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ON THE SPHERES CARRYING AN ALMOST CONTINGENT STRUCTURE

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1. **Introduction.** It is well-known that odd dimensional spheres carry a normal contact structure [6]. Blair, Ludden and Yano [2] have recently studied a more general structure whose non-trivial example is an even dimensional sphere. Recently, the present author introduced the notion of almost contingent structures [8] with a view to develop a unified theory of various existing structures on a differentiable manifold. It is the purpose of this paper to show that even as well as odd dimensional spheres carry an almost contingent structure. In the sequel, each manifold introduced is C^∞, arcwise connected and satisfies the second axiom of countability.

2. **Almost contingent metric structure.**[1] Consider a differentiable manifold V_n on which there exists a vector valued function J, q vector fields, q 1-forms ξ_a and η^b over V_n such that

$$J^2 = \lambda^2 I + \varepsilon \sum_{a=1}^{q} \eta^a \otimes \xi_a \qquad (1 \le a, b, \ldots, \le q) \tag{2.1}$$

$$\eta^a(\xi_b) = \delta^a_b, \text{ rank } J = p, \; p+q = n, \tag{2.2}$$

where we assume that λ and ε are non zero constants and $\lambda^2+\varepsilon=0$.

In the above case, we say that V_n is endowed with an almost contingent structure. The following identities follow from (2.1) and (2.2).

$$J\xi_a = 0, \eta^a \circ J = 0 \qquad \forall a. \tag{2.3}$$

By assumption, V_n admits positive definite Riemannian metrics of class C^∞. In particular, one can choose a metric m with respect to which each ξ_a is a unit vector field. Explicitly, we have

$$\eta^a = m(\xi_a, .). \tag{2.4}$$

THEOREM 2.1. *Given V_n with an almost contingent structure, there exists a positive definite Riemannian metric g such that*

$$\eta^a(X) = g(\xi_a, X) \qquad \forall a, \tag{2.5}$$

$$g(JX, JY) + \varepsilon \sum_a \eta^a(X)\eta^a(Y) = \varepsilon g(X, Y). \tag{2.6}$$

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(1) In this paper, V_n belongs only to the second class. For details on classification of Vn, we refer to [9].

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Proof. Let m be a Riemannian metric over V_n satisfying (2.4). If we set

$$g(X, Y) = \tfrac{1}{2}[m(X, Y)+\frac{1}{\varepsilon} m(JX, JY)+\sum_a \eta^a(X)\eta^a(Y)], \tag{2.7}$$

then by virtue of (2.1)∼(2.4), one can easily show that (2.5) and (2.6) are true.

In the above case, we say that V_n is endowed with an almost contingent metric structure and g is called an associated Riemannian metric. If we set

$$\Omega(X, Y) = g(JX, Y), \tag{2.8}$$

then it is easy to show that the tensor Ω is skew-symmetric. Consequently, the rank p of J is even. We call Ω the associated fundamental 2-form of the given structure.

Operating (2.1) by J and then using (2.3) we get $J^3=\lambda^2 J$. If we put $\lambda^2 l=J^2$ and $\lambda^2 m=-J^2+\lambda^2 I$, I denoting the identity operator, then it is easy to show [8] that the operators l and m applied to the tangent space M_x^c are complementary projection operators which define distributions L and M respectively such that dim $L=p$ and dim $M=q$. Converse is trivial. Further, one can easily verify that L and M are orthogonal with respect to g and $g(Ju, Jv)=\lambda^2 g(u, v)$ for every $u, v \in L$. Consequently, we can choose in L $p=2r$ mutually orthogonal unit vectors $(e_1, \ldots, e_r; e_{r+1}, \ldots, e_{2r})=(e_{a_1}; e_{a_2})$ and a basis $(e_{2r+1}, \ldots, e_n)=(e_{a_3})$ for M such that $(e_{a_1}; e_{a_2}; e_{a_3})$ forms an orthogonal frame for $M_x^c(1\le a_1, b_1, \ldots, \le r; r+1\le a_2, b_2, \ldots, \le 2r; 2r+1\le a_3, b_3, \ldots, \le n)$. Let us say that this frame is adapted to the almost contingent metric structure if $Je_{a_1}=i\,\lambda e_{a_2}$, $Je_{a_2}=-i\,\lambda e_{a_1}$ and $Je_{a_3}=0$. J and g will have the following components with respect to this adapted frame.

$$g = \begin{pmatrix} E_r & 0 & 0 \\ 0 & E_r & 0 \\ 0 & 0 & 0_q \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & i\,\lambda E_r & 0 \\ -i\,\lambda E_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.9}$$

E_r denoting the $r\times r$ unit matrix. With respect to another adapted frame the orthogonal transformation matrix T will be of the form

$$T = \begin{pmatrix} A_r & B_r & 0 \\ -B_r & A_r & 0 \\ 0 & 0 & 0_q \end{pmatrix}. \tag{2.10}$$

Thus, it can be shown that the set $M_x^c(V_n)$ of the adapted frames relative to the different points of V_n has a natural structure of a principal fibre bundle with the base space V_n and the structure group $U(r)\times 0(q)$. The converse is trivial. This leads to the following.

THEOREM 2.2. *V_n admits an almost contingent metric structure iff its structure group* [5] *is* $U(r)\times 0(q)$.

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3. **Submanifolds of codimension q of an almost product manifold.** Let V_{n+q} be an $(n+q)$-dimensional almost hermitian manifold, that is, V_{n+q} carries a tensor field F of type (1, 1) such that

$$F^2 = I \tag{3.1}$$

and a metric G satisfying

$$G(FX, FY) = -G(X, Y). \tag{3.2}$$

Suppose that V_n is a submanifold with unit normals $C_1, \ldots, C_q$ and induced metric g. Thus, if B denotes the differential of the imbedding and X and Y tangent vector fields on V_n, then

$$G(BX, BY) = g(X, Y).$$

$$G(C_a, C_b) = \delta_{ab}, \qquad G(BX, C_a) = 0. \tag{3.3}$$

We choose $C_1, \ldots, C_q$ in such a way that $n+q$ vectors $B_1, \ldots, B_n, C_1, \ldots, C_q$ give the positive orientation of V_n, where $B_1, \ldots, B_n$ are vectors of V_n. The transforms FBX and FC_a can be expressed as

$$FBX = BJX+i \sum_a \eta^a(X)C_a, \tag{3.4}$$

$$FC_a = -i\ B\xi_a+\sum_b \alpha_{ab}C_b, \qquad \forall a, \tag{3.5}$$

where J is a vector valued function, each ξ_a is a vector field, each η^a is a 1-form and α_{ab} are $q(q-1)/2$ functions on V_n such that $\alpha_{aa}\equiv 0$. Operating (3.4) and then (3.5) by F and manipulating, we find

$$J^2 = I-\sum_a \eta^a \otimes \xi_a. \tag{3.6}$$

$$\eta^a \circ J = \sum_b \alpha_{ab}\eta^b, \qquad J\xi_a = \sum_b \alpha_{ba}\xi_b. \tag{3.7}$$

$$\eta^c(\xi_a) = \delta^c_a+\sum_b \alpha_{ab}\alpha_{cb}. \tag{3.8}$$

On the other hand, from (3.3) we get

$$g(JX, JY)+g(X, Y) = \sum_a \eta^a(X)\eta^a(Y). \tag{3.9}$$

In the above case we say that V_n is endowed with an almost contingent metric structure in the broad sense ($\lambda=1$, $\varepsilon=-1$). Consequently, we have proven the following theorem.

THEOREM 3.1. *A submanifold V_n of codimension q of an almost product metric manifold admits an almost contingent structure in the broad sense.*

COROLLARY 1. *A hypersurface of an almost product manifold carries an almost contingent structure* ($\lambda=1$, $\varepsilon=-1$ *and* $q=1$).

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Now let us apply the Gauss-Weingarten equations

$$(\nabla_{BX})Y = \sum_a h_a(X, Y)C_a, \tag{3.10}$$

$$\nabla_{BX}C_a = -BH_aX + \sum_b l_{ab}(X)C_b, \qquad l_{aa} \equiv 0, \tag{3.11}$$

where h_a, H_a and l_{ab} are the second fundamental forms, corresponding Weingarten maps and the third fundamental forms respectively. Let us assume that V_{n+q} is a Kählerian manifold, that is, $\nabla F=0$. Differentiating (3.4) covariantly along V_n, using above mentioned assumption, (3.10) and (3.11), we get the following differential equations.

$$(\nabla_X J)Y = i \sum_a [\eta^a(Y)H_aX - h_a(X, Y)\xi_a], \tag{3.12}$$

$$(\nabla_X \overset{a}{\eta})Y = i\left[h_a(X, JY) + \sum_b \alpha_{ab}h_b(X, Y)\right] + \sum_b l_{ab}(X)\eta^b(Y). \tag{3.13}$$

An almost product structure F is said to be integrable [1] iff the Nijenhuis torsion $[F, F]=0$.

DEFINITION 1. We say that an almost contingent structure J is normal if J satisfies

$$S_J \equiv [J, J] - \sum_a d\eta^a \otimes \xi_a = 0. \tag{3.14}$$

A lengthy computation of $[F, F]$ indicates that J is normal iff F is integrable. Furthermore, if F is hermitian or Kählerian then J will be called Grayan or Sasakian respectively.

Using (3.12) and (3.13) in (3.14), we find

$$\begin{aligned} S_J(X, Y) = {} & i\sum_a [\eta^a(X)(JH_a - H_aJ)Y - \eta^a(Y)(JH_a - H_aJ)X] \\ & + \sum_a \sum_b [l_{ab}(X)\eta^b(Y) - l_{ab}(Y)\eta^b(X)]\xi_a = 0 \end{aligned} \tag{3.15}$$

Let us assume that the connection induced in the normal bundle of V_n is flat, that is, we can choose each C_a in such a way that each $l_{ab} \equiv 0$. This leads to the following theorem:

THEOREM 3.2. *A submanifold V_n of codimension q of an almost product manifold admits a normal almost contingent structure*[2] *if* $\nabla F=0$, *each Ha commutes with J and the connection induced in the normal bundle is flat.*

Above theorem holds, in particular, for a totally umbilical or a totally geodesic submanifold [4].

(2) In the sequel, we shall drop the words "in the broad sense" unless any confusion arises.

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EXAMPLES. A plane or a sphere of codimension q in a Euclidean space.

REMARK. A special case of this theorem appears in [3] where the dimension of a plane or a sphere is necessarily even.

For a totally umbilical submanifold such that each $l_{ab}\equiv 0$, we have, for suitably chosen unit normals C_a,

$$h_a(X, Y) = h_a g(X, Y), \qquad \forall a$$

and consequently (3.13) becomes

$$(\nabla_X \eta^a)Y = i[h_a g(X, JY)+\sum_b \alpha_{ab}h_b g(X, Y)].$$

These equations give

$$(\nabla_X \eta^a)Y+(\nabla_Y \eta^a)X = i\sum_b \alpha_{ab}h_b g(X, Y), \tag{3.16}$$

which shows that each ξ_a defines infinitesimal conformal transformation in V_n.

4. **Examples.** (a) Odd-dimensional spheres.

We have seen in §3 corollary 1 that a hypersurface of an almost product manifold carries an almost contingent structure ($\lambda=1$, $\varepsilon=-1$ and $q=1$). In this case, (3.10)$\sim$(3.15) will reduce to the following:

$$(\nabla_{BX})Y = h(X, Y)C, \nabla_{BX}C = -BHX, \quad \text{where} \quad C_1 = C \quad \text{and} \quad h_1 = h \tag{4.1}$$

$$(\nabla_X J)Y = i\,[\eta(Y)HX-h(X, Y)\xi], \tag{4.2}$$

$$(\nabla_X \eta)Y = i\,h(X, JY) \tag{4.3}$$

$$S_J(X, Y) = i\,[\eta(X)(JH-HJ)Y-\eta(Y)(JH-HJ)X]. \tag{4.4}$$

Thus, the following theorem can easily be proved.

THEOREM 4.1. *An almost contingent structure on a hypersurface of an almost product Kählerian manifold is normal iff H commutes with J.*

DEFINITION 2. We say that a normal almost contingent structure is Sasakian if

$$(\nabla_X J)Y = \eta(Y)X-i\,g(X, Y)\xi \tag{4.5}$$

Let us assume that V_n carries a Sasakian structure. Using (4.5) in (4.2), we get

$$i[\eta(Y)HX-h(X, Y)\xi] = \eta(Y)X-i\,g(X, Y)\xi$$

Operating this result with η and simplifying, we get

$$h(X, Y) = g(X, Y)+\mu\eta(X)\eta(Y), \tag{4.6}$$

μ being a scalar field in V_n. Conversely, if the second fundamental tensor h of V_n is given by (4.6), then for a suitable choice of μ, one can easily verify that V_n is Sasakian. Thus we have the following

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THEOREM 4.2. *The induced Grayan structure in a hypersurface V_n in a Kählerian manifold V_{n+1} is Sasakian iff h satisfies* (4.6).

COROLLARY 2. *A totally umbilical hypersurface with positive constant mean curvature in a Kählerian manifold has a Sasakian structure by means of the induced metric.*

This corollary says that an odd-dimensional sphere has a Sasakian structure. Thus, we have shown that an almost contingent structure ($q=1$) carries an odd-dimensional sphere.

In particular, let V_{n+1} be replaced by R^{n+1} regarded as a Kähler manifold and S^n be a unit sphere embedded in V_{n+1}. If x is the position vector in R^{n+1} determining S^n, then $x \cdot x=1$ and $x \cdot x_i=0$, $x_i=\partial_i x$, $i=1, 2, \ldots, n$. The metric tensor g of S^n being given by $x_i \cdot x_j$. Now the mean curvature vector or outward normal C may be identified with x. Hence we have $0=\nabla_j(x_i \cdot x)=(h_{ji}C) \cdot C+x_i \cdot x_j$ and therefore $h=-g$. Thus, V_n can be replaced by an odd-dimensional unit sphere S^n.

(b) Even-dimensional spheres. Let us assume that $q=2$ and V_{n+2} is replaced by R^{n+2} regarded as a Kähler manifold where n in this case is even. Furthermore, let S^n be the unit sphere embedded in R^{n+2} and R^{n+1} be the hypersurface[3] of R^{n+2}. We also assume that R^{n+1} is a cosymplectic manifold [6]. If C_1 is the outer normal to S^n in R^{n+1} and C_2 the normal to R^{n+1} in R^{n+2}, then $0=\nabla_j(x_j \cdot x)=(h_{ji1}C_1+h_{ji2}C_2) \cdot C_1+x_i \cdot x_j$ and $0=\nabla_j C_2=-H^i_{j2}x_i-l_{j12}C_1$. Hence $h_1=-g$, $h_2=0$ and $l_{12}=0$. Thus, V_n can be replaced by an even-dimensional unit sphere S^n. As we have seen in section 3, the induced structure on S^n has $\alpha_{12}=G(FC_1, C_2)$, that is, α_{12} is the cosine of the angle between FC_1 and C_2. Thus, we have shown that even as well as odd dimensional spheres carry an almost contingent structure.

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REFERENCES

1. A. G. Walker, *Almost product structure*, Proceedings of the third symposium in pure mathematics of the A.M.S., **3** (1961), 94–100.

2. D. E. Blair, G. D. Ludden and K. Yano, *Induced structures on submanifolds*, Kōdai Math. Rep. **22** (1970), 188–198.

3. K. Yano and M. Okumura, *On (f, g, u, v, λ)-structures*, Kōdai Math. Sem. Rep. **22** (1970), 401–423.

4. M. Okumura, *Totally umbilical submanifolds of a Kählerian manifold*, J. Math. Soc. Japan **19** (1967), 317–327.

5. D. E. Blair, *Geometry of manifolds with structural group $U(n)\times O(s)$*, J. Diff Geom. **4** (1970), 155–167.

(3) It is not difficult to see that R^{n+1}, as a hypersurface of R^{n+2}, carries an almost contingent structure for $q=1$.

https://doi.org/10.4153/CMB-1975-039-3 Published online by Cambridge University Press

6. G. D. Ludden, *Submanifolds of cosymplectic manifolds*, J. Diff. Geom., **4** (1970), 237–244.

7. Y. Tashiro, *On contact structure of hypersurfaces in complex manifolds* 1, Tohoku Math. J., **15** (1963), 62–78.

8. K. L. Duggal, *On a unified theory of differentiable structures*, I: *almost contingent structures*, Tensor, N.S. **25** (1972), 303–308.

9. ——, *On a unified theory of differentiable structures*, II: *Existence theorems*, Tensor, N.S. **29** (1975).

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