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ON A QUESTION OF COLIN CLARK CONCERNING THREE PROPERTIES OF CONVEX SETS(1)

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Let C be a convex subset of a normed linear space E. The following properties of C were studied by Clark [2]:

(1) C is *of finite width;* that is, C lies between two parallel closed hyperplanes;

(2) C is *of finite width in some direction;* that is, for some line L in E there is a finite upper bound for the lengths of C's intersections with lines parallel to L;

(3) C is *partially bounded;* that is, there is a finite upper bound for the radii of balls contained in C.

Plainly (1)$\Rightarrow$(2)$\Rightarrow$(3). Clark [2] proved the reverse implications when E is finite-dimensional, and asked what happens in the infinite-dimensional case. We show here that both of the reverse implications are invalid in an arbitrary infinite-dimensional separable Banach space, even if C is required to be closed. When C has nonempty interior, (2)$\Rightarrow$(1) but (3) $\not\Rightarrow$ (2). All of the necessary background for understanding the constructions and arguments can be found in [1] and [3].

Let us first consider an arbitrary infinite-dimensional normed linear space E. Let B be a Hamel basis for E, linearly ordered so as to have no last element, and let C denote the convex cone consisting of all points x of E such that the last nonzero coordinate in x's canonical expression in terms of B is positive. Then C has property (3) but not property (2). Any dense hyperplane in E has property (2) but not property (1). These examples are dense rather than closed.

If E is the space (l^p) for $1 \leq p < \infty$, it is easily verified that the closed convex cone consisting of all nonnegative members of E has property (3) but not property (2).

Now we will show that if E is an arbitrary infinite-dimensional separable Banach space, then E contains a closed convex cone which has property (2) but not property (1) and a closed convex body which has property (3) but not property (2). Let H be a closed hyperplane through the origin in E, let U denote the unit ball of E, and let q be a point of $E \sim H$. Let $u_1, u_2, \ldots$ be a sequence dense in $U \cap H$, and let X denote the closed convex hull of the set

$$\{0, -u_1, u_1, -u_2/2, u_2/2, \ldots, -u_n/n, u_n/n, \ldots\}.$$

Being the closed convex hull of a compact set, X is itself compact. As H is not

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locally compact, a simple category argument shows that the linear subspace $RX=[0, \infty[X$ is not all of H.

Now let K denote the closed convex cone consisting of all nonnegative multiples of points of $q+X$; that is, $K=[0, \infty[(q+X)$. As X lies in a proper subspace of H, K lies in a proper subspace of E and hence has zero width in many directions. In particular, K has property (2). However, K does not have property (1). If it had, there would be a nonzero continuous linear functional f on E such that the numbers $\inf fK$ and $\sup fK$ are both finite. As $[0, \infty[K=K$, it follows that $fK=\{0\}$, whence $f(q)=0, f(u_n)=0$ for all n, $fH=\{0\}$, and f is the zero functional.

To obtain a closed convex body C which has property (3) but not property (2), let $C=K+U$, the set of all points at distance ≤ 1 from K. We will show that C does not contain any ball of radius $>\rho$, where $\rho>0$ is such that the ball $(2/\rho)U$ is contained in the vector sum $(U \cap H)+[-q, q]$. Consider an arbitrary ball $B \subset C$. In order to show that the radius of B does not exceed ρ, it suffices to produce a continuous linear functional f on E such that $\|f\|\geq 1$ and the variation of f on B is at most 2ρ. Note that there exists $\tau \varepsilon]0, \infty[$ such that

$$(*) \qquad B \subset [0, \tau](q+X)+U.$$

By the standard separation theorem, in conjunction with the fact that the origin is not interior to the compact convex set X relative to H, there exists a continuous linear functional g on H such that $\sup g(U \cap H)=1$ but $\sup gX<\rho/(2\tau)$. Let the linear functional f on E be defined by the condition that $f=g$ on H and $f(q)=0$. Then $\|f\|\geq 1$ and $\sup f(2/\rho)U\leq 1$, so $\|f\|\leq \rho/2$. As the variation of f on each of the sets $[0, \tau](q+X)$ and U is at most ρ, it follows from (*) that f's variation on B is at most 2ρ. Hence C has property (3).

To prove that the set $C=K+U$ lacks property (2), we show that for each $h \in H$ and each $\mu\geq 0$, C contains a translate of the segment $[0, h+\mu q]$. Note first that the set $[0, \infty[X$ is dense in H, and hence there exists $x \in X$, $\alpha>0$, and $u \in U \cap H$ such that $h=\alpha x+u$. From the definition of C it then follows that C includes the points αq and $\alpha q+h$, and with $\alpha q+h \in C$ and $[0, \infty[q \subset C$ it follows from the closedness and convexity of C that $\alpha q+h+\mu q \in C$. But then $[\alpha q, \alpha q+h+\mu q] \subset C$, and C lacks property (2).

It remains only to show that if E is a topological linear space and C is a convex subset of E such that $\operatorname{int} C\neq\phi$ and C has property (2), then C also has property (1). We assume without loss of generality that C is closed. (Note that property (2) can be formulated in an arbitrary real vector space, as it involves only a family of mutually parallel segments.)

By property (2), there is a line L through the origin in E and there is a point q of $L\sim\{0\}$ such that if L' is any translate of L, then the intersection of L' with the closed convex body C is contained in a translate of the segment $[0, q]$. In particular, there is a point p of $E\sim C$ such that the ray $p+[0, \infty[q$ intersects the interior of C, and then, by a standard separation theorem, there is a closed hyperplane H

through p such that C lies in the closed halfspace $H+[0, \infty[q$. We assume without loss of generality that p is the origin, so that E is the direct sum, both algebraically and topologically, of its subspaces H and Rq. Let π denote the projection of E onto H, whence πC is a convex set whose interior S relative to H is nonempty. For each point $s \in S$, let $\xi(s)$ and $\eta(s)$ denote respectively the smallest and the largest values of τ for which $s+\tau q \in C$. Then ξ is a convex function on the relatively open convex set S and η is a concave function on S, with $0 \leq \xi < \eta \leq \xi + 1$. It follows that ξ and η are both continuous, with $\eta - 1 \leq \xi$. Now consider the sets

$$A = \{s+\lambda q : s \in S, \lambda < \eta(s)-1\} \quad \text{and} \quad B = \{s+\lambda q : s \in S, \lambda > \xi(s)\}.$$

Being disjoint open convex sets, they are separated in E by a closed hyperplane J, and it can be verified that C lies between the hyperplanes J and $J+q$.

Added in Proof. It is proved above that every infinite-dimensional separable Banach space (a) contains a closed convex body that is partially bounded but not of finite width in any direction and (b) contains a closed convex set that is of infinite width but is of finite width in some direction. After the present paper had been accepted for publication, there appeared a paper of Thorp and Whitley [4], submitted more than a year before the present paper, which establishes (a) for all infinite-dimensional separable Banach spaces and (b) for all infinite-dimensional Banach spaces having Schauder bases. They also observe that a Banach space X has property (a) or property (b) if X has a quotient space with the stated property.

References

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