .$$

Thus the first N integers contain N members of the two sequences. Hence the first N + 1 integers contain N + 1 members and this (N + 1)st member must be the integer N + 1. Therefore each integer occurs precisely once and the sequences are complementary.

THEOREM 2. The set of winning pairs for the k-Wythoff game comprises the k sets

(8) $$\{(u_{i,n}, v_{i,n})\} \qquad , n = 0, 1, 2, 3, \ldots ,$$

for $i = 0, 1, 2, \ldots, k-1$.

Proof. We deal separately with the three properties which characterize the set of winning pairs.

Proof of property (a): If (x, y) is not a pair of type (8) we must show that it can be reduced to such a pair by a single move. Clearly the order of the piles is immaterial and we can assume $x \leqslant y$. Let $y - x \equiv i \bmod k$ so that

(9) $$y = x + i + rk,$$

where $r \geqslant 0$ and $0 \leqslant i \leqslant k-1$.

By Theorem 1 either:

(i) $x = 0$;

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(ii) $x = u_{i,n}$, $n \geqslant 1$; or

(iii) $x = v_{k-i,n-1}$, $n \geqslant 1$.

(i) In this case

$$x = u_{i,o} = 0 \ , \ y = i + rk.$$

If $r = 0$ the pair (x, y) is the pair $(u_{i,o}, v_{i,o})$ which is of type (8). If $r > 0$ the pile y can be reduced by rk matches to give this pair.

(ii) We distinguish three cases:

(α) $y > v_{i,n}$;

(β) $y = v_{i,n}$;

(γ) $y < v_{i,n}$.

(α) $y = x + i + rk = u_{i,n} + i + rk = v_{i,n} + (r-n)k$. Thus the pile y can be reduced by $r-n > 0$ multiples of k to yield the pair $(u_{i,n}, v_{i,n})$.

(β) The pair (x, y) is already of type (8).
(γ) In this case $r < n$. Take s matches from each pile so that $x-s = u_{i,n}-s = u_{i,r}$. Then $y-s = x + i + rk-s = u_{i,r} + i + rk = v_{i,r}$, and the pair (x, y) has been reduced to $(u_{i,r}, v_{i,r})$ where $r \geqslant 0$.

(iii) Now

$$y \geqslant x = v_{k-i,n-1} > u_{k-i,n-1} ,$$

so that

$$y = x + i + rk = v_{k-i,n-1} + i + rk$$

$$= u_{k-i,n-1} + k-i + (n-1)k + i + rk$$

$$= u_{k-i,n-1} + (n + r)k ,$$

and the move is obvious.

Proof of property (b): We must show that a move on a pair of type (8) produces a pair not in the set. Now

$$v_{i,n} \equiv u_{i,n} + i \bmod k$$

and this congruence is preserved by any move, since

$$v_{i,n}-s \equiv u_{i,n} + i-s \bmod k ,$$

$$v_{i,n}-rk \equiv u_{i,n} + i \bmod k ,$$

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and $$v_{i,n} \equiv u_{i,n} + i - rk \mod k .$$

The only pairs in the set (8) which satisfy this congruence are pairs of the types $(u_{i,n'}, v_{i,n'})$ and $(u_{k-i,n'}, v_{k-i,n'})$, the latter because

$$v_{k-i,n'} \equiv u_{k-i,n'} + k - i \mod k ,$$

i.e., $$u_{k-i,n'} \equiv v_{k-i,n'} + i \mod k .$$

Suppose a move on the pair $(u_{i,n}, v_{i,n})$ gives the pair $(u_{i,n'}, v_{i,n'})$. Then $n' < n$, $u_{i,n'} < u_{i,n}$ and $v_{i,n'} < v_{i,n}$ so that the move must have removed s matches from each pile. By (6)

$$v_{i,n} - s = u_{i,n} - s + nk + i ,$$

i.e., $$v_{i,n'} = u_{i,n'} + nk + i ,$$

which is impossible by (6).

On the other hand suppose the move yields the pair $(u_{k-i,n'}, v_{k-i,n'})$. Since $u_{i,n} < v_{i,n}$ for all i and n (except $u_{o,o} = v_{o,o} = 0$), the move must have removed a multiple of k matches from the pile $v_{i,n}$ so that

$$u_{i,n} = v_{k-i,n'}$$

and $$v_{i,n} - rk = u_{k-i,n'} .$$

But the former equation violates the complementarity property of the sequences $\{u_{i,n}\}$ and $\{v_{k-i,n'}\}$.

Thus no move on a pair $(u_{i,n}, v_{i,n})$ will produce a pair in the set (8).

Proof of property (c): All the winning pairs on which no legitimate move can be made are clearly

$$(0,0) , (0,1) , \ldots , (0,k-1) ;$$

that is, $(u_{o,o}, v_{o,o})$, $(u_{1,o}, v_{1,o})$, ... , $(u_{k-1,o}, v_{k-1,o})$, and these are included in the set (8).

This completes the proof of Theorem 2.

Following are short tables of winning pairs for the first few k-Wythoff games. They were constructed with the use of Theorem 1, thus obviating the direct evaluation of (3) and (4).

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k = 2:

n	$u_{o,n}$	$v_{o,n}$	$u_{1,n}$	$v_{1,n}$
0	0	0	0	1
1	1	3	2	5
2	2	6	3	8
3	4	10	4	11
4	5	13	6	15
5	7	17	7	18
6	8	20	9	22
7	9	23	10	25
8	11	27	12	29
9	12	30	13	32
10	14	34	14	35
11	15	37	16	39
12	16	40	17	42

k = 3:

n	$u_{o,n}$	$v_{o,n}$	$u_{1,n}$	$v_{1,n}$	$u_{2,n}$	$v_{2,n}$
0	0	0	0	1	0	2
1	1	4	1	5	2	7
2	2	8	3	10	3	11
3	3	12	4	14	4	15
4	5	17	5	18	6	20
5	6	21	6	22	7	24
6	7	25	8	27	8	28
7	9	30	9	31	9	32
8	10	34	10	35	11	37
9	11	38	12	40	12	41
10	13	43	13	44	13	45
11	14	47	14	48	15	50
12	15	51	16	53	16	54

k = 4:

n	$u_{o,n}$	$v_{o,n}$	$u_{1,n}$	$v_{1,n}$	$u_{2,n}$	$v_{2,n}$	$u_{3,n}$	$v_{3,n}$
0	0	0	0	1	0	2	0	3
1	1	5	1	6	1	7	2	9
2	2	10	2	11	3	13	3	14
3	3	15	4	17	4	18	4	19
4	4	20	5	22	5	23	5	24
5	6	26	6	27	6	28	7	30
6	7	31	7	32	8	34	8	35
7	8	36	8	37	9	39	9	40
8	9	41	10	43	10	44	10	45
9	11	47	11	48	11	49	12	51
10	12	52	12	53	12	54	13	56
11	13	57	13	58	14	60	14	61
12	14	62	15	64	15	65	15	66

It will be observed from the remarks in part (b) of the previous proof that once i has been determined by the initial piles a player need only know two sequences of winning pairs, corresponding to i and k-i. (He need only know one sequence if i = 0 or i = k/2 and k is even). For example, if k = 3 and subscripts W and L denote winning and losing pairs, possible sequences of moves are:

$$(14,48)_W \rightarrow (11,48)_L \rightarrow (11,3)_W, \text{ i.e., } (3,11)_W ;$$
$$(14,48)_W \rightarrow (14,27)_L \rightarrow (8,27)_W .$$

3. <u>Concluding</u> <u>remarks</u>. Setting i = 0 in Theorem 1, $\{u_{o,n}\}$ and $\{v_{k,n-1}\}$ are complementary. That is

$$[n(1 + 1/A)] \qquad \text{and} \qquad [n(1 + A)]$$

are complementary for n = 1,2,3, ... This is true for any irrational A > 0, as was first observed by S. Beatty [2], and the proof of Theorem 1 may be used without modification.

A theorem similar to Theorem 1 has been proved by T. Skolem [3].

Denoting the general complementary sequences of Beatty by

$$u_n = [n(1 + 1/\alpha)] \quad \text{and} \quad v_n = [n(1 + \alpha)],$$

the values (5) of α belong to a certain class of quadratic surds for which

$$u_{v_n} = u_n + v_n .$$

We hope to publish elsewhere an account of this and other results connected with Beatty sequences.

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SOME PROPERTIES OF BEATTY SEQUENCES I*

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1. Introduction. Two sequences of natural numbers are said to be complementary if they contain all the positive integers without repetition or omission. S. Beatty [1] observed that the sequences

(1) $$u_n = [n(1 + 1/\alpha)] , \quad n = 1, 2, 3, \ldots ,$$

(2) $$v_n = [n(1 + \alpha)] , \quad n = 1, 2, 3, \ldots ,$$

(where square brackets denote the integral part function) are complementary if and only if $\alpha > 0$ and α is irrational. We call the pair (1), (2) Beatty sequences of argument α .

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