A CHARACTERISATION AND TWO EXAMPLES OF RIESZ OPERATORS

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1. Introduction. A Riesz operator is a bounded linear operator on a Banach space which possesses a Riesz spectral theory. These operators have been studied in [5] and [6]. In §2 of this paper we characterise Riesz operators in terms of their resolvent operators. In [6] it was shown that every Riesz operator on a Hilbert space can be decomposed into the sum of compact and quasi-nilpotent parts. §3 contains an example to show that these parts cannot, in general, be chosen to commute. In §4 the eigenset of a Riesz operator is defined. It is a sequence of quadruples each of which consists of an eigenvalue, the corresponding spectral projection, index and nilpotent part. This sequence satisfies certain obvious conditions, and the question arises of the existence of a Riesz operator which has such a sequence as its eigenset. We give an example of an eigenset which has no corresponding Riesz operator.

Our nomenclature will be that of [5] and [6]. Let us recall that X is a Banach space and that $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{QN}$ and $\mathfrak{R}$ denote the subsets of linear operators on X which are bounded, compact, quasi-nilpotent and Riesz, respectively. If $T \in \mathfrak{B}$, then $\sigma(T)$ is the spectrum of T, $\rho(T)$ is its resolvent set and $r(T)$ is its spectral radius. If Y is a subspace of X which is invariant under T, then $T|Y$ denotes the restriction of T to Y. The resolvent operator

$$R(z;T) = (zI-T)^{-1}$$

is defined and bounded for $z \in \rho(T)$. If x is a non-zero element of X, then the one-dimensional subspace generated by x will be written $[x]$.

2. The resolvent characterisation. In this section the underlying space is a Banach space. Ruston [3] has characterised Riesz operators as follows.

Ruston Condition. *Let $K \in \mathfrak{B}$. Then $K \in \mathfrak{R}$ if and only if*

$$\lim_{n\to\infty} \left\{ \inf_{C \in \mathfrak{C}} \| K^n - C \| \right\}^{1/n} = 0.$$

$\mathfrak{C}$ is a uniformly closed ideal of $\mathfrak{B}$. Hence the quotient algebra $\mathfrak{B}/\mathfrak{C}$ is a Banach algebra. The Ruston condition is equivalent to the requirement that the coset $K^c = K + \mathfrak{C}$ be quasi-nilpotent in $\mathfrak{B}/\mathfrak{C}$.

We shall require a lemma on analytic functions of compact operators.

Lemma. *Let D be a connected open set in the complex plane containing a non-empty open set U. If $F(z)$: $D \to \mathfrak{B}$ is analytic in D and if $F(z) \in \mathfrak{C}$ for $z \in U$, then $F(z) \in \mathfrak{C}$ for $z \in D$.*

Proof. (Due to A. Lebow). $F(z)$ is analytic in D; hence so also is its image $F(z)^c$ in $\mathfrak{B}/\mathfrak{C}$. Now

$$F(z)^c = 0 \qquad (z \in U),$$

by hypothesis. But D is connected. Therefore, by the principle of analytic continuation ([**1**], p. 202),

$$F(z)^c = 0 \quad (z \in D).$$

This proves the lemma.

RESOLVENT CONDITION. *Let $K \in \mathfrak{B}$. Then $K \in \mathfrak{R}$ if and only if*

$$R(z; K) = C(z) + B(z) \qquad (z \in \rho(K)),$$

where $C(z) \in \mathfrak{C}$ for $z \in \rho(K)$ and $B(z)$ is an entire function of z^{-1}.

Proof. We show that the Ruston and resolvent conditions are equivalent.

Let K satisfy the resolvent condition. Then, for $|z| > r(K)$, $R(z; K)$ and $B(z)$ have expansions in powers of z^{-1}; thus so also has $C(z)$. Therefore

$$\sum_0^\infty z^{-n-1} K^n = \sum_0^\infty z^{-n-1} C_n + \sum_0^\infty z^{-n-1} B_n \qquad (|z| > r(K)),$$

where $\| B_n \|^{1/n} \to 0$ as $n \to \infty$.

Formula (9.3.5.1) on p. 199 of [**1**] shows that $C_n \in \mathfrak{C}$ for each n. Also

$$K^n = C_n + B_n \qquad (n \geqq 0). \tag{1}$$

Hence K satisfies the Ruston condition.

Conversely, let K satisfy the Ruston condition. Then we can find sequences $\{C_n\}$ in $\mathfrak{C}$ and $\{B_n\}$ with $\| B_n \|^{1/n} \to 0$ as $n \to \infty$ which satisfy equation (1). Take $|z| > r(K)$. Then

$$R(z; K) = \sum_0^\infty z^{-n-1} K^n = \sum_0^\infty z^{-n-1} C_n + \sum_0^\infty z^{-n-1} B_n.$$

The convergence of these last two series follows from the condition on the sequence $\{B_n\}$. Thus, for $|z| > r(K)$,

$$R(z; K) = C(z) + B(z),$$

where $B(z)$ is an entire function of z^{-1} and $C(z) \in \mathfrak{C}$. $C(z)$ is defined and analytic for $z \in \rho(K)$. If we show that $\rho(K)$ is connected, an application of the lemma will complete the proof.

To do this, let μ be a fixed positive real number. We can choose a positive integer n such that $\| B_n \| < \mu^n$. Then, for $|\lambda| \geqq \mu$, $A_n = I - \lambda^{-n} B_n$ is a homeomorphism and

$$\lambda^n I - K^n = \lambda^n I - B_n - C_n = A_n(\lambda^n I - A_n^{-1} C_n).$$

Thus $\lambda^n \in \sigma(K^n)$ if and only if $\lambda^n \in \sigma(A_n^{-1} C_n)$. But $A_n^{-1} C_n \in \mathfrak{C}$. Therefore, if $\lambda^n \in \sigma(K^n)$, then λ^n is an eigenvalue of $A_n^{-1} C_n$, and hence is an eigenvalue of K^n. Thus, if $\lambda \in \sigma(K)$ and $|\lambda| \geqq \mu$, then λ^n is an eigenvalue of K^n. It can now be shown that there are only a finite number of points λ in $\sigma(K)$ with $|\lambda| \geqq \mu$. (See the proof of Lemma 3.5 in [3].) As μ is arbitrary, it follows that the non-zero spectrum of K is discrete. Hence $\rho(K)$ is connected.

3. First example. Let X be a Hilbert space. By [6], Theorem 7.5, if $K \in \mathfrak{R}$, then $K = C+Q$, where $C \in \mathfrak{C}$ and $Q \in \mathfrak{Q}\mathfrak{N}$. This decomposition is not unique. We shall construct a K for which C and Q cannot be chosen to commute.

Take X to be l^2 and let $\{e_j\}_1^\infty$ be an orthonormal basis for X. Define linear operators C and Q on X as follows:

$$Ce_j = j^{-1} e_j \qquad (j \geqq 1),$$

and

$$\left.\begin{aligned} Qe_{2j-1} &= e_{2j}, \\ Qe_{2j} &= 0 \end{aligned}\right\} \qquad (j \geqq 1).$$

Clearly $C \in \mathfrak{C}$ and $Q^2 = 0$. Put $K = C+Q$; then $K^2 \in \mathfrak{C}$ and hence $K \in \mathfrak{R}$. However, $K \notin \mathfrak{C}$, since

$$Ke_{2j-1} = (2j-1)^{-1} e_{2j-1} + e_{2j} \qquad (j \geqq 1).$$

Hence $\{Ke_{2j-1}\}_1^\infty$ does not contain a convergent subsequence. A simple calculation shows that the non-zero eigenvalues and the corresponding eigenspaces of K are given by the formulae:

$$\lambda_j = j^{-1} \qquad (j \geqq 1),$$

and

$$\left.\begin{aligned} E_{2j-1} &= [e_{2j-1} + (2j-1)(2j)\, e_{2j}], \\ E_{2j} &= [e_{2j}] \end{aligned}\right\} \qquad (j \geqq 1). \tag{2}$$

Suppose now that $K = C_1 + Q_1$, where $C_1 \in \mathfrak{C}$, $Q_1 \in \mathfrak{Q}\mathfrak{N}$ and $C_1 Q_1 = Q_1 C_1$. It follows that $KQ_1 = Q_1 K$. Hence if $x \in E_j$ for any j, then

$$K(Q_1 x) = Q_1(Kx) = Q_1(j^{-1}x) = j^{-1}(Q_1 x)$$

and therefore $Q_1 x \in E_j$. Thus E_j is invariant under Q_1. Since E_j is a one-dimensional subspace, $\sigma(Q_1 \mid E_j)$ consists of one eigenvalue. Hence

$$\sigma(Q_1 \mid E_j) \subset \sigma(Q_1) = \{0\},$$

and so $Q_1 \mid E_j$ is the zero operator. Formula (2) shows that $Q_1 e_j = 0$ for each j. Accordingly $Q_1 = 0$ and $K = C_1 \in \mathfrak{C}$, which is a contradiction.

4. Second Example. Let X be an infinite dimensional Hilbert space. Let $K \in \mathfrak{R}$ have an infinite spectrum (the case of a finite spectrum is trivial). Then, associated with K, we have the following sequence of quadruples:

$$\left\{\lambda_j, \quad P_j, \quad \nu_j, \quad Q_j\right\}_1^\infty,$$

where (i) λ_j are complex numbers tending to zero as $j \to \infty$,

(ii) P_j are projections on X with finite dimensional ranges N_j, such that

$$P_j P_k = P_k P_j = 0 \qquad (j \neq k),$$

(iii) ν_j are integers greater than or equal to one,

(iv) Q_j are nilpotent operators on N_j with order of nilpotence equal to ν_j.

Of course λ_j are the eigenvalues of K, P_j the corresponding spectral projections, ν_j the indices of the eigenvalues and

$$K \mid N_j = \lambda_j I_j + Q_j,$$

where I_j is the unit operator on N_j. This sequence of quadruples we call the *eigenset* of K.

We now formulate a general problem: Given a sequence of quadruples satisfying conditions (i)–(iv), does there exist a $K \in \mathfrak{R}$ with this sequence as its eigenset? This appears to be a difficult problem. Some of its complexities are illustrated in Hamburger's work [2].

We simplify matters by requiring that $\nu_j = 1$ for each j, and hence that each $Q_j = 0$. If the sequence $\{\| P_j \|\}_1^\infty$ is bounded, it is easy to see that the problem has an affirmative solution. For then, by a theorem due to Lorch and Mackey which is proved in [4], there exists an invertible A in $\mathfrak{B}$ such that

$$A^{-1} P_j A = W_j \qquad (j \geqq 1),$$

where W_j is a self-adjoint projection. The W_j's obviously satisfy condition (ii). The series $\sum_1^\infty \lambda_j W_j$ is uniformly convergent to an element C of $\mathfrak{C}$. Thus $ACA^{-1} \in \mathfrak{C}$ and this has the required properties.

Our example shows that, if the sequence $\{P_j\}_1^\infty$ is not uniformly bounded, there may exist no such Riesz operator. Again take $\nu_j = 1$ for each j, and take X to be l^2. Let $x \in l^2$ and define the sequence $\{P_j\}_1^\infty$ by the equations

$$\begin{aligned} x &= (x_1, x_2, x_3, x_4, \ldots), \\ P_1 x &= (x_1 - \mu_1 x_2, 0, 0, 0, \ldots), \\ P_2 x &= (\mu_1 x_2, x_2, 0, 0, \ldots), \\ P_3 x &= (0, 0, x_3 - \mu_2 x_4, 0, \ldots), \\ P_4 x &= (0, 0, \mu_2 x_4, x_4, \ldots) \end{aligned}$$

and so on, where the μ_j are complex numbers. This sequence clearly satisfies (ii). Consider the operator $\lambda_1 P_1 + \lambda_2 P_2$ restricted to the subspace $N_1 \oplus N_2$. We have

$$\begin{aligned} \| \lambda_1 P_1 + \lambda_2 P_2 \mid N_1 \oplus N_2 \|^2 &= \sup_{(x_1, x_2) \neq (0,0)} \frac{|\lambda_1 x_1 + (\lambda_2 - \lambda_1)\mu_1 x_2|^2 + |\lambda_2 x_2|^2}{|x_1|^2 + |x_2|^2} \\ &\geqq |(\lambda_2 - \lambda_1)\mu_1|^2. \end{aligned}$$

Similarly

$$\left\| \sum_1^{2n} \lambda_j P_j \,\Big|\, \bigoplus_1^{2n} N_j \right\| \geqq \left| (\lambda_{2n} - \lambda_{2n-1})\mu_n \right| \qquad (n \geqq 1). \tag{3}$$

Now choose $\lambda_j = j^{-1}$ and $\mu_j = j^3$ for each j. Then the right hand side of equation (3) tends to infinity with n.

Suppose that there exists $K \in \mathfrak{R}$ with the required properties. Each Q_j is zero and hence

$$K \,\Big|\, \bigoplus_1^{2n} N_j = \sum_1^{2n} \lambda_j P_j \,\Big|\, \bigoplus_1^{2n} N_j \qquad (n \geqq 1).$$

Thus

$$\left\| \sum_1^{2n} \lambda_j P_j \,\Big|\, \bigoplus_1^{2n} N_j \right\| \leqq \| K \| \qquad (n \geqq 1),$$

which gives an obvious contradiction.

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