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On the Horizontal Monotonicity of $|\Gamma(s)|$

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Abstract. Writing $s = \sigma + it$ for a complex variable, it is proved that the modulus of the gamma function, $|\Gamma(s)|$, is strictly monotone increasing with respect to σ whenever $|t| > 5/4$. It is also shown that this result is false for $|t| \leq 1$.

1 Introduction

The gamma function was introduced by Euler [6] in 1729 and has been the object of much study since then. An account of this function set against a historical framework is available in [4] and [13]. Its behaviour along the real axis is well known and in particular Gauss [7, p. 147] found that $\Gamma(\sigma)$ is monotone increasing on the range $\sigma > 1.461632145$. The behaviour of the modulus of the gamma function on vertical strips in the right half plane has been studied in considerable detail. A case in point is the following result of M. Lerch [8, p. 15]:

$$|\Gamma(\sigma + it)| = \lambda \frac{\Gamma(1+\sigma)}{\sqrt{\sigma^2 + t^2}} \sqrt{\frac{2\pi t}{e^{\pi t} - e^{-\pi t}}}$$

for some $\lambda \in (1, \sqrt{1+t^2})$. Results of this type were motivated by studies in the distribution of primes and the closely related Riemann zeta function. For the digamma function $\Psi(s)$, defined by $\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$, Landau gives a few results of this type in his book [10, pp. 317–318, 368–369], one of which is the following:

$$\Re(\Psi(i\omega)) = \log \omega + O(\omega^{-1}), \quad \omega \gg 1.$$

In sharp contrast, there is very little available in the literature concerning the behaviour of the gamma function along horizontal directions in the complex plane. Due to the poles of $\Gamma(\sigma)$ at $\sigma = 0, -1, -2, \dots$, $|\Gamma(\sigma)|$ becomes infinite at all these values and thus is, so to speak, extremely non-monotone along the (negative) real axis. It therefore seems somewhat surprising that this irregular behaviour vanishes as soon as one moves a small distance off the real axis, a property of the function that does not appear to have been observed in the literature. The purpose of this short note is to prove the following.

Theorem 1.1 *Let $s = \sigma + it$, with $|t| \geq 5/4$. Then $|\Gamma(s)|$ is strictly monotone increasing with respect to σ.*

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We remark that the lower bound $5/4$ given in Theorem 1.1 is close to, but not the best possible. Using MAPLE, it seems that the best lower bound is approximately given by $|t| > 1.04794998$, and in Section 3 it is shown that monotonicity fails for $|t| = 1$. We also remark that the same property does not seem to hold for the digamma function $\Psi(s)$, in particular for σ sufficiently large negative $|\Psi(\sigma + it)|$ oscillates with respect to σ, no matter how large $|t|$ is taken. A recent paper of Alzer [2] deals with a monotonicity property for the Hurwitz zeta function, and an earlier paper [1] deals with monotonicity of the gamma function, in both cases along the real axis.

In the following section we prove an elementary lemma that enables one to detect monotonicity of the modulus of any holomorphic function f. We also state a few results from the theory of the gamma function that we need for the proof of Theorem 1.1 given in Section 3.

Figure 1.1 gives a beautiful illustration of Theorem 1.1, showing the poles at $s = 0, -1, -2, -3, -4$ along the real axis and how the modulus becomes monotone increasing as $|t|$ becomes somewhat larger than 1 (*e.g.*, the bottom curve in the figure represents $|\Gamma(\sigma - 2.5 \cdot i)|$). It is taken from the book of Jahnke and Emde [9], written well before the age of computer graphics.

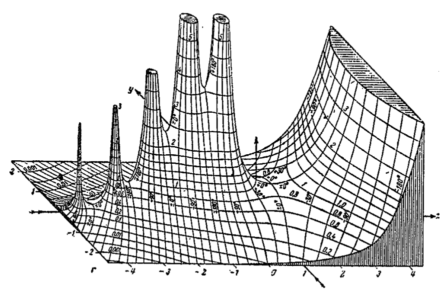

Figure 1.1: The Modulus of the Gamma Function

2 Preparatory Lemma and Some Results on Γ and Ψ

We shall use the standard notation $s = \sigma + it$ $(\sigma, t \in \mathbb{R})$ throughout. The following lemma is useful for measuring the rate of change of $|f(s)|$ with respect to σ. We shall assume that f is defined on an open subdomain Ω in the complex plane. Further, we shall assume that Ω is convex so that the sections $\Omega_t = \{\sigma : \sigma + it \in \Omega\}$ are open intervals.

Lemma 2.1 *Suppose the function f is holomorphic at s and $f(s) \neq 0$, then*

$$\Re\Big(\frac{f'(s)}{f(s)}\Big) = \frac{1}{|f(s)|} \cdot \frac{\partial |f(s)|}{\partial \sigma} \quad .$$

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Proof Writing $f = u + iv$, a direct computation gives

$$|f(s)|\frac{\partial|f(s)|}{\partial\sigma} = uu_\sigma + vv_\sigma. \tag{2.1}$$

On the other hand,

$$\frac{f'(s)}{f(s)} = \frac{u_\sigma + iv_\sigma}{u + iv} = \frac{uu_\sigma + vv_\sigma - ivu_\sigma + iuv_\sigma}{|f(s)|^2},$$

from which the desired result follows upon taking real parts and using (2.1). ■

Corollary 2.2 *For $f(s) \neq 0$, either*

$$\operatorname{sgn}\Big(\frac{\partial|f(s)|}{\partial\sigma}\Big) = \operatorname{sgn}\Big(\Re\Big(\frac{f'(s)}{f(s)}\Big)\Big) \quad \textit{or} \quad \frac{\partial|f(s)|}{\partial\sigma} = \Re\Big(\frac{f'(s)}{f(s)}\Big) = 0.$$

From this corollary, one can show that $|f|$ is increasing in Ω_t, assuming it has no zeros there, by simply showing $\Re(f'/f) > 0$ on Ω, and similarly for $|f|$ decreasing.

We conclude this section by consolidating a few basic facts about the functions Γ and Ψ. Details may be found in many references on the subject of which we mention only [3–5, 11, 13, 15].

Theorem 2.3 (i) *$\Gamma(s)$ is a non-vanishing meromorphic function with simple poles at zero and the negative integers. It satisfies $\Gamma(\bar{s}) = \overline{\Gamma(s)}$ and the functional equation*

$$\Gamma(s+1) = s\Gamma(s). \tag{2.2}$$

(ii) *The digamma function satisfies the functional relation $\Psi(s+1) = \Psi(s) + \frac{1}{s}$.*
(iii) $\Re\Psi(s+1) = \Re\Psi(s) + \frac{\sigma}{|s|^2}$.
(iv) *The function $\Psi(s)$ may be developed as an Euler–Maclaurin series:*

$$\Psi(s) = \log s - \frac{1}{2s} + R_0'(s), \quad \sigma \geq 0, \tag{2.3}$$

where the (Stieltjes) remainder $R_0'(s)$ satisfies:

$$|R_0'(s)| \leq \sec^3\Big(\frac{\theta}{2}\Big) \cdot \frac{1}{12|s|^2}, \quad \sigma \geq 0, \tag{2.4}$$

θ being the principal argument of s.

Proof Equation (2.2) is the standard functional relation for Γ; the rest of (i) are standard properties found in all references; (ii) follows from (2.2) by logarithmic differentiation, and (iii) follows from (ii) by taking the real parts. For (iv) see [5, 12] or the original manuscript of Stieltjes [14]. ■

Corollary 2.4 *If $\Re\Psi(\sigma+it) > 0$, $0 \leq \sigma \leq 1$, then the same is true for $-\infty < \sigma \leq 1$.*

Proof One may rewrite the result in Theorem 2.3(iii), replacing σ by $\sigma - 1$, as

$$\Re\Psi(\sigma - 1 + it) = \Re\Psi(\sigma + it) - \frac{\sigma - 1}{(\sigma - 1)^2 + t^2}.$$

Since $-(\sigma - 1)/|s|^2 \geq 0$ here, it is then clear that once the result holds for some $\sigma \in [0, 1]$ it also holds for $\sigma - 1, \sigma - 2$, etc. ■

3 Proof of the Main Theorem

We now come to the proof of Theorem 1.1. By Corollary 2.2 we must show that $\Re\Psi(s) > 0$ for $t \geq 5/4$, and by Corollary 2.4 it suffices to show this for $0 \leq \sigma$. So consider the upper quarter plane $0 \leq \sigma$, $5/4 \leq t$. Then also $0 < \theta \leq \pi/2$. From (2.4), and taking into account that $\sec^3(\theta/2) \leq 2\sqrt{2}$ here, this gives

$$|\Re R_0'(s)| \leq |R_0'(s)| \leq \frac{\sqrt{2}}{6|s|^2}.$$

Equation (2.3) for $\Psi(s)$ gives

$$\Re\Psi(s) = \log|s| - \frac{\sigma}{2|s|^2} + \Re(R_0'(s)).$$

So to show $\Re\Psi(s) > 0$, it suffices to show that in the given quarter plane

$$\log|s| > \frac{\sigma}{2|s|^2} + \frac{\sqrt{2}}{6|s|^2} = \frac{3\sigma + \sqrt{2}}{6|s|^2}. \tag{3.1}$$

Since the left-hand side of (3.1) increases with t while the right-hand side decreases, it suffices to show that (3.1) holds for $t = 5/4$, that is,

$$\frac{1}{2}\log(\frac{25}{16} + \sigma^2) > \frac{3\sigma + \sqrt{2}}{6(\sigma^2 + \frac{25}{16})}.$$

With minor simplification this reduces to showing that the function

$$f(\sigma) := (\sigma^2 + \frac{25}{16}) \cdot \log(\sigma^2 + \frac{25}{16}) - \sigma - \frac{\sqrt{2}}{3}$$

is strictly positive, $\sigma \in \mathbb{R}$, which is an exercise in elementary calculus. Indeed it is trivial that $f''(\sigma) > 0$, whereby f has at most a single (absolute) minimum. Numerical analysis shows $f'(\sigma) = 0$ for the single value $\sigma_0 = 0.3302990452\dots$, and $f(\sigma_0) = .0571293279\cdots > 0$, whence $f > 0$. This completes the proof for $t \geq 5/4$, and the result also follows for $t \leq -5/4$, since $\Gamma(\bar{s}) = \overline{\Gamma(s)}$ implies $|\Gamma(\bar{s})| = |\Gamma(s)|$. ∎

It seems of interest to also provide the following alternate argument, which is more direct, elementary, and does not use the delicate Stieltjes estimate for the remainder $R_0'(s)$. However, it will be given only for $|t| > 2$, since further refinements of this bound by this method seem troublesome. We begin by recalling (see [13]) the series development for $\Psi(s)$:

$$\Psi(s) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+s}\right), \tag{3.2}$$

where $\gamma = .577215665\dots$ denotes Euler's constant. From (3.2) follows

$$\Re(\Psi(s)) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{n+\sigma}{(n+\sigma)^2 + t^2}\right), \tag{3.3}$$

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As observed in the proof of Theorem 1.1 in Section 3, we need to examine the sign of $\Re(\Psi(s))$ when $|t| \geq 2$ and $\sigma \geq 0$. Separating out the $n = 0$ term from (3.3) we get

$$\Re(\Psi(s)) = 1 - \gamma - \frac{\sigma}{\sigma^2 + t^2} + \sum_{n=1}^{\infty} \Big(\frac{\sigma^2 + t^2 + (n\sigma - n - \sigma)}{(n+1)((n+\sigma)^2 + t^2)} \Big). \tag{3.4}$$

Now if $\sigma \geq 1$, then the terms under the summation sign in (3.4) are positive. As for the remaining terms, $1 - \gamma \geq 0.4$, while $\sigma/(\sigma^2 + t^2) \leq \frac{1}{2t} \leq 1/4$. Turning to the case $0 \leq \sigma \leq 1$ we separate out the $n = 1$ term as well, and we get

$$\Re(\Psi(s)) = \tag{3.5}$$

$$1 - \gamma - \frac{\sigma}{\sigma^2 + t^2} + \frac{1}{2}\Big(\frac{\sigma^2 + t^2 - 1}{(\sigma+1)^2 + t^2} \Big) + \sum_{n=2}^{\infty} \Big(\frac{\sigma^2 + t^2 + (n\sigma - n - \sigma)}{(n+1)((n+\sigma)^2 + t^2)} \Big).$$

The first three terms together exceed $3/20$; the fourth term exceeds $3/16$; the summands for $n = 2, 3, 4$ are positive. Assuming that the summands for $n = 2, 3, \dots, N-1$ are positive for some $N \geq 5$, the modulus of the sum in (3.5) from $n = N$ onwards is less than

$$\sum_{n=5}^{\infty} \frac{n-4}{(n+1)(n^2+4)} < \frac{\pi^2}{6} - \Big(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \Big) < \frac{10}{6} - \Big(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \Big) < \frac{1}{4},$$

and so $\Re(\Psi(s)) > 1/3 - 1/4 > 0$, completing the proof. ■

Remarks 3.1 (i) It is clear from (3.4) and Corollary 2.2 that $|\Gamma(\sigma + it)|$, for any value of t, is *eventually* monotone increasing in σ.

(ii) The lower bound $5/4 = 1.25$ could be sharpened by using more terms of the Euler–Maclaurin series for $\Psi(s)$. For example, taking (cf. [12, p. 159])

$$\Psi(s) = \log(s) - \frac{1}{2s} - \frac{1}{12s^2} + R_2', \quad |R_2'| < \frac{1}{20|s|^4},$$

will eventually lead to a lower bound of 1.073, using the same ideas as in the above proof.

(iii) Taking $|t| = 1$ and $\sigma = 1/2$ in (3.3), we see after a little simplification that

$$\Re\Psi(\frac{1}{2} + i) = -\gamma + \sum_{n=0}^{\infty} \frac{3 - 2n}{(n+1)(4n^2 + 4n + 5)}.$$

Thus

$$\begin{aligned} \Re\Psi(\frac{1}{2} + i) &= -\gamma + \frac{3}{5} + \frac{1}{26} - \sum_{n=2}^{\infty} \frac{2n-3}{(n+1)(4n^2 + 4n + 5)} \\ &< -\gamma + \frac{3}{5} + \frac{1}{26} - \sum_{n=2}^{8} \frac{2n-3}{(n+1)(4n^2 + 4n + 5)} = -.00386345 < 0, \end{aligned}$$

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and so (also taking (i) into account), $|\Gamma(\sigma + i)|$ fails to be monotone in σ. The same conclusion in fact holds for any fixed t with $|t| \leq 1$. Indeed, from (3.3) one sees that decreasing $|t|$ also decreases $\Re\Psi(s)$, so $\Re\Psi(1/2 + it) < 0$ remains true. This demonstrates that the results in Theorem 1.1 are quite sharp.

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