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TRANSLATION INVARIANT LINEAR FUNCTIONALS ON SEGAL ALGEBRAS

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Let $S(G)$ be a Segal algebra on an infinite compact Abelian group G. We study the existence of many discontinuous translation invariant linear functionals on $S(G)$. It is shown that if G/C_G contains no finitely generated dense subgroups, then the dimension of the linear space of all translation invariant linear functionals on $S(G)$ is greater than or equal to 2^c and there exist 2^c discontinuous translation invariant linear functionals on $S(G)$, where c and C_G denote the cardinal number of the continuum and the connected component of the identity in G, respectively.

Throughout this note G will denote an infinite compact Abelian group with the normalised Haar measure λ_G, and $L^p(G)$ $(1 \leqslant p \leqslant \infty)$ will denote the Lebesgue space with respect to λ_G. The space of all continuous functions on G will be denoted by $C(G)$. We shall also use the symbols c and C_G to denote the cardinal number of the continuum and the connected component of the identity in G, respectively.

Roelcke, Asam, S.Dierolf and P. Dierolf [9, Theorem 4] proved that if G is a torsion group, then the dimension of the linear space of all translation invariant linear functionals on $C(G)$ is greater than or equal to 2^c. This result in particular implies that $C(G)$ admits 2^c discontinuous translation invariant linear functionals for any infinite compact Abelian torsion group G. The existence of discontinuous translation invariant linear functionals on $L^2(G)$ was studied by Meisters [6]. Recall that a compact Abelian group is called polythetic if it contains a finitely generated dense subgroup (see [2, 6]). Meisters, together with Larry Baggett, proved that $L^2(G)$ has discontinuous translation invariant linear functionals provided that G/C_G is not polythetic [6, Corollary to Theorem 6]. The purpose of this note is to indicate how the methods in [9] may be improved to establish a theorem which strengthens and generalises the above two results.

For a function f on G and $a \in G$, we define the a-translate $\tau(a)f$ of f by $(\tau(a)f)(x) = f(x-a)$ $(x \in G)$. Recall that, by definition, a Segal algebra on G is a dense subalgebra $S(G)$ of the convolution algebra $L^1(G)$ such that

(i) $S(G)$ is a Banach algebra under some norm $\|\cdot\|_S$ and $\|f\|_S \geqslant \|f\|_{L^1}$ for all $f \in S(G)$;

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(ii) $S(G)$ is translation invariant (that is, $\tau(a)f \in S(G)$ for all $f \in S(G)$ and all $a \in G$) and for each $f \in S(G)$ the mapping $a \to \tau(a)f$ of G into $S(G)$ is continuous;

(iii) $\|\tau(a)f\|_S = \|f\|_S$ for all $f \in S(G)$ and all $a \in G$.

(For fundamental results on Segal algebras, we refer to [**7**, **8**, **12**].) We say that a linear functional Φ on a Segal algebra $S(G)$ is translation invariant if $\Phi(\tau(a)f) = \Phi(f)$ for all $f \in S(G)$ and all $a \in G$. In this note we shall be concerned with translation invariant linear functionals on Segal algebras on G. Henceforth we shall use the abbreviation TILF for "translation invariant linear functional" and denote by TILF $(S(G))$ the linear space of all TILF's on $S(G)$.

Let us now state our theorem.

Theorem. *Let G be a compact Abelian group and let $S(G)$ be a Segal algebra on G. If G/C_G is not polythetic, then the dimension of the linear space* TILF $(S(G))$ *is greater than or equal to 2^c and there exist 2^c discontinuous TILF's on $S(G)$.*

To prove our Theorem, we require some preliminary notation and lemmas. $\widehat{G}$ will denote the (discrete) dual group of a compact Abelian group. (We use 1_G to denote the trivial character of G.) For $f \in L^1(G)$, $\widehat{f}$ denotes the Fourier transform of f. For a Segal algebra $S(G)$, we denote by $\triangle(S(G))$ and $S(G)_0$ the linear subspace of $S(G)$ generated by $\{f - \tau(a)f : f \in S(G), a \in G\}$ and the closed linear subspace $\{f \in S(G) : \widehat{f}(1_G) = 0\}$ of $S(G)$, respectively. Then it is clear that $S(G)_0$ contains $\triangle(S(G))$.

Lemma 1. *Let G be a compact Abelian group and let $S(G)$ be a Segal algebra on G. Then the closure $\overline{\triangle(S(G))}$ in $S(G)$ equals $S(G)_0$ and every continuous TILF on $S(G)$ is a scalar multiple of the Haar integral.*

Proof: For a subset E of $L^1(G)$, we denote by $\overline{E}^{L^1}$ the closure of E in the L^1-norm. Since $\overline{\triangle(L^1(G))}^{L^1} = L^1(G)_0$ ([4, Lemma 1.1]) and $S(G)$ is dense in $L^1(G)$, we have

$$\overline{\triangle(S(G))}^{L^1} = \overline{\triangle(L^1(G))}^{L^1} = L^1(G)_0.$$

Notice that $\overline{\triangle(S(G))}$ is a closed ideal of $S(G)$. Thus it follows from [**12**, Theroem 4.3] that

$$\overline{\triangle(S(G))} = \overline{\overline{\triangle(S(G))}}^{L^1} \cap S(G).$$

Hence we have

$$\begin{aligned} S(G)_0 = L^1(G)_0 \cap S(G) &= \overline{\triangle(S(G))}^{L^1} \cap S(G) \\ &\subseteqq \overline{\overline{\triangle(S(G))}}^{L^1} \cap S(G) = \overline{\triangle(S(G))}. \end{aligned}$$

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Since the converse inclusion relation is clear, we conclude that $\overline{\Delta(S(G))} = S(G)_0$. Let Φ be a continuous TILF on S(G). Then, of course, $\Phi \equiv 0$ on $\Delta(S(G))$ and hence on $\overline{\Delta(S(G))}$. Since $\overline{\Delta(S(G))} = S(G)_0$ and $S(G)_0$ has codimension one, either Φ is identically zero or the kernel of Φ coincides with $S(G)_0$. In either case Φ is a scalar multiple of the Haar integral. This completes the proof. ☐

LEMMA 2. *Let G be an infinite metrisable compact Abelian group and let $S(G)$ be a Segal algebra. Then there exists a family $\{h_r\}_{r>1}$ (indexed by real numbers r with $r > 1$) of functions in $S(G)$ with the following properties:*

(i) *$\widehat{h}_r(1_G) = 0$ for every $r > 1$,*

(ii) *$\{\gamma \in \widehat{G} : \widehat{h}(\gamma) = 0\}$ is finite for every nonzero function h in the linear space generated by $\{h_r\}_{r>1}$.*

PROOF: Since $\widehat{G}$ is countably infinite, we denote $\widehat{G}$ by $\{\gamma_0 = 1_G, \gamma_1, \gamma_2, \ldots, \gamma_n, \ldots\}$. For each $r > 1$, we define a function h_r in $S(G)$ by

$$h_r = \sum_{n=1}^{\infty} n^{-r} \|\gamma_n\|_S^{-1} \gamma_n.$$

See, for example, Theorem 4.2 of [**12**]. (Note that the series of the right side converges in $S(G)$.) It is easy to see that $\widehat{h}_r(1_G) = 0$ and $\widehat{h}_r(\gamma_n) = n^{-r} \|\gamma_n\|_S^{-1}$ for all $n \geqslant 1$. Thus (i) holds. To see (ii), let $h = \sum_{j=1}^{m} c_j h_{r_j}$ be a nonzero function in the linear space generated by $\{h_r\}_{r>1}$, where c_j $(1 \leqslant j \leqslant m)$ is a nonzero complex number and $1 < r_1 < r_2 < \ldots < r_m$. Since

$$\begin{aligned}
\left|\widehat{h}(\gamma_n)\right| &= \left|\sum_{j=1}^{m} c_j \widehat{h}_{r_j}(\gamma_n)\right| \\
&= \left|\sum_{j=1}^{m} c_j n^{-r_j} \|\gamma_n\|_S^{-1}\right| \\
&= n^{-r_1} \|\gamma_n\|_S^{-1} \left|\sum_{j=1}^{m} c_j n^{r_1 - r_j}\right| \\
&\geqslant n^{-r_1} \|\gamma_n\|_S^{-1} \left(|c_1| - \sum_{j=2}^{m} |c_j| \, n^{r_1 - r_j}\right)
\end{aligned}$$

for all $n \geqslant 1$, we have $\widehat{h}(\gamma_n) \neq 0$ for all sufficiently large positive integers n and hence (ii) holds. This completes the proof. ☐

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Let us now turn to the proof of the Theorem. We shall show that the dimension of the linear space $S(G)_0/\Delta(S(G))$ is greater than or equal to c. This immediately implies that

$$\dim \mathrm{TILF}(S(G)) \geqslant 2^c.$$

Since the linear space of all continuous TILF's on $S(G)$ has dimension one by Lemma 1, we also obtain that there exist 2^c discontinuous TILF's on $S(G)$.

We first consider the case where G is metrisable and not polythetic. Let $\{h_r\}_{r>1}$ be a family of functions in $S(G)$ as in Lemma 2 and let X denote the linear subspace of $S(G)$ generated by $\{h_r\}_{r>1}$. Then, by Lemma 2 (i), X is included in $S(G)_0$. We also have

$$X \cap \Delta(S(G)) = \{0\}.$$

To see this, suppose that there exist $f_1, f_2, \ldots, f_n \in S(G)$ and $a_1, a_2, \ldots, a_n \in G$ such that

$$f = \sum_{j=1}^{n} (f_j - \tau(a_j) f_j)$$

is nonzero and is contained in X. Then, by Lemma 2 (ii), there exist only finitely many $\gamma_1, \gamma_2, \ldots, \gamma_m \in \widehat{G} \setminus \{1_G\}$ such that $\widehat{f}(\gamma_k) = 0$ for $k = 1, 2, \ldots, m$. Choose $b_1, b_2, \ldots, b_m \in G$ such that $\gamma_k(b_k) \neq 1$ for $k = 1, 2, \ldots, m$ and denote by H the closed subgroup of G generated by $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$. (If $\{\gamma \in \widehat{G}: \widehat{f}(\gamma) = 0\} = \{1_G\}$, then we simply consider the closed subgroup H of G generated by $\{a_1, \ldots, a_n\}$.) Since G is not polythetic, H is proper in G and hence there exists $\gamma \in \widehat{G} \setminus \{1_G\}$ such that $\gamma(x) = 1$ for all $x \in H$. Then we have

$$0 \neq \widehat{f}(\gamma) = \sum_{j=1}^{n} \left(1 - \overline{\gamma(a_j)}\right) \widehat{f}_j(\gamma) = 0.$$

But this is a contradiction, and hence $X \cap \Delta(S(G)) = \{0\}$ as desired. Thus we obtain

$$\dim S(G)_0/\Delta(S(G)) \geqslant \dim X.$$

Since $\dim X = c$ by Lemma 2 (ii), we conclude that

$$\dim S(G)_0/\Delta(S(G)) \geqslant c.$$

We next turn to the general case. Since G/C_G is not polythetic, there exists a closed subgroup H of G such that G/H is metrisable and not polythetic ([2], Lemma 5.2). Notice that we can define a bounded linear operator T_H from $L^1(G)$ onto $L^1(G/H)$ as follows:

$$T_H(f)(x + H) = \int_H f(x + \xi) d\lambda_H(\xi) \quad (f \in L^1(G), x \in G).$$

By [8, Section 13, Theorem 1], the image of $S(G)$ under T_H is a Segal algebra on G/H. Let us denote by $S(G/H)$ this Segal algebra. Then it can be easily verified that the image of $S(G)_0$ under T_H coincides with $S(G/H)_0$ and that $\Delta(S(G)) = T_H^{-1}(\Delta(S(G/H)))$. Thus $S(G)_0/\Delta(S(G))$ is linearly isomorphic with $S(G/H)_0/\Delta(S(G/H))$. Since G/H is metrisable and not polythetic, we have

$$\dim S(G)_0/\Delta(S(G)) = \dim S(G/H)_0/\Delta(S(G/H)) \geqslant c.$$

This completes the proof of the Theorem.

REMARKS. (a) If G is an infinite compact Abelian torsion group, then G is totally disconnected and not polythetic and hence G satisfies the assumption of our Theorem. Of course, there exist compact and totally disconnected Abelian groups which are neither torsion nor polythetic. For instance, the direct product $\prod_{p\in\mathcal{P}} Z(p)$ is a typical example, where $\mathcal{P}$ denotes the set of all prime numbers and $Z(p)$ is the finite cyclic group of order p.

(b) Lemma 1 also remains valid for any locally compact Abelian group. To see this, we have only to repeat the proof of Lemma 1 with a locally compact Abelian group G.

(c) It is well-known that $C(G)$ and $L^p(G)$ $(1 \leqslant p < \infty)$ are Segal algebras on G for any compact Abelian group G. Our Theorem for these Segal algebras improves and strengthens [9, Theorem 4] and [6, Corollary to Theorem 6]. For a number of examples of Segal algebras other than $C(G)$ and $L^p(G)$ $(1 \leqslant p < \infty)$, we refer to [12, Examples 4.12].

(d) If G is an infinite compact Abelian group and if G/C_G is polythetic, then there exist Segal algebras $S(G)$ on G such that every TILF on $S(G)$ is automatically continuous. Indeed, for such G's, Johnson [2, Theorem 5.2] proved that $L^2(G)_0 = \Delta(L^2(G))$ and hence every TILF on $L^2(G)$ is continuous. (For some related results, see [1, 10, 11].) On the contrary, it is shown by Saeki [11, Theorem 1*] that if G is a noncompact, σ-compact, locally compact Abelian group, then any Segal algebra on G admits uncountably many discontinuous TILF's. Our Theorem complements this result of Saeki. The question of the existence of discontinuous TILF's on some special Segal algebras is also studied in [3, 4, 5].

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