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THE DIRICHLET PROBLEM FOR THE SUBELLIPTIC LAPLACIAN ON THE HEISENBERG GROUP II

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0. Introduction. Let H_3 be the Heisenberg group in three dimensions, Δ the fundamental subelliptic laplacian on H_3 (see Section 1 for notations and definitions) and U be an open subset of H_3. If φ is a continuous function on the boundary ∂U of U, the Dirichlet problem is thus,

$$(1) \quad \begin{aligned} \Delta f &= 0 \quad \text{in } U \\ f|_{\partial U} &= \varphi \quad \text{on } \partial U. \end{aligned}$$

In [3], p. 104, it was asserted by the first author that, when ∂U is regular (see Section 1 for this definition), the problem (1) has a solution continuous on $\bar{U}$ and a probabilistic formula was given. In [3], we prove that our probabilistic formula gives a solution of the so called "martingale problem" associated to Δ on U (see [5] for this notion). But it appears that the connection between the solution in the martingale problem sense and the true solution is not at all clear in the subelliptic case; in particular it is not obvious at all that the probabilistic formula is a C^2 function. The purpose of this work is to give a complete proof of the existence of the classical solution of (1); we have to use a fairly involved argument relying heavily on the probabilistic tools of [3] and on the explicit formulas of [4]. The conclusion is that, in this case, the solution of the Dirichlet problem in the weak sense of the martingale problem is also the true solution of the Dirichlet problem in the classical sense. For more general subelliptic or degenerate elliptic operators, we do not know the relation between the "true problem" and the "martingale problem". We thank Paul Krée for having called our attention to the fact that martingale problems do not give true solutions for Dirichlet problems.

1. Notations and definitions. a) We briefly recall the notations of [3] and [4], H_3 is the Heisenberg group of dimension 3, $H_3 = \mathbf{C} \times \mathbf{R}$ with coordinates (z, t) $z = x + iy$. This is a nilpotent Lie group for the multiplicative law

$$(2) \qquad (z, t) \cdot (z', t') = (z + z', t + t' + 2 \operatorname{Im} z\bar{z}').$$

The fundamental left invariant vector fields on H_3 are

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$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$$

$$Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

$$T = \frac{\partial}{\partial t}$$

and the Heisenberg subelliptic laplacian is

$$(3) \qquad \Delta = X^2 + Y^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4y\frac{\partial^2}{\partial x \partial t} - 4x\frac{\partial^2}{\partial y \partial t} + 4|z|^2\frac{\partial^2}{\partial t^2}.$$

b) We also define the diffusion process associated to $\frac{1}{2}\Delta$ as

$$(X(s),\ Y(s),\ T(s)).$$

where $X(s)$ and $Y(s)$ are two independent brownian motions starting from 0 at $s = 0$ and

$$T(s) = 2\int_0^s (Y(s)dX(s) - X(s)dY(s))$$

is the Paul Lévy area integral.

If $g_0 = (x_0, y_0, t_0)$ is a given point in H_3, the diffusion starting at point g_0 at time $s = 0$ is exactly

$$(5) \qquad g_\omega(s) = g_0 \cdot (X(s),\ Y(s),\ T(s)).$$

2. Regular points and statement of the results. a) If U is a given open subset of H_3, and g_0 is a point in $\bar{U}$, S_U will denote the first exist time of U of $g_\omega(s)$ starting from g_0. We say that $g_0 \in \partial U$ is a *regular point* if S_U is 0 almost surely and we say that U is a *regular open subset* if all points $g_0 \in \partial U$ are regular points.

b) Let us also recall the definition of Bony [**1**]. A point $g_0 \in \partial U$ is *very regular* if essentially the plane generated by the directions X and Y is not the tangent plane to ∂U at g_0 (assuming that ∂U is a C^1 surface). A characteristic point of U is not very regular. In [**1**], Bony proves that the Dirichlet problem is well posed if all points of ∂U are very regular. Also in [**1**], the regularity of harmonic measures are proved in this case of very regular open set.

Unfortunately in our situation those results are totally useless because if U is any bounded open set such that U is topologically a sphere, then there exist at least two points of ∂U which are not very regular (by Poincaré Bendixson theory for example).

c) Define as usual $|z|^2 = x^2 + y^2$ and

$$(6) \qquad \rho(z, t) = (|z|^4 + t^2)^{1/4}.$$

The *Koranyi ball of radius* ρ_0 *and center* 0 is the set

$$B(0, \rho_0) = \{g \in H_3 \,|\rho(g) \leqq \rho_0\}.$$

The *Koranyi ball of radius* ρ_0 *and center* g_0 is

$$B(g_0, \rho_0) = g_0 \cdot B(0, \rho_0).$$

The *Koranyi sphere* is the boundary of the Koranyi ball.

It is obvious that the points $(x = 0, y = 0, t = \pm\rho_0^2)$ are characteristic points of $\partial B(0, \rho_0)$ and that the other points of $\partial B(0, \rho_0)$ are very regular. Nevertheless, it is proved in **[3]** that these two characteristic points are regular.

d) The theorem we shall prove is the following.

THEOREM. *Let U be a regular open subset of* H_3. *Then the Dirichlet problem*

$$(1) \quad \begin{cases} \Delta f = 0 & \text{in } U \\ f|_{\partial U} = \varphi & \text{on } \partial U \end{cases}$$

has a unique solution continuous on $\bar{U}$, *if* φ *is a bounded continuous function on* ∂U.

The main steps of the proof are the following:

1) We define the diffusion $g_\omega(t)$ with generator $\frac{1}{2}\Delta$ as in Section 1 and for $g_0 \in U$

$$(7) \quad f(g_0) = E(\varphi(g_\omega(S_U))\,|g_\omega(0) = g_0)$$

where S_U is the first exit time of U and $E(\ldots|g_\omega(0) = g_0)$ is the conditional expectation knowing that $g_\omega(0) = g_0$ (the diffusion starts at $s = 0$ from g_0).

2) In **[3]**, it has been proved that

$$f(g) \to \varphi(g_0) \quad \text{if } g \to g_0 \text{ and } g_0 \in \partial U,$$

so that f takes the value φ on ∂U continuously; this is due to the fact that all points of ∂U are regular points.

3) The problem is to prove that $f(g)$ is a C^2 function of g in U and satisfies $\Delta f = 0$ in the usual sense. It satisfies this equation in the weak sense of "martingale problem" but that tells nothing (as usual with generalized solutions). To see this main step we shall prove the following lemmas.

LEMMA 1. *If f is defined in U by formula* (7), *then for any* $g_0 \in U$, *any* $\rho_0 > 0$ *such that* $\overline{B(g_0, \rho_0)} \subset U$, *we have*

$$(8) \quad f(g_0) = E(f(g_\omega(S_{B(g_0,\rho_0)}))\,|g_\omega(0) = g_0)$$

where $S_{B(g_0,\rho_0)}$ *is the first exit time of the diffusion process from the Koranyi ball* $B(g_0, \rho_0)$.

LEMMA 2. *Let* $B(0, \rho_0)$ *be the Koranyi ball of center* 0 *and of radius* ρ_0. *Then the law of* $g(S_{B(0,\rho_0)})$ *knowing that* $g_\omega(0) = 0$ *is*

$$(9) \quad \mathrm{Prob}(g_\omega(S_{B(0,\rho_0)}) \in d\sigma | g_\omega(0) = 0) = \frac{1}{2\pi\rho_0^2} \frac{|z|^2 d\sigma(z, t)}{\|\nabla\rho^4\|}$$

where $d\sigma(z, t)$ *is the euclidean area element of* $\partial B(0, \rho_0)$,

$$\|\nabla\rho^4\| = (16|z|^6 + 4t^2)^{1/2}$$

is the euclidean length of the gradient of $\rho^4(z, t)$.

LEMMA 3. *Let* $B(g_0, \rho_0)$ *be the Koranyi ball of center* g_0 *and radius* ρ_0. *Let* l_{g_0} *be left translation by* g_0. *Then the law of* $g_\omega(S_{B(g_0,\rho_0)})$ *knowing that* $g_\omega(0) = g_0$ *is the image by* l_{g_0} *of the law* (9), *i.e., it is*

$$(10) \quad \theta(\rho_0, g_0, dg) = \frac{1}{2\pi\rho_0^2} |z \circ l_{g_0}^{-1}|^2 (l_{g_0})_* \left(\frac{d\sigma(z, t)}{\|\nabla\rho^4\|}\right).$$

LEMMA 4. *Let f be a bounded function in a domain D in* H_3 *such that for any* $g_0 \in D$, *any* $\rho_0 < \epsilon$ *sufficiently small, we have the mean value property*:

$$(11) \quad f(g_0) = \int_{\partial B(g_0,\rho_0)} f(g)\theta(\rho_0, g_0, dg).$$

Then f is a C^∞ *function in D and satisfies*

$$\Delta f = 0 \quad \text{in } D.$$

3. Proof of lemma 1. It is a standard application of Markov property. If $\mathscr{B}_S$ is the σ-algebra of events prior to S where S is a stopping time of the diffusion $g_\omega(s)$, then, by conditioning by $\mathscr{B}_{S_{B(g_0,\rho_0)}}$ we obtain

$$f(g_0) = E(E(\varphi(g_\omega(S_U))\,|\mathscr{B}_{S_{B(g_0,\rho_0)}})\,|g_\omega(0) = g_0).$$

But it is clear that $S_{B(g_0,\rho_0)} \leqq S_U$: so Markov property gives

$$E(\varphi(g_\omega(S_U))\,|\mathscr{B}_{S_{B(g_0,\rho_0)}}) = E(\varphi(g_\omega(S_U))\,|\,(g_\omega, (0) = g_\omega(S_{B(g_0,\rho_0)}))).$$

Remark. In case U unbounded, we have supposed that φ is bounded.

4. Identification of the laws of exit function at exit time from a Koranyi ball. a) Lemma 3 is a trivial consequence of Lemma 2 because everything is left invariant.

b) So we are reduced to the case of $B(0, \rho_0)$ and the diffusion starts at 0 at time $s = 0$. It is clear that the law of $g_\omega(S_{B(0,\rho_0)})$ will have a cylindrical symmetry around the t axis. So if $\mu(dg)$ is this law, we have only to compute

$$\int \alpha(\,|Z|,\ t)\mu(dg)$$

for $\alpha(\,|Z|,\ t)$ a continuous function on $\partial B(0,\ \rho_0)$ having cylindrical symmetry around the t axis.

c) Now, in **[4]**, we have proved that there is a solution of the Dirichlet problem

$$\begin{cases} \Delta\beta = 0 & \text{in } B(0,\ \rho_0) \\ \beta|_{\partial B(0,\rho_0)} = \alpha(\,|Z|,\ t) & \text{on } \partial B(0,\ \rho_0) \end{cases}$$

for any continuous data on $\partial B(0,\ \rho_0)$ with cylindrical symmetry. Then β is C^2 in $B(0,\ \rho_0)$ and we can apply the Ito formula to $\beta(g_\omega(s))$ for every ω and $s < S_{B(0,\rho_0)}$; we obtain by noticing that $\Delta\beta = 0$ that $\beta(g_\omega(s))$ is a martingale, and positive:

$$(12) \quad \beta(0) = E(\beta(g_\omega(S_{B(0,\rho_0)}))\ |g_\omega(0) = 0).$$

But using the formula for the Poisson-Kernel that we have explicitly obtained in **[4]** we can see on the other hand that:

$$(13) \quad \beta(0) = \frac{1}{2\pi\rho_0^2} \int_{\partial B(0,\rho_0)} \alpha(\,|z|,\ t) \frac{|z|^2 d\sigma(z,t)}{\|\nabla\rho^4\|}.$$

By comparing (12) and (13) we obtain Lemma 2.

5. Functions having the mean value property (11). a) Suppose that f is a C^2 function with the mean value property (11) and let us verify that $\Delta f = 0$. Because Δ and the mean value operator are left invariant, it is sufficient to check this at point 0. We have for any $\epsilon > 0$ small

$$(14) \quad f(0) = \frac{1}{2\pi\epsilon^2} \int_{\partial B(0,\epsilon)} f(z,\ t) \frac{|z|^2 d\sigma(z,\ t)}{\|\nabla\rho^4\|}$$

$$(15) \quad f(z,\ t) = f(0) + 1^{\text{st}} \text{ order term} + Q(f)(0) + o(\epsilon^2).$$

If we replace f inside (14) by its value (15), we see that all first order terms and second order terms with mixed derivatives will give 0 so we obtain

$$\begin{aligned} 0 &= \frac{1}{\epsilon^2}\left[\frac{1}{2\pi\epsilon^2} \int_{\partial B(0,\epsilon)} f(z,\ t) \frac{|z|^2 d\sigma(z,\ t)}{\|\nabla\rho^4\|} - f(0)\right] \\ &= \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(0) + \frac{\partial^2 f}{\partial y^2}(0)\right]\left(\frac{1}{2\pi\epsilon^4} \int_{\partial B(0,\epsilon)} \frac{x^2|z|^2 d\sigma(z,\ t)}{\|\nabla\rho^4\|}\right) \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(0)\right)\left(\frac{1}{2\pi\epsilon^4} \int_{\partial B(0,\epsilon)} t^2|z|^2 \frac{d\sigma(z,\ t)}{\|\nabla\rho^4\|}\right) + O(\epsilon). \end{aligned}$$

But it is seen by rescaling that the integral with t^2 is $O(\epsilon^2)$ and the integral with x^2 is $O(1)$ so we obtain

$$(\Delta f)(0) = \frac{\partial^2 f}{\partial x^2}(0) + \frac{\partial^2 f}{\partial y^2}(0) = 0.$$

b) To prove the regularity of f having mean value property (11), we proceed as in [**2**] Chapter IV, (2$^{\text{nd}}$ proof of the converse of the mean value property). Take D_α to be the set of points g_0 of D such that

$$\overline{B(g_0, \rho)} \subset D \quad \text{for } \rho \leqq \alpha.$$

We take a smooth function $\chi(\rho^4)$ which is 0 for $\rho^4 > \alpha^4$, positive and multiply (11) for fixed g_0 by

$$\chi(\rho(g_0^{-1}g)^4) d(\rho(g_0^{-1}g))^4$$

and integrate in ρ^4 from 0 to $+\infty$.

But we have

$$dxdydt = d(\rho^4(g)) \wedge \frac{d\sigma(g)}{\|\nabla\rho^4\|}.$$

Because $dxdydt$ is invariant by l_{g_0}

$$dxdydt = d(\rho^4(g_0^{-1}g)) \wedge (l_{g_0})_* \left(\frac{d\sigma(g)}{\|\nabla\rho^4\|}\right).$$

So we obtain

$$Cf(g_0) = \int_{H_3} \chi(\rho^4(g_0^{-1}g)) \,|z \circ l_{g_0^{-1}}(g)|^2 f(g) dxdydt$$

where C is a constant.

But the kernel in the right hand side is C^∞ of g_0 and so $f(g_0)$ is also C^∞.

6. End of the proof of the theorem. We define f in U by formula (7). Lemma 1 tells us that f satisfies the mean value property with respect to the exist measure of the diffusion process $g_\omega(t)$ from Koranyi balls. Lemmas 2 and 3 identify these exit measures to the measures $\theta(\rho_0, g_0, d_g)$. So f has the mean value property (11). By Lemma 4, f is C^∞ and satisfies the equation $\Delta f = 0$ in U.

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