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SOME EXTREME RAYS OF THE POSITIVE PLURIHARMONIC FUNCTIONS

FRANK FORELLI

1. Introduction.

1.1. We will denote by $\mathbf{B}$ the open unit ball in $\mathbf{C}^n$, and we will denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$. Let

$$N(\mathbf{B}) = \{g : g \in H(\mathbf{B}), \operatorname{Re} g > 0, g(0) = 1\}.$$

Thus $N(\mathbf{B})$ is convex (and compact in the compact open topology). We think that the structure of $N(\mathbf{B})$ is of interest and importance. Thus we proved in [**1**] that if

$$(1.1) \qquad f(z) = \sum_1^n z_j^2,$$

if

$$(1.2) \qquad g = (1 + f)/(1 - f),$$

and if $n \geqq 2$, then g is an extreme point of $N(\mathbf{B})$. We will denote by $E(\mathbf{B})$ the class of all extreme points of $N(\mathbf{B})$. If $n = 1$ and if (1.2) holds, then as is well known $g \in E(\mathbf{B})$ if and only if

$$(1.3) \qquad f(z) = cz$$

where $c \in \mathbf{T}$.

Let $\oplus_1^N V_k$ be an orthogonal decomposition of $\mathbf{C}^n$ into complex subspaces of positive dimension, and define $\pi : \mathbf{C}^N \times \mathbf{C}^n \to \mathbf{C}^n$ by

$$\pi(\mu, w) = \sum_1^N \mu_k w_k = (\mu_1 w_1, \ldots, \mu_N w_N)$$

where

$$w = \sum_1^N w_k = (w_1, \ldots, w_N), \quad w_k \in V_k.$$

Let $f \in H(\mathbf{B})$ and let

$$f_\alpha(w) = \int_{\mathbf{T}^N} \bar{\mu}^\alpha f(\pi(\mu, w)) d\mu$$

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where $\alpha \in \mathbf{Z}^N$. Then $f_\alpha = 0$ if $\alpha \notin \mathbf{N}^N$, and we have

$$(1.4) \quad f = \sum_{\alpha \geqq 0} f_\alpha$$

where by $\alpha \geqq 0$ we mean $\alpha \in \mathbf{N}^N$. Furthermore $f_\alpha \in H_\alpha$ where by

$$H_\alpha = H_\alpha\left(\bigoplus_1^N V_k\right)$$

we mean the class of all polynomials $\varphi(w)$ in $\mathbf{C}^n$ such that

$$\varphi(\pi(\mu, w)) = \mu^\alpha \varphi(w)$$

if $\mu \in \mathbf{C}^N$. For example if $n = 4$ and $N = 2$, then

$$z_1^2 z_2^2(z_3^2 + z_4^2) + (z_1^2 + z_2^2)^2 z_3 z_4 \in H_{(4,2)}.$$

If $j \in \mathbf{N}$, then we will denote by H_j the class of all polynomials in $\mathbf{C}^n$ that are homogeneous of degree j. Thus if $\varphi \in H_\alpha$, then $\varphi \in H_{|\alpha|}$ where by $|\alpha|$ we mean $\sum_1^N \alpha_k$. If $N = 1$, then $\alpha = |\alpha|$ and $H_\alpha = H_{|\alpha|}$, whereas if $N = n$ (in which case each V_k is of dimension 1), then H_α is the class of all monomials cz^α in $\mathbf{C}^n$.

Let $X \subset \mathbf{C}^n$. There is the following property which may or may not hold.

1.1.1. If $\varphi \in \bigcup_1^\infty H_j$ and if $\varphi = 0$ on X, then $\varphi = 0$.

If the property 1.1.1 holds, then we will say that X is *thick* in $\mathbf{C}^n$.

We will denote by $\mathbf{S}$ the unit sphere in $\mathbf{C}^n$. Thus

$$\mathbf{S} = \left\{z : z \in \mathbf{C}^n, \sum_1^n z_j \bar{z}_j = 1\right\} = \partial \mathbf{B}.$$

If $\varphi \in \bigcup_1^\infty H_j$, then we let

$$||\varphi|| = \sup\{|\varphi(z)| : z \in \mathbf{S}\}$$

and we let

$$X_\varphi = \{z : z \in \mathbf{S}, |\varphi(z)| = ||\varphi||\}.$$

We will denote by $\mathbf{N}_+$ the class of all positive integers.

In this paper we consider polynomials f in H_γ and ask if the Cayley transform of f is extreme in $N(\mathbf{B})$. We have the following sufficient condition.

1.2. Theorem. *Let* $f \in H_\gamma$, $||f|| = 1$, $\gamma \in \mathbf{N}_+^N$. *Then*

$$(1 + f)/(1 - f) \in E(\mathbf{B})$$

if the components of γ *are relatively prime and if* X_f *is thick in* $\mathbf{C}^n$.

1.3. The function (1.1) (if $n \geqq 3$) stresses the fact that the condition on γ is not necessary (in this case $N = 1$ and $\gamma = 2$). We do not know if the condition on X_f is necessary. The title of this paper refers to the fact that if γ and X_f satisfy the conditions of Theorem 1.2, then $\mathrm{Re}\,[(1 + f)/(1 - f)]$ generates an extreme ray in the positive pluriharmonic functions on **B**. Although Theorem 1.2 probably does not tell us much about the extreme points of $N(\mathbf{B})$ (if $n \geqq 2$), it is just about all that is known, and its proof, although not difficult, is lengthy.

The following proposition (whose proof we omit) provides polynomials f which satisfy the conditions of Theorem 1.2.

1.4. Proposition. *Let f_k be a homogeneous polynomial of positive degree in V_k, $1 \leqq k \leqq N$, and let*

$$f(w) = \prod_1^N f_k(w_k).$$

Then X_f is thick in $\mathbf{C}^n$ if X_{f_k} is thick in V_k, $1 \leqq k \leqq N$.

1.5. Let f and γ be as in Theorem 1.2. If $N = n$ (in which case f is a monomial) and if $(1 + f)/(1 - f)$ is extreme in $N(\mathbf{B})$, then by Theorem 1.2 of [**2**], the components of γ are relatively prime. Thus in this case (by Proposition 1.4) $(1 + f)/(1 - f)$ is extreme if and only if the components of γ are relatively prime.

Let us denote by Clos $E(\mathbf{B})$ the closure of $E(\mathbf{B})$ in the compact open topology. If $n = 1$, then by (1.3), $E(\mathbf{B}) =$ Clos $E(\mathbf{B})$. Furthermore for every n, $1 \in N(\mathbf{B})$, but $1 \notin E(\mathbf{B})$. There is the following corollary of Theorem 1.2.

1.6. Corollary. *If $n \geqq 2$, then* $1 \in$ Clos $E(\mathbf{B})$; *hence* $E(\mathbf{B}) \neq$ Clos $E(\mathbf{B})$.

We will omit the proof.

2. Lemmas and propositions which are preparatory to the proof of Theorem 1.2.

2.1. We recall that if $\lambda, \mu \in \mathbf{C}$ and if $\lambda \neq 1$, then

$$\text{Re}\,[(1 + \lambda + 2\mu)/(1 - \lambda)] = (1 - |\lambda|^2 + 2\,\text{Re}\,[(1 - \bar{\lambda})\mu])/|1 - \lambda|^2. \tag{2.1}$$

If $f \in H(\mathbf{B})$, then we will denote by A_f the class of all φ in $H(\mathbf{B})$ such that

$$|f|^2 + 2\,\text{Re}\,(\bar{f}\varphi) \leqq 1 + 2\,\text{Re}\,\varphi \tag{2.2}$$

on **B**. Thus A_f is convex.

If Y is a compact Hausdorff space, then we will denote by $M_+(Y)$ the class of all Radon measures on Y. Thus if $\sigma \in M_+(Y)$ and $E \subset Y$, then $\sigma(E) \geqq 0$.

We recall that if $\sigma \in M_+(\mathbf{T}^N)$, then $\hat{\sigma} : \mathbf{Z}^N \to \mathbf{C}$ is defined by

$$\hat{\sigma}(\alpha) = \int \bar{\mu}^\alpha d\sigma(\mu).$$

If $\gamma \in \mathbf{Z}^N$, then we let

$$G_\gamma = \{\mu : \mu \in \mathbf{T}^N, \mu^\gamma = 1\}.$$

We recall the following fact from the theory of $M_+(\mathbf{T}^N)$.

2.2. PROPOSITION. *Let* $\sigma \in M_+(\mathbf{T}^N)$, *let* $\hat{\sigma}(0) = 1$, *and let* $\gamma \in \mathbf{Z}^N$. *If* $|\hat{\sigma}(\gamma)| = 1$, *then* $\sigma \in M_+(\bar{\lambda}G_\gamma)$ *where* $\lambda^\gamma = \hat{\sigma}(\gamma)$, $\lambda \in \mathbf{T}^N$.

2.3. PROPOSITION. *Let* $g \in N(\mathbf{B})$. *Thus*

$$g = 1 + 2 \sum_1^\infty g_j$$

where $g_j \in H_j$. *Furthermore let* $k \in \mathbf{N}_+$, *let* $f \in H_k$, *and let* $||f|| = 1$. *If* X_f *is thick in* $\mathbf{C}^n$ (*in which case* $k \geqq 2$ *if* $n \geqq 2$) *and if* $g_k = f$, *then*

$$g = (1 + f + 2\varphi)/(1 - f) \tag{2.3}$$

where φ *is a polynomial of degree* $\leqq k - 1$, $\varphi(0) = 0$, *and* $\varphi \in A_f$.

Proof. Let $z \in X_f$ and define $h : \mathbf{D} \to (0, \infty)$ by $h(\mu) = \operatorname{Re} g(\mu z)$. Thus

$$h(\mu) = 1 + 2 \operatorname{Re} \sum_1^\infty g_j(z)\mu^j.$$

Since h is harmonic and $\geqq 0$,

$$h(\mu) = \hat{\sigma}(0) + 2 \operatorname{Re} \sum_1^\infty \hat{\sigma}(j)\mu^j$$

where $\sigma \in M_+(\mathbf{T})$. We have $\hat{\sigma}(0) = 1$ and $\hat{\sigma}(k) = f(z)$, hence by Proposition 2.2 (with $N = 1$ and $\gamma = k$), $\sigma \in M_+(\bar{\lambda}G_k)$ where $\lambda^k = f(z)$. Thus if $j, m \in \mathbf{Z}$, then

$$\hat{\sigma}(j + km) = \int \bar{\mu}^{km}\bar{\mu}^j d\sigma(\mu) = \int \lambda^{km}\bar{\mu}^j d\sigma(\mu) = f(z)^m \hat{\sigma}(j). \tag{2.4}$$

We let $g_0 = 1$. Thus if $j, m \in \mathbf{N}$ and if $z \in X_f$, then by (2.4)

$$g_{j+km}(z) = f(z)^m g_j(z).$$

Furthermore $g_{j+km} - f^m g_j \in H_{j+km}$; hence by the thickness of X_f, $g_{j+km} = f^m g_j$. We have

$$g = 1 + 2 \sum_{j=1}^k \sum_{m=0}^\infty g_{j+km} = 1 + 2 \sum_1^k g_j \sum_0^\infty f^m$$
$$= 1 + 2\left(f + \sum_1^{k-1} g_j\right)/(1 - f),$$

thus if $\varphi = \sum_1^{k-1} g_j$, then (2.3) holds.

By (2.3) and the identity (2.1), $\varphi \in A_f$ which completes the proof of Proposition 2.3. (We remark that a special case of Proposition 2.3 is proved in [**1**].)

2.4. PROPOSITION. *Let* $k \in \mathbf{N}$, *let* $k \geqq 2$, *let* φ *be a polynomial in* $\mathbf{C}^n$ *of degree* $\leqq k - 1$, *and let* $\varphi(0) = 0$. *Thus* $\varphi = \sum_1^{k-1} \varphi_j$ *where* $\varphi_j \in H_j$. *Furthermore let*

$f \in H_k$. *If* $\varphi \in A_f$, *then*

(2.5) $\quad |\varphi_j - \bar{\varphi}_{k-j} f| \leqq 1 - f\bar{f}$

on $\mathbf{B} = \mathbf{B} \cup \mathbf{S}$.

Proof. Let

$$g = (1 - f\bar{f}) + 2 \operatorname{Re} [\varphi(1 - \bar{f})].$$

If $(\mu, z) \in \mathbf{T} \times \mathbf{C}^n$, then

(2.6) $\quad g(\mu z) = [1 - f(z)\bar{f}(z)] + 2 \operatorname{Re} \sum_1^{k-1} [\varphi_j(z) - \bar{\varphi}_{k-j}(z) f(z)]\mu^j.$

If $(\mu, z) \in \mathbf{T} \times \overline{\mathbf{B}}$, then by the definition (2.2) of A_f, $g(\mu z) \geqq 0$, hence by (2.6), the inequality (2.5) holds.

2.5. We recall that if α is a *multi-index*, i.e. if $\alpha \in \mathbf{N}^N$, then by $|\alpha|$ we mean $\sum_1^N \alpha_k$. We will omit the proof (which is straightforward) of the following proposition.

2.6. Proposition. *Let* $\alpha, \beta, \gamma \in \mathbf{N}^N$ *and let* $|\alpha| < |\gamma|$.

a. *Let* $\mu^\alpha = 1$ *if* $\mu \in G_\gamma$. *Then* $\alpha = 0$.

b. *Let* $\mu^\alpha = \bar{\mu}^\beta$ *if* $\mu \in G_\gamma$. *If* $0 < |\beta| \leqq |\gamma|$, *then* $\alpha + \beta = \gamma$.

c. *Let* $\mu^\alpha = \mu^\beta$ *if* $\mu \in G_\gamma$. *If* $|\beta| < |\gamma|$, *then* $\alpha = \beta$.

Let the components of γ *be relatively prime and let* $0 < |\alpha| < |\gamma|$ (*thus* $N \geqq 2$).

d. *Then* α *and* γ *are linearly independent over* $\mathbf{R}$.

e. *If we define* $\phi : \mathbf{T}^N \to \mathbf{T}^2$ *by* $\phi(\mu) = (\mu^\alpha, \mu^\gamma)$, *then* $\phi(\mathbf{T}^N) = \mathbf{T}^2$.

f. *If we define* $\phi : G_\gamma \to \mathbf{T}$ *by* $\phi(\mu) = \mu^\alpha$, *then* $\phi(G_\gamma) = \mathbf{T}$.

2.7. Lemma. *Let* f *and* γ *be as in Theorem* 1.2. *Furthermore let* $\alpha, \beta \in \mathbf{N}^N$, *let* $0 < |\alpha| < |\gamma|$, *let* $\alpha + \beta = \gamma$, *and let* $\varphi_\alpha + \varphi_\beta \in A_f$ *where* $\varphi_\alpha \in H_\alpha$, $\varphi_\beta \in H_\beta$. *If the components of* γ *are relatively prime and if* X_f *is thick in* $\mathbf{C}^n$, *then* $\varphi_\alpha = \varphi_\beta = 0$.

Proof. a. If $|\alpha| \neq |\beta|$, and if $z \in X_f$, then by Proposition 2.4,

(2.7) $\quad \varphi_\alpha(z) = \bar{\varphi}_\beta(z) f(z)$.

b. If $|\alpha| = |\beta|$, if $z \in X_f$, and if $\mu \in G_\gamma$, then by Proposition 2.4,

$$\mu^\alpha \varphi_\alpha(z) + \mu^\beta \varphi_\beta(z) = [\bar{\mu}^\alpha \bar{\varphi}_\alpha(z) + \bar{\mu}^\beta \bar{\varphi}_\beta(z)] f(z),$$

hence

$$\mu^\alpha \varphi_\alpha(z) + \bar{\mu}^\alpha \varphi_\beta(z) = \bar{\mu}^\alpha \bar{\varphi}_\alpha(z) f(z) + \mu^\alpha \bar{\varphi}_\beta(z) f(z).$$

Thus by Proposition 2.6f,

(2.8) $\quad \varphi_\alpha(z) = \bar{\varphi}_\beta(z) f(z)$

if $z \in X_f$.

c. We have (by the definition of A_f)

(2.9) $\quad |f|^2 + 2 \operatorname{Re} [\bar{f}(\varphi_\alpha + \varphi_\beta)] \leqq 1 + 2 \operatorname{Re} [\varphi_\alpha + \varphi_\beta]$

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on $\overline{\mathbf{B}}$. On X_f we have by (2.7) and (2.8),

$$(2.10)\quad |f|^2 + 2\,\mathrm{Re}\,[\bar{f}(\varphi_\alpha + \varphi_\beta)] = 1 + 2\,\mathrm{Re}\,[\varphi_\alpha + \varphi_\beta].$$

Let $z \in X_f$. Then $z = (t_1w_1, \ldots, t_Nw_N)$ where $0 \leqq t_k \leqq 1$, $w_k \in V_k \cap \mathbf{S}$, and

$$\sum_1^N t_k^2 = 1.$$

Let $t = (t_1, \ldots, t_N)$ and $w = (w_1, \ldots, w_N)$. Then $z = \pi(t, w)$.

Let $x \in \mathbf{R}^N$, $x \cdot x = 1$. Then $\pi(x, w) \in \mathbf{S}$; hence by (2.9),

$$(2.11)\quad (1 - Ax^{2\gamma}) + (Dx^\alpha + Ex^\beta - Bx^{\gamma+\alpha} - Cx^{\gamma+\beta}) \geqq 0$$

where $A = |f(w)|^2$, $B = 2\,\mathrm{Re}\,[\bar{f}(w)\varphi_\alpha(w)]$, $C = 2\,\mathrm{Re}\,[\bar{f}(w)\varphi_\beta(w)]$, $D = 2\,\mathrm{Re}\,\varphi_\alpha(w)$, and $E = 2\,\mathrm{Re}\,\varphi_\beta(w)$.

Let us define $\rho, \tau, \sigma : \mathbf{R}^N \to \mathbf{R}$ by $\rho(x) = x \cdot x$, $\tau(x) = 1 - Ax^{2\gamma}$, and

$$\sigma(x) = Dx^\alpha + Ex^\beta - Bx^{\gamma+\alpha} - Cx^{\gamma+\beta}.$$

Thus if $\rho(x) = 1$, then by (2.11),

$$(2.12)\quad (\tau + \sigma)(x) \geqq 0.$$

Furthermore by (2.10), $(\tau + \sigma)(t) = 0$, hence by (2.12),

$$(2.13)\quad \nabla(\tau + \sigma)(t) \parallel \nabla\rho(t).$$

We have $\tau(x) = 1 - |f(\pi(x, w))|^2$, hence $\nabla\tau(t) \parallel \nabla\rho(t)$. Thus by (2.13), $\nabla\sigma(t) \parallel \nabla\rho(t)$.

We have

$$(2.14)\quad x_k\partial\sigma/\partial x_k = Dx^\alpha\alpha_k + Ex^\beta\beta_k - Bx^{\gamma+\alpha}(\gamma_k + \alpha_k) - Cx^{\gamma+\beta}(\gamma_k + \beta_k).$$

If $\nabla\sigma(t) = \lambda\,\nabla\rho(t)$ where $\lambda \in \mathbf{R}$, then $(\partial\sigma/\partial x_k)(t) = 2\lambda t_k$, hence by (2.14),

$$(2.15)\quad Dt^\alpha\alpha + Et^\beta\beta - Bt^{\gamma+\alpha}(\gamma + \alpha) - Ct^{\gamma+\beta}(\gamma + \beta) = 2\lambda T$$

where $T = (t_1^2, \ldots, t_N^2)$. We have

$$t^{\gamma+\alpha}B = 2\,\mathrm{Re}\,[\bar{f}(z)\varphi_\alpha(z)],$$

$$t^{\gamma+\beta}C = 2\,\mathrm{Re}\,[\bar{f}(z)\varphi_\beta(z)],\quad t^\alpha D = 2\,\mathrm{Re}\,\varphi_\alpha(z),$$

$t^\beta E = 2\,\mathrm{Re}\,\varphi_\beta(z)$, hence by (2.7), (2.8), and (2.15),

$$(2.16)\quad [\mathrm{Re}\,\varphi_\alpha(z)]\beta + [\mathrm{Re}\,\varphi_\beta(z)]\alpha = \chi T$$

(where $\chi = -\lambda/2$).

If $\mu \in \mathbf{T}^N$, then

$$\pi(\mu, z) = (t_1\mu_1w_1, \ldots, t_N\mu_Nw_N),$$

hence by (2.16)

$$(2.17)\quad (\mathrm{Re}\,[\mu^\alpha\varphi_\alpha(z)])\beta + (\mathrm{Re}\,[\mu^\beta\varphi_\beta(z)])\alpha = \chi(\mu)T.$$

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By Proposition 2.6e (and the fact that $\alpha + \beta = \gamma$), there is a μ in $\mathbf{T}^N$ such that $\mu^\alpha \varphi_\alpha(z) = |\varphi_\alpha(z)|$ and $\mu^\beta \varphi_\beta(z) = |\varphi_\beta(z)|$. Likewise there is a λ in $\mathbf{T}^N$ such that $\lambda^\alpha \varphi_\alpha(z) = |\varphi_\alpha(z)|$ and $\lambda^\beta \varphi_\beta(z) = -|\varphi_\beta(z)|$. Then by (2.17), and the fact that $|\varphi_\alpha(z)| = |\varphi_\beta(z)|$, we have

$$2|\varphi_\alpha(z)|\beta = [\chi(\mu) + \chi(\lambda)]T$$
$$2|\varphi_\alpha(z)|\alpha = [\chi(\mu) - \chi(\lambda)]T.$$

Thus if $\varphi_\alpha(z) \neq 0$, then α and β are linearly dependent; hence α and γ are linearly dependent which contradicts Proposition 2.6d.

Thus $\varphi_\alpha(z) = \varphi_\beta(z) = 0$ if $z \in X_f$; hence $\varphi_\alpha = \varphi_\beta = 0$ which completes the proof of Lemma 2.7.

2.8. LEMMA. *Let f and γ be as in Theorem* 1.2. *Furthermore let $\alpha \in \mathbf{N}^N$, let $0 < |\alpha| < |\gamma|$, and let $\varphi_\alpha \in A_f$ where $\varphi_\alpha \in H_\alpha$. If the components of γ are relatively prime and if X_f is thick in $\mathbf{C}^n$, then $\varphi_\alpha = 0$.*

Proof. If $|\gamma| - |\alpha| \neq |\alpha|$, and if $z \in X_f$, then by Proposition 2.4, $\varphi_\alpha(z) = 0$, hence $\varphi_\alpha = 0$.

Let $|\gamma| - |\alpha| = |\alpha|$. If $z \in X_f$ and if $\mu \in G_\gamma$, then by Proposition 2.4

$$\mu^\alpha \varphi_\alpha(z) = \bar{\mu}^\alpha \bar{\varphi}_\alpha(z) f(z) = \bar{\mu}^\alpha \varphi_\alpha(z).$$

Thus by Proposition 2.6f, $\varphi_\alpha(z) = 0$, hence $\varphi_\alpha = 0$.

2.9. LEMMA. *Let f and γ be as in Theorem* 1.2. *Furthermore let φ be a polynomial of degree $\leqq |\gamma| - 1$, let $\varphi(0) = 0$, and let $\varphi \in A_f$. If the components of γ are relatively prime and if X_f is thick in $\mathbf{C}^n$, then $\varphi = 0$.*

Proof. Let $|\gamma| \geqq 2$ (in which case $N \geqq 2$) and let

$$I = \{\alpha : \alpha \in \mathbf{N}^N, 0 < |\alpha| < |\gamma|\}.$$

We have (see (1.4))

$$\varphi = \sum_{\alpha \in I} \varphi_\alpha$$

where $\varphi_\alpha \in H_\alpha$. If $\mu \in \mathbf{T}^N$, then

$$\varphi(\pi(\mu, w)) = \sum_{\alpha \in I} \varphi_\alpha(\pi(\mu, w)) = \sum_{\alpha \in I} \mu^\alpha \varphi_\alpha(w).$$

Thus if $\sigma \in M_+(\mathbf{T}^N)$, then

$$\int \varphi(\pi(\bar{\mu}, w)) d\sigma(\mu) = \sum_{\alpha \in I} \hat{\sigma}(\alpha) \varphi_\alpha(w),$$

hence if $\sigma \in M_+(G_\gamma)$ and if $\sigma(G_\gamma) = 1$, then by (2.2)

$$(2.18) \quad \sum_{\alpha \in I} \hat{\sigma}(\alpha) \varphi_\alpha \in A_f.$$

Let $\beta \in I$ and let

$$d\sigma(\mu) = (1 + \operatorname{Re} \mu^\beta) d\mu, \quad \mu \in G_\gamma.$$

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Let $\alpha \in I$. Then by Proposition 2.6a, b, and c, $\hat{\sigma}(\alpha) = 1/2$ if $\alpha + \beta = \gamma$ or if $\alpha = \beta$. Otherwise $\hat{\sigma}(\alpha) = 0$. If $\gamma - \beta \in \mathbf{N}^N$, then we write $\beta \leqq \gamma$. Thus if $\beta \leqq \gamma$, then by (2.18),

$$\tfrac{1}{2}\varphi_{\gamma-\beta} + \tfrac{1}{2}\varphi_\beta \in A_f.$$

Likewise if $\beta \nleqq \gamma$, then $\frac{1}{2}\varphi_\beta \in A_f$. Thus if $\beta \leqq \gamma$, then by Lemma 2.7, $\varphi_\beta = 0$, and if $\beta \nleqq \gamma$, then by Lemma 2.8, $\varphi_\beta = 0$ which completes the proof of Lemma 2.9.

3. The proof of Theorem 1.2. Let f and γ be as in Theorem 1.2, and let $g = (1 + f)/(1 - f)$. It is to be proved that if the components of γ are relatively prime and if X_f is thick in $\mathbf{C}^n$, then g is an extreme point of $N(\mathbf{B})$. Let $h \in C(\mathbf{B})$. If $g + h \in N(\mathbf{B})$, then $h \in H(\mathbf{B})$ and $h(0) = 0$, hence $h = 2\sum_1^\infty h_j$ where $h_j \in H_j$. Thus if

$$g = 1 + 2\sum_1^\infty g_j$$

where $g_j \in H_j$, then

$$g + h = 1 + 2\sum_1^\infty (g_j + h_j). \tag{3.1}$$

We have $g_{|\gamma|} = f$. Let $\psi = h_{|\gamma|}$ and let $z \in \mathbf{S}$. If $\mu \in \mathbf{D}$, then by (3.1)

$$1 + 2\,\mathrm{Re}\sum_1^\infty [g_j(z) + h_j(z)]\mu^j > 0,$$

hence $|f(z) + \psi(z)| \leqq 1$. Likewise if $g - h \in N(\mathbf{B})$, then $|f(z) - \psi(z)| \leqq 1$, hence $\psi(z) = 0$ if $z \in X_f$, hence $\psi = 0$. Thus by Proposition 2.3

$$g + h = (1 + f + 2\varphi)/(1 - f)$$

where φ is a polynomial of degree $\leqq |\gamma| - 1$, $\varphi(0) = 0$, and $\varphi \in A_f$. By Lemma 2.9, $\varphi = 0$, hence $g + h = g$, hence $h = 0$. Thus $g \in E(\mathbf{B})$.

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The University of Wisconsin – Madison,
Madison, Wisconsin