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ON CERTAIN PROPERTIES OF SUBNORMAL SUBGROUPS

JENNIFER WHITEHEAD

1. Introduction.

1.1. *Main results.* Let G be a group generated by two subnormal subgroups H and K. Denoting the class of nilpotent groups by $\mathfrak{N}$, and the limit of the lower central series by $G^{\mathfrak{N}}$, Wielandt showed in [**14**], for groups with a composition series that

$$(*) \qquad G^{\mathfrak{N}} = H^{\mathfrak{N}}K^{\mathfrak{N}} \quad \text{and} \quad H^{\mathfrak{N}}K = KH^{\mathfrak{N}}.$$

More recently, Stonehewer has shown in [**13**] that (*) remains true when H and K are minimax groups or have finite rank, provided that $H/H^{\mathfrak{N}}$ and $K/K^{\mathfrak{N}}$ are nilpotent. Here, we obtain a further generalization of Wielandt's result, removing the condition that the lower central series of H and K terminate. If $\gamma_r(G)$ denotes the r-th term of the lower central series of G, and Max-*sn* denotes the class of groups having the maximal condition on subnormal subgroups, then we state:

THEOREM A. *Suppose that a group G is generated by two subnormal subgroups H and K, and let $H, K \in$ Max-sn. Then, given any two positive integers r_1 and r_2, there exists a positive integer r such that*

$$\gamma_r(G) \leqq \langle \gamma_{r_1}(H), \gamma_{r_2}(K) \rangle.$$

We say G satisfies Min-*n* if G satisfies the minimal condition on normal subgroups. Then any group G satisfying Min-*n* has the property that the lower central series terminates after finitely many steps. If G is generated by two subnormal subgroups H and K, we do not know that $H/H^{\mathfrak{N}}$ and $K/K^{\mathfrak{N}}$ are nilpotent, since the condition Min-*n* is not necessarily inherited even by normal subgroups (see Robinson [**9**, p. 153]). However, we are able to prove more generally:

THEOREM B. *Let G satisfy* Min-*n and let G be generated by two subnormal subgroups H and K. Then for any two positive integers r_1 and r_2*

$$G^{\mathfrak{N}} \leqq \gamma_{r_1}(H)\gamma_{r_2}(K).$$

We note that it is not possible to find any general theorem of this type, since the join of two nilpotent subgroups need not be nilpotent (see Zassenhaus [**16**, Appendix D, Exercise 23]).

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Let $\mathfrak{X}$ be a class of groups. We define the *$\mathfrak{X}$-residual* $G^{\mathfrak{X}}$ to be the intersection of all normal subgroups of G whose factor groups in G are $\mathfrak{X}$-groups. Let $L\mathfrak{N}$ denote the class of locally nilpotent groups. For the proof of Theorem B we shall examine the locally nilpotent residual of a locally finite group which is the join of two subnormal subgroups. We obtain the following criterion for the permutability of the locally nilpotent residuals of two subnormal subgroups, and we note that this is analogous to the case of nilpotent residuals proved as Theorem A in [**13**].

THEOREM C. *Let $G = \langle H, K\rangle$ where H, K are subnormal subgroups of G and suppose that $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}}\rangle$. Then $G^{L\mathfrak{N}} = H^{L\mathfrak{N}}K^{L\mathfrak{N}}$, $H^{L\mathfrak{N}}K = KH^{L\mathfrak{N}}$ and $HK^{L\mathfrak{N}} = K^{L\mathfrak{N}}H$ provided that $H/H^{L\mathfrak{N}} \in L\mathfrak{N}$.*

Our second result on locally nilpotent residuals generalizes Wielandt's theorem (*) to the class of locally finite groups.

THEOREM D. *Let G be a locally finite group, generated by two subnormal subgroups H and K. Then $G/G^{L\mathfrak{N}}$, $H/H^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}} \in L\mathfrak{N}$ (this is well known), $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}}\rangle = H^{L\mathfrak{N}}K^{L\mathfrak{N}}$, and $H^{L\mathfrak{N}}K = KH^{L\mathfrak{N}}$.*

Finally, we examine a criterion for subnormality in soluble groups. In [**15**], Wielandt found conditions in a finite group which imply the subnormality of a subgroup, and has also extended some of these results to groups satisfying the maximal condition. Similar results have been proved by Hartley and Peng for groups satisfying the minimal condition [**4**], and by Peng [**8**] for soluble-by-finite groups and groups satisfying maximal and minimal conditions on abelian subgroups. Let $[h, k]$ denote the commutator $h^{-1}k^{-1}hk$ and define inductively $[h, {}_nk] = [[h, {}_{n-1}k]k]$. Then we obtain the following condition for a finitely generated subgroup of a soluble group to be subnormal:

THEOREM E. *Let G be a soluble group and let $H \leqq G$ such that $H = \langle h_i | 1 \leqq i \leqq m\rangle$. Suppose there exists a fixed integer n such that $[g, {}_nh_i] \in H$ for all $g \in G$, $1 \leqq i \leqq m$.* If

(i) *G is polycyclic, or*

(ii) *H is* Min-*by-nilpotent,*

then H is subnormal in G.

We prove part (i) first for finite soluble groups and then extend to polycyclic groups. We do not know whether the result remains true in the finite case when the condition of solubility is removed, and we leave this as an open question.

1.2. *Notation.* If H is a subgroup of G, we define $H_0 = G$ and inductively $H_{i+1} = H^{H_i}$ for $i \geqq 0$. H_i is called the i-th *normal closure of H in G*. If H is subnormal in G we write H sn G; and if the index of subnormality is at most n, then we write $H \lhd^n G$. Hence $H \lhd^n G$ if and only if $H = H_n$. We define $[H, K] = \langle [h, k]; h \in H, k \in K\rangle$ and $[H, {}_nK] = [[H, {}_{n-1}K], K]$. Let X be any subgroup of G, then $\mathrm{Core}_G(X)$ denotes the largest normal subgroup of G contained in X.

The closure operations we shall use are S, S_n, Q, R_0, R, N_0, $\acute{N}$ and are defined as follows:

$\mathfrak{X} = S\mathfrak{X}$	if every subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.
$\mathfrak{X} = S_n\mathfrak{X}$	if every subnormal subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.
$\mathfrak{X} = Q\mathfrak{X}$	if every homomorphic image of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.
$\mathfrak{X} = R_0\mathfrak{X}$	if whenever for normal subgroups N_1, N_2 of G with $G/N_1 \in \mathfrak{X}$, $G/N_2 \in \mathfrak{X}$ then $G/(N_1 \cap N_2) \in \mathfrak{X}$.
$\mathfrak{X} = R\mathfrak{X}$	if $N_\lambda \lhd G$ and $G/N_\lambda \in \mathfrak{X} (\lambda \in \Lambda)$ always imply $G/(\bigcap_{\lambda \in \Lambda} N_\lambda) \in \mathfrak{X}$.
$\mathfrak{X} = N_0\mathfrak{X}$	if the product of any pair of normal $\mathfrak{X}$-subgroups is an $\mathfrak{X}$-group.
$G \in \acute{N}\mathfrak{X}$	if G can be generated by its ascendant $\mathfrak{X}$-subgroups.

We say that $\mathfrak{X}$ is *coalescent* if and only if in any group the join of a pair of subnormal $\mathfrak{X}$-subgroups is always a subnormal $\mathfrak{X}$-subgroup. $\mathfrak{A}$ denotes the class of abelian groups; $L\mathfrak{F}$ and $L\mathfrak{F}_\Pi$ denote the classes of locally finite and locally finite Π-groups for a set of primes Π, respectively.

2. Proofs of Theorems C and D.

2.1. *Preliminary lemmas.* Our first result examines the effect of homomorphisms on the $\mathfrak{X}$-residual of a group G. We omit the proof which is straightforward.

LEMMA 2.1. *Let $\mathfrak{X} = Q\mathfrak{X}$ and suppose that G is a group such that $G/G^{\mathfrak{X}} \in \mathfrak{X}$. Then if θ is any homomorphism of G, we have $(G\theta)^{\mathfrak{X}} = (G^{\mathfrak{X}})\theta$ and $G\theta/(G\theta)^{\mathfrak{X}} \in \mathfrak{X}$.*

We next examine the $\mathfrak{X}$-residual of the join of a subnormal and a normal subgroup.

LEMMA 2.2. *Let $\mathfrak{X} = N_0\mathfrak{X} = S_n\mathfrak{X} = Q\mathfrak{X}$. Suppose that $G = HK$ where H sn G, $K \lhd G$ and that $K/K^{\mathfrak{X}} \in \mathfrak{X}$, $H/H^{\mathfrak{X}} \in \mathfrak{X}$. Then $G/G^{\mathfrak{X}} \in \mathfrak{X}$ and $G^{\mathfrak{X}} = H^{\mathfrak{X}}K^{\mathfrak{X}}$.*

Proof. It follows easily from the definition of $G^{\mathfrak{X}}$ and the S_n-closure of $\mathfrak{X}$ that $H^{\mathfrak{X}}, K^{\mathfrak{X}} \leqq G^{\mathfrak{X}}$.

Since $K^{\mathfrak{X}} \lhd G$, by Lemma 2.1 we may assume $K^{\mathfrak{X}} = 1$ and show $H^{\mathfrak{X}} = G^{\mathfrak{X}}$. Let $H \lhd^m G$ and use induction on m. If $m = 1$, $H^{\mathfrak{X}} \lhd G$ and $G/H^{\mathfrak{X}}$ is generated by two normal $\mathfrak{X}$-subgroups. By the N_0-closure of $\mathfrak{X}$ we have $G/H^{\mathfrak{X}} \in \mathfrak{X}$. So $G^{\mathfrak{X}} = H^{\mathfrak{X}}$ and $G/G^{\mathfrak{X}} \in \mathfrak{X}$.

Suppose $m > 1$. Then $H_1 = H(H_1 \cap K)$ and $H_1 \cap K \in S_n\mathfrak{X} = \mathfrak{X}$. Since $H \lhd^{m-1} H_1$, by induction $H_1^{\mathfrak{X}} = H^{\mathfrak{X}}$ and $H_1/H_1^{\mathfrak{X}} \in \mathfrak{X}$. Then by case $m = 1$ the result follows.

It is not hard to show that if $\mathfrak{X}$ is an S and R_0-closed class of finite groups then the class of $L\mathfrak{X}$-groups is R-closed relative to the class of locally finite groups. Using this fact, we examine the $L\mathfrak{X}$-residual of a locally finite group G, and relate it to the $\mathfrak{X}$-residuals of its finite subgroups.

LEMMA 2.3. *Let* $G \in L\mathfrak{F}$. *Let* $\mathfrak{X} = \langle S, R_0, Q\rangle\mathfrak{X} \leqq \mathfrak{F}$. *Then we have* $G^{L\mathfrak{X}} = \langle F^{\mathfrak{X}} | F$ *is a finite subgroup of* $G\rangle$.

Proof. By the above we have $G/G^{L\mathfrak{X}} \in L\mathfrak{X}$.

Let $R = \langle F^{\mathfrak{X}} | F$ is a finite group of $G\rangle$. Then $R \lhd G$ and if F is any finite subgroup of G, $FR/R \in Q\mathfrak{X} = \mathfrak{X}$. Hence if K/R is any finite subgroup of G/R then $K/R \in \mathfrak{X}$. Therefore $G/R \in L\mathfrak{X}$ and it follows that $G^{L\mathfrak{X}} \leqq R$.

However, if F is any finite subgroup of G, it is easy to see that $F^{\mathfrak{X}} \leqq G^{L\mathfrak{X}}$. So $R \leqq G^{L\mathfrak{X}}$ and we have equality.

2.2. *Proof of Theorem C.* We now turn to the problem of permutability of locally nilpotent residuals. Following the methods of Stonehewer in [**13**], we use the following result of Brewster [**2**] on the permutability of subnormal subgroups.

LEMMA 2.4. (Brewster). *Suppose that H and K are subnormal subgroups of a group G, that* $G = \langle H, K\rangle$ *and that for all finite* $c \geqq 1$ $G = HK\gamma_c(G)$. *Then* $G = HK$.

We proceed with the proof of Theorem C. Let $M = G^{L\mathfrak{N}}$ and for some integer $c \geqq 1$ let $N = \gamma_c(M)$. Then $M \lhd G$ and using Lemma 2.1 we may apply Lemma 2.2 to the product

$$\frac{HM}{N} = \left(\frac{HN}{N}\right)\left(\frac{M}{N}\right).$$

Since $(M/N)^{L\mathfrak{N}} = 1$, we obtain $(HM/N)^{L\mathfrak{N}} = (HN/N)^{L\mathfrak{N}} = (H^{L\mathfrak{N}}N)/N$ and therefore $(H^{L\mathfrak{N}}N)/N \lhd HM/N$.

So $H^{L\mathfrak{N}}N \lhd M$ and we have $M = H^{L\mathfrak{N}}K^{L\mathfrak{N}}\gamma_c(M)$. As this is true for all finite $c \geqq 1$, by Lemma 2.4 $M = H^{L\mathfrak{N}}K^{L\mathfrak{N}}$.

The last statement of the theorem now follows easily since $KH^{L\mathfrak{N}} = KK^{L\mathfrak{N}}H^{L\mathfrak{N}} = KG^{L\mathfrak{N}} = G^{L\mathfrak{N}}K = H^{L\mathfrak{N}}K$. Similarly $HK^{L\mathfrak{N}} = K^{L\mathfrak{N}}H$.

2.3. *Proof of Theorem D.* We generalize the methods used by Wielandt in [**14**], but note that the proof itself is independent of the finite case. If Π is a set of primes, we denote the Π-residual of a locally finite group G by $O^{\Pi}(G)$ and note that $G/O^{\Pi}(G)$ is a locally finite Π-group.

We shall need the following results:

LEMMA 2.5. *Let X and Y be subgroups of a locally finite group such that* $G = \langle X, Y\rangle$ *and* Y sn G. *Suppose that every finite subgroup of X lies in a subnormal subgroup of G in X.*

If $O^{\Pi}(Y) \leqq X \cap Y$ *then* $O^{\Pi}(G) \leqq X$.

Proof. Let $Y \lhd^m G$ and proceed by induction on m.

Case 1. $m = 1$. Then $Y \lhd G$ and by Lemma 2.3

(**) $\quad O^{\Pi}(G) = \langle O^{\Pi}(F) | F$ is a finite subgroup of $G\rangle$.

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Let F be any finite subgroup of G. Then $F \leqq F_1$ a finite subgroup, where $F_1 = \langle X \cap F_1, Y \cap F_1 \rangle$. By hypothesis, there exists a subnormal subgroup X_1 of G, such that $X \cap F_1 \leqq X_1 \leqq X$. Hence $F_1 = \langle X_1 \cap F_1, Y \cap F_1 \rangle$ where $X_1 \cap F_1$ sn F_1, $Y \cap F_1 \lhd F_1$. By Lemma 2.2 we have

$$O^{\Pi}(F_1) = O^{\Pi}(X_1 \cap F_1)O^{\Pi}(Y \cap F_1) \leqq X.$$

So $O^{\Pi}(F) \leqq X$ for all finite subgroups F of G and so by (**) $O^{\Pi}(G) \leqq X$ as required.

Case 2. $m > 1$. Let $Y_1 = Y^X$. If F is any finite subgroup of $Y_1 \cap X$, then by hypothesis there exists a subnormal subgroup S of G such that $F \leqq S \leqq X$. Then $F \leqq S \cap Y_1 \leqq Y_1 \cap X$ where $S \cap Y_1$ sn G. Hence the group $\bar{Y} = \langle Y_1 \cap X, Y \rangle$ satisfies the hypotheses of the Lemma, and $Y \lhd^{m-1} \bar{Y}$. By induction we have $O^{\Pi}(\bar{Y}) \leqq X \cap Y$.

Let $N = O^{\Pi}(\bar{Y})$, $\bar{X} = X \cap Y_1$. Then $N \leqq \bar{X}$, $\bar{X}/N$ is a Π-group and N has no non-trivial Π-quotients. Therefore $N = O^{\Pi}(\bar{X}) \lhd \langle X, \bar{Y} \rangle = G$.

Now $Y_1/N = \langle Y^x N/N | x \in X \rangle$ and so Y_1/N is generated by subnormal $L\mathfrak{F}_{\Pi}$-subgroups. By the $\acute{N}$-closure of the class $L\mathfrak{F}_{\Pi}$ (see Robinson [**9**, p. 57 Theorem 2.31]), $Y_1/N \in L\mathfrak{F}_{\Pi}$ and applying case 1 to G/N we obtain $O^{\Pi}(G/N) \leqq X/N$. Using Lemma 2.1 we have $O^{\Pi}(G) \leqq X$.

LEMMA 2.6. *Let G be a locally finite group. Let A, B be subnormal subgroups of G, and let $J = \langle A, B \rangle$. Then if F is any finite subgroup of J, there exists a subnormal subgroup S of G such that $F \leqq S \leqq J$.*

Proof. This follows immediately from Roseblade and Stonehewer [**12**, Theorem A].

It is now an easy consequence of Lemmas 2.5 and 2.6 to prove:

THEOREM 2.1. *Let G be a locally finite group generated by two subnormal subgroups H and K. Then*

$$O^{\Pi}(G) = \langle O^{\Pi}(H), O^{\Pi}(K) \rangle.$$

Proof. It is easy to show that $\langle O^{\Pi}(H), O^{\Pi}(K) \rangle \leqq O^{\Pi}(G)$. To show the opposite inclusion, by Lemma 2.6 the group $\langle O^{\Pi}(H), O^{\Pi}(K) \rangle$ satisfies the hypotheses of the group X in Lemma 2.5. Hence, applying this Lemma to $\langle O^{\Pi}(H), K \rangle$ with $Y = K$ we have:

$$O^{\Pi}(\langle O^{\Pi}(H), K \rangle) \leqq \langle O^{\Pi}(H), O^{\Pi}(K) \rangle.$$

Similarly, applying Lemma 2.5 to G with $X = \langle O^{\Pi}(H), K \rangle$ and $Y = H$ we obtain

$$O^{\Pi}(G) \leqq \langle O^{\Pi}(H), K \rangle.$$

Hence $O^{\Pi}(G) \leqq \langle O^{\Pi}(H), O^{\Pi}(K) \rangle$ and we have equality.

We are now ready to prove Theorem D. We note first that by the R-closure of the class $L\mathfrak{N}$ relative to the class $L\mathfrak{F}$, we have $G/G^{L\mathfrak{N}}$, $H/H^{L\mathfrak{N}}$ and $K/K^{L\mathfrak{N}}$ are locally nilpotent. By Theorem C, it is enough to prove that $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}}\rangle$. It follows easily that $\langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}}\rangle \leqq G^{L\mathfrak{N}}$.

Now let p be any prime, and in Theorem 2.1 let $\Pi = \{p\}$ and apply to the group G generated by H and K.

Then $O^p(G) = \langle O^P(H), O^p(K)\rangle$.

Since $K/K^{L\mathfrak{N}}$ is the direct product of p-groups, denoting the set of primes different from p by p', we have $O^{p'}(O^p(K)) \leqq K^{L\mathfrak{N}}$. Let $L = O^p(G)$. Applying Theorem 2.1 to the group L with $\Pi = \{p'\}$ we have

$$O^{p'}(L) = \langle O^{p'}(O^p(H)), O^{p'}(O^p(K))\rangle.$$

Then $G^{L\mathfrak{N}}/(\text{Core}_G\langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}}\rangle)$ is an $L\mathfrak{F}_{p'}$-group.

Since p was any prime, this is true for all primes p. Hence $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}}\rangle$.

3. Proof of Theorem B. We shall use the fact that for a group G satisfying the minimal condition and generated by two subnormal subgroups H and K, $G^{\mathfrak{N}} = \langle H^{\mathfrak{N}}, K^{\mathfrak{N}}\rangle = H^{\mathfrak{N}}K^{\mathfrak{N}}$. This is not hard to show, adapting the proof given by Wielandt in [**14**] for groups with a composition series.

If G is a group, we denote by $G^{(n)}$ the n-th term of the derived series of G. Starting on the proof of Theorem B, there exist positive integers s_1 and s_2 such that

$$H^{(s_1)} \leqq \gamma_{r_1}(H) \quad \text{and} \quad K^{(s_2)} \leqq \gamma_{r_2}(K).$$

By Roseblade [**11**], there exists an integer s such that

$$G^{(s)} \leqq H^{(s_1)}K^{(s_2)} \leqq \gamma_{r_1}(H)\gamma_{r_2}(K)$$

and hence, without loss of generality, we may factor by $G^{(s)}$ and assume G is soluble. Since G satisfies Min-n, G is locally finite (see Baer [**1**]). By Theorem D, the factors $G/G^{L\mathfrak{N}}$, $H/H^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}}$ are locally nilpotent and

$$G^{L\mathfrak{N}} \leqq H^{L\mathfrak{N}}K^{L\mathfrak{N}} \leqq \gamma_{r_1}(H)\gamma_{r_2}(K).$$

Therefore, we may assume $G^{L\mathfrak{N}} = 1$ and that $G \in L\mathfrak{N}$. So G satisfies the minimal condition (see McLain [**6**]), and by the remark above $G^{\mathfrak{N}} = H^{\mathfrak{N}}K^{\mathfrak{N}} \leqq \gamma_{r_1}(H)\gamma_{r_2}(K)$.

We introduce some notation before proving a corollary to the Theorem. If H and K are subgroups of a group G, then the *permutizer* $P_H(K)$ of K in H is defined in [**12**] to be the largest subgroup of H which permutes with K.

Corollary. *Let $G = \langle H, K\rangle$ satisfy* Min-n *and let H and K be subnormal subgroups of G. Then there exists a positive integer δ such that $\gamma_\delta(H) \leqq P_H(K)$.*

Proof. The proof of this is analogous to that given for the case of the derived series in [**11**] (see [**11**, corollary to Theorem B]).

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4. Proof of Theorem A. We shall need a theorem of Mal'cev [**7**] which states that any subgroup of a polycyclic group is equal to the intersection of all the subgroups of finite index that contain it.

If G is any group, and r an integer we define G^r by $G^r = \langle g^r;\, g \in G\rangle$.

We now begin the proof of Theorem A and note as in Theorem B we may assume G is soluble. By the subnormal coalescence of the class Max-*sn* (see [**10**]) we have that $G \in$ Max-*sn*, and so G is polycyclic. Let G have derived length d, and let $X = \langle \gamma_{r_1}(H), \gamma_{r_2}(K)\rangle$. We use induction on d. If $d = 1$, then the Theorem is trivially true. So assume $d > 1$ and let N be the last but one term of the derived series of G. By induction, there exists a positive integer r_3 such that $\gamma_{r_3}(G) \leqq NX$. By [**13**, Lemma 2], there exist positive integers t_1, t_2 such that

$$\gamma_{t_1}(NH) \leqq \gamma_{r_1}(H) \quad \text{and} \quad \gamma_{t_2}(NK) \leqq \gamma_{r_2}(K).$$

By replacing r_1, r_2 by t_1, t_2 respectively, we may assume that $N \leqq H \cap K$, and $X \lhd XN$.

Let $L = \gamma_{r_3}(G)$. Then $L' \leqq (XN)' \leqq X$ and since we may assume without loss of generality that $\mathrm{Core}_G(X) = 1$ we have $L' = 1$.

Let $M = XL$. Since M is polycyclic, there exists $F \lhd M$, $X \leqq F$ such that M/F is finite and F/X is torsion-free. Let $|M : F| = n$. Then $M^n \leqq F$ and $|M : M^nX|$ is finite and M^nX/X is torsion-free.

By Mal'cev's theorem, there exists $F_0 \leqq G$ such that $|G : F_0| < \infty$ and $M^nX = F_0 \cap M$. By Wielandt's theorem (*) we have that there exists an integer r_4 such that

$$\gamma_{r_4}(G) \leqq X\, \mathrm{Core}_G(F_0).$$

Let $r_5 = \max\{r_3, r_4\}$. Then $\gamma_{r_5}(G) \leqq X(F_0 \cap M) \leqq XM^n$.

Hence we may suppose that $L = \gamma_{r_5}(G)$ and that M/X is torsion-free. In a similar way, we can show for all primes p, there exist integers $r(p)$ such that $\gamma_{r(p)}G \leqq XM^p$. Let s be the rank of G. Then M/XM^p has order less than or equal to p^s. Then $\gamma_{r_5+s}(G) \leqq \bigcap_p XM^p$.

Now $(M/X)^p = XM^p/X$ and since M/X is a torsion-free abelian group of finite rank $\bigcap\, (M/X)^p = X$.

Hence $\bigcap_p XM^p = X$ and $\gamma_{r_5+s}(G) \leqq X$. So, substituting $r = r_5 + s$, the theorem is proved.

5. Proof of Theorem E.

5.1. *The finite case.* We would like to thank Dr. Brian Hartley for suggesting the following result and for indicating the method of proof.

THEOREM 5.1. *Let G be a finite soluble group. Let $H = \langle h_i | 1 \leqq i \leqq m\rangle$. Then H is subnormal in G if and only if $[g, {}_nh_i] \in H$ for all $g \in G$, $n = |G|$, $i = 1, \ldots, m$.*

To prove this we introduce two concepts. Let G be any group and let $a \in G$.

We define subgroups $X_r(a)$ for integers $r \geqq 0$ by

$$X_r(a) = \langle a, [g, {}_r a]; g \in G \rangle.$$

Then $X_r(a) \geqq X_{r+1}(a)$ for all integers $r \geqq 0$.

Following Wielandt, we define the subnormalizer of a subgroup X to be the intersection of all subnormal subgroups of G containing X. We denote the subnormalizer of X by $S_G(X)$. In general $S_G(X)$ need not be subnormal in G. For let G be the infinite dihedral group

$$D_\infty = \langle x, y | x^{-1}yx = y^{-1}, x^2 = 1 \rangle$$

and let $X = \langle x \rangle$ and $Y = \langle y \rangle$.

Then $x^{-1}y_1x = y_1^{-1}$ for all $y_1 \in Y$ and so $X_r(x) = \langle y^{2^r}, x \rangle$. Then $X_r(x) \lhd X_{r-1}(x)$ and so $X_r(x)$ sn G for $r \geqq 0$. But if $S_G(X)$ sn G, then $X_m(x) \leqq S_G(X)$ for some integer m. Therefore, the chain

$$\ldots \leqq X_i(x) \leqq X_{i-1}(x) \leqq \ldots \leqq \langle x^G \rangle \leqq G$$

terminates after finitely many steps. Since $X_r(x) < X_{r-1}(x)$ for all positive integers r, we have a contradiction.

Theorem 5.1 follows from:

LEMMA 5.1. *Let G be a finite soluble group and let $a \in G$. Then if $n = |G|$, we have*

$$S_G(\langle a \rangle) = X_r(a) \quad \text{for all } r \geqq n.$$

Proof. Since G is finite of order n, $S_G(\langle a \rangle)$ is subnormal in G, and the subnormal index is bounded by n. Hence $X_r(a) \leqq S_G(\langle a \rangle)$ for all $r \geqq n$. So it is enough to show that $X_r(a)$ sn G. Suppose for a contradiction that this is false, and let G be a minimal counterexample.

Choose an integer $r \geqq n$, and assume $X = X_r(a)$ is not subnormal in G. Let $G_1 = \langle a^G \rangle$. If $G_1 < G$, by the minimality of G the subgroup

$$X_1 = \langle a, [g_1, {}_m a]; g_1 \in G_1 \rangle \quad \text{where } m = |G_1|$$

is subnormal in G_1. Hence X_1 sn G and $X \leqq X_1$. But since $m = |G_1|$, $X_1 \leqq X$. Hence $X = X_1$, and X is subnormal in G, which is a contradiction. So $G = \langle a^G \rangle$. Let N be a minimal normal subgroup of G. Then XN/N sn G/N and XN sn G. Therefore $XN = G$. Since $N \in \mathfrak{A}$, $X \cap N \lhd G$. If $X \cap N = N$ then $N \leqq X$ and $G = X$. So $X \cap N = 1$ and X is a maximal subgroup of G.

Now $C_G(a) \leqq N_G(X) = X$. Hence $C_N(a) = 1$. Let N_1 be a minimal a-invariant subgroup of N. Then $[N_1, a] = N_1 = [N_1, {}_r a] \leqq X$. This contradiction shows that X sn G.

Proof of Theorem 5.1. We have $H = \langle h_i | 1 \leqq i \leqq m \rangle = \langle X_n(h_i) | 1 \leqq i \leqq m \rangle$ where $n = |G|$. By Lemma 5.1, we know that each $X_n(h_i)$ is subnormal in G where $1 \leqq i \leqq m$, and hence H is subnormal in G. Obviously the converse holds.

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5.2. *Proof of Theorem* E. To prove part (i) of Theorem E we use the finite case and a result due to Kegel [**5**] which states that a subgroup X of a polycyclic group G is subnormal in G if and only if X is subnormal in G modulo normal subgroups of finite index in G. Since the hypotheses of Theorem E remain true on taking homomorphic images of G, Theorem E (i) easily follows.

To prove Theorem E (ii) we use the following lemmas:

LEMMA 5.2. *Let G be any group. Let $X \leqq G$ and let N be a normal abelian subgroup of G. Let $a \in X$ and suppose that $\bar{X} = \langle a, [g_1, {}_r a] | g_1 \in XN \rangle$ and $XN = \bar{X}N$. Then $\bar{X} \lhd^r XN$.*

Proof. Without loss of generality, we may suppose that $G = XN$. Since $\bar{X} \cap N \lhd G$ we may also assume $\bar{X} \cap N = 1$.

Let $N_1 = N_N(\bar{X})$. Then $N_1 \lhd G$. By induction define N_i for $i > 1$ by

$$N_i/N_{i-1} = N_{N/N_{i-1}}(\bar{X}N_{i-1}/N_{i-1}).$$

We shall show that $N = N_r$.

Since $C_G(a) \leqq N_G(\bar{X})$ we have $C_N(a) \leqq N_1$ and $[n, {}_r a] \in N \cap \bar{X} = 1$. Hence $[n, {}_{r-1}a] \in N_1$. Similarly, factoring G by N_1, we obtain $[n, {}_{r-2}a] \in N_2$. By a simple induction, $N = N_r$,

$$\bar{X} \lhd \bar{X}N_1 \lhd \ldots \lhd \bar{X}N_r = G \quad \text{and} \quad \bar{X} \lhd^r G.$$

LEMMA 5.3. *Let G be a soluble group and let $a \in G$. Suppose for some integer m, $X_m(a) = X_{m+r}(a)$ for all $r \geqq 0$.*

Then $X_m(a)$ sn *G.*

Proof. We use induction on d, the derived length of G. If $d = 1$, then the lemma is trivially true. Hence assume $d > 1$ and let N be the last but one term of the derived series of G.

Let $X = X_m(a)$. By induction, XN/N sn G/N and XN sn G. Suppose $XN \lhd^s G$.

Let $\bar{X} = \langle a, [xn, {}_m a]; x \in X, n \in N \rangle$.

Then $X_{s+m}(a) \leqq \bar{X}$. By hypothesis, $X = X_{s+m}(a)$. Hence $X \leqq \bar{X} \leqq X$ and $X = \bar{X}$. So $XN = \bar{X}N$ and by Lemma 5.2 $X = \bar{X}$ sn XN.

We are now able to prove Theorem E (ii). By hypothesis, $X_n(h_i) \leqq H$ for $1 \leqq i \leqq m$. Since H is Min-by-nilpotent, for each $h_i \in H$ we can find an integer $k \geqq n$ such that $X_k(h_i) = X_{k+r}(h_i)$ for all $r \geqq 0$. Let t be the largest such k.

Then $X_{t+r}(h_i) = X_t(h_i)$ for all $r \geqq 0$, $1 \leqq i \leqq m$.

By Lemma 5.3, $X_t(h_i)$ sn G, and by the subnormal coalescence of the class of minimax groups (see for example [**3**]) we have H is subnormal in G as required.

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Texas A & M University,
College Station, Texas

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