Canad. Math. Bull. Vol. **48** (2), 2005 pp. 161–174

Hankel Convolution Operators on Spaces of Entire Functions of Finite Order

Jorge J. Betancor

Abstract. In this paper we study Hankel transforms and Hankel convolution operators on spaces of entire functions of finite order and their duals.

1 Introduction

Our objective in this note is to study Hankel transforms and Hankel convolution operators of entire functions of finite order and their duals. Our investigation is inspired by the ideas developed by Ehrenpreis [5]. However we need to introduce new procedures to prove the results in the Hankel setting. Some of the arguments used here are simpler than the one considered in [5, §5]. Moreover our procedures can be used to prove the results in [5, §5] about the usual convolution operators and Fourier transforms of entire functions of finite order.

The Hankel transform is defined as follows

$$h_\mu(\phi)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\phi(x)x^{2\mu+1}\,dx,$$

where ϕ is, for instance, a function in the Lebesgue space $L^1(x^{2\mu+1}dx)$. Here J_μ represents the Bessel function of the first kind and order μ. Throughout this paper μ will be greater than $-\frac{1}{2}$.

The Hankel convolution operations were investigated by Haimo [8] and Hirschman [9] in the Lebesgue space $L^p(x^{2\mu+1}dx)$, $1 \le p \le \infty$. If $f, g \in L^1(x^{2\mu+1}dx)$, then the Hankel convolution $f\#_\mu g$ of f and g is defined by

$$(f\#_\mu g)(x) = \int_0^\infty f(y)\,{}_\mu\tau_x(g)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dx, \quad x \in (0,\infty),$$

where the Hankel translated ${}_\mu\tau_x(g)$ of g by $x \in (0,\infty)$ is given by

$${}_\mu\tau_x(g)(y) = \int_0^\infty D_\mu(x,y,z)g(z)\frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz, \quad y \in (0,\infty),$$

Received by the editors April 25, 2003.
Partially supported by Consejería de Educación, Gobierno Atónomo de Canarias, Proyecto PI 2001/039
AMS subject classification: 46F12.
Keywords: Hankel transform, convolution, entire functions, finite order.

and ${}_\mu\tau_0(g) = g$. The function D_μ is the Delsarte kernel given by

$$D_\mu(x,y,z) = (2^\mu\Gamma(\mu+1))^2 \int_0^\infty (xt)^{-\mu} J_\mu(xt)(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} J_\mu(zt)t^{2\mu+1}\,dt,$$
$$x, y, z \in (0,\infty).$$

The Hankel transform h_μ is related with the Hankel convolution and the Hankel translations of order μ by the following formulas (see [9])

$$h_\mu(f\#_\mu g) = h_\mu(f)h_\mu(g),$$

and

$$h_\mu({}_\mu\tau_x g) = 2^\mu\Gamma(\mu+1)(x.)^{-\mu} J_\mu(x.)h_\mu(g), \quad x \in (0,\infty),$$

that are valid when, for instance, f and g are in $L^1(x^{2\mu+1}dx)$.

The study of the Hankel convolutions and Hankel translations on spaces of entire functions was started by Belhadj and Betancor [1]. They extended the definition of the Hankel translation to the complex plane. By $\mathcal{H}_e$ we denote the space of even and entire functions. According to [1] (see also [4]), if $f(z) = \sum_{n=0}^\infty a_n z^{2n}$, $z \in \mathbf{C}$, is in $\mathcal{H}_e$ and $w \in \mathbf{C}$, the Hankel translation ${}_\mu\tau_w f$ is defined by

$${}_\mu\tau_w(f)(z) = \sum_{n=0}^\infty a_n \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\mu+1)\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} w^{2(n-k)} z^{2k}, \quad z \in \mathbf{C}.$$

Thus ${}_\mu\tau_w$ defines a continuous linear mapping from $\mathcal{H}_e$ into itself. The Hankel convolution $T\#_\mu f$ of $T \in \mathcal{H}_e'$, the dual space of $\mathcal{H}_e$, and $f \in \mathcal{H}_e$ is defined by

$$(T\#_\mu f)(z) = \langle T, {}_\mu\tau_z(f)\rangle, \quad z \in \mathbf{C}.$$

In this note we analyze the Hankel translation and the Hankel convolution on even and entire functions of finite order. Let $a > 1$. We represent by $Z_{a,e}$ the space of all the even and entire functions that have order $\le a$. That is, an even and entire function f is in $Z_{a,e}$ if, and only if, for every $\varepsilon > 0$,

$$|f(z)| = O(\exp(|z|^{a+\varepsilon})), \text{ as } z \to \infty.$$

The space $Z_{a,e}$ is endowed with the topology associated with the family $\{p_{a,\varepsilon}\}_{\varepsilon>0}$ of seminorms, where, for every $\varepsilon > 0$,

$$p_{a,e}(f) = \sup_{z\in\mathbf{C}} \exp(-|z|^{a+\varepsilon})|f(z)|, \quad f \in Z_{a,e}.$$

Thus $Z_{a,e}$ is a Fréchet space. The dual of $Z_{a,e}$ is denoted, as usual, by $Z_{a,e}'$.

By $Q_{a,e}$ we represent the space of even and entire functions having order less than a.

For every $z \in \mathbf{C}$, the function $(zt)^{-\mu} J_\mu(zt)$, $t \in \mathbf{C}$, is in $Z_{a,e}$. Then we define the Hankel transform $h_\mu(T)$ of $T \in Z'_{a,e}$ as follows

$$h_\mu(T)(z) = 2^\mu \Gamma(\mu+1)\langle T(t), (zt)^{-\mu} J_\mu(zt)\rangle, \quad z \in \mathbf{C}.$$

We establish that h_μ is a one-to-one mapping from $Z'_{a,e}$ onto $Q_{a',e}$. Here and in the sequel, a' denotes the exponent conjugate to a. Also we prove that for every $z \in \mathbf{C}$, the Hankel translation ${}_\mu\tau_z$ defines a continuous linear mapping from $Z_{a,e}$ into itself. If $T \in Z'_{a,e}$ and $f \in Z_{a,e}$, the Hankel convolution $T\#_\mu f$ of T and f is defined by

$$(T\#_\mu f)(z) = \langle T, {}_\mu\tau_z f\rangle, \quad z \in \mathbf{C}.$$

We prove that T defines a continuous Hankel convolution operator from $Z_{a,e}$ into itself. Moreover, if $T \neq 0$, then the Hankel convolution operator generated by T on $Z_{a,e}$ is surjective.

Throughout this paper, by C we denote a suitable positive constant not necessarily the same in each occurrence.

2 Hankel Transforms and Hankel Convolutions on the Space $Z'_{a,e}$

First we analyze the Hankel transform on the space $Z'_{a,e}$.

Let $a > 1$. For every $w \in \mathbf{C}$, the function f_w defined by

$$f_w(z) = 2^\mu \Gamma(\mu+1)(zt)^{-\mu} J_\mu(zt), \quad z \in \mathbf{C},$$

is in $Z_{a,e}$. Indeed, let $w \in \mathbf{C}$. It is clear that f_w is an even and entire function. Moreover, according to [6, (5.3.b)] and [5, (52)′], we have that

$$|f_w(z)| \le Ce^{|zw|} \le C\exp\Big(\frac{1}{a}|z|^a + \frac{1}{a'}|w|^{a'}\Big), \quad z \in \mathbf{C}.$$

Let $T \in Z'_{a,e}$. We define the Hankel transform $h_\mu T$ of T through

$$h_\mu(T)(w) = 2^\mu \Gamma(\mu+1)\langle T(z), (zw)^{-\mu} J_\mu(zw)\rangle, \quad w \in \mathbf{C}.$$

By [5, Proposition 3], we can write

$$h_\mu(T)(w) = 2^\mu \Gamma(\mu+1)\sum_{n=0}^{\infty} \frac{(-1)^k w^{2k}}{2^{\mu+2k}\Gamma(\mu+k+1)}\langle T(z), z^{2k}\rangle, \quad w \in \mathbf{C}. \tag{2.1}$$

Hence $h_\mu(T)$ is an even and entire function.

On the other hand, since $T \in Z'_{a,e}$ there exist $C, \varepsilon > 0$ such that

$$|\langle T, f\rangle| \le C\sup_{z\in\mathbf{C}} \exp(-|z|^{a+\varepsilon})|f(z)|, \quad f \in Z_{a,e}.$$

In particular, for every $w \in \mathbf{C}$, we have

$$\begin{aligned}|h_\mu(T)(w)| &\le C \sup_{z\in\mathbf{C}} \exp(-|z|^{a+\varepsilon})|(zw)^{-\mu} J_\mu(zw)| \\ &\le C \sup_{z\in\mathbf{C}} \exp(-|z|^{a+\varepsilon} + |z||w|) \\ &\le C \sup_{z\in\mathbf{C}} \exp\Big(-|z|^{a+\varepsilon} + \frac{1}{a+\varepsilon}|z|^{a+\varepsilon} + \frac{1}{a'-\eta}|w|^{a'-\eta}\Big) \\ &\le C \exp\Big(\frac{1}{a'-\eta}|w|^{a'-\eta}\Big).\end{aligned}$$

Here a' is the conjugate exponent of a and $0 < \eta < a' - 1$ such that $\frac{1}{a+\varepsilon} + \frac{1}{a'-\eta} = 1$. Hence $h_\mu(T)$ is in the linear space $Q_{a',e}$ that consists of all the even and entire functions of order less than a'.

Suppose now that $h_\mu(T) = 0$. Then according to (2.1), $\langle T(z), z^{2k}\rangle = 0$, for every $k \in \mathbf{N}$. By [5, Proposition 3] the linear space span$\{z^{2k} : k \in \mathbf{N}\}$ generated by $\{z^{2k} : k \in \mathbf{N}\}$ is dense in $Z_{a,e}$. Hence $\langle T, f\rangle = 0$, for each $f \in Z_{a,e}$.

The results that we have just proved can be summarized in the following.

Proposition 2.1 *The Hankel transform h_μ is a one-to-one mapping from $Z'_{a,e}$ into $Q_{a',e}$, for every $a > 1$.*

Our next objective is to see that $h_\mu(Z'_{a,e}) = Q_{a',e}$.

Next we analyze the behaviour of the Bessel operator $\Delta_\mu = z^{-2\mu-1}Dz^{2\mu+1}D$ on $Z_{a,e}$.

Proposition 2.2 *Let $a > 1$. The Bessel operator Δ_μ defines a continuous linear mapping from $Z_{a,e}$ into itself.*

Proof Suppose that $f(z) = \sum_{n=0}^\infty a_n z^{2n}$, $z \in \mathbf{C}$, is in $Z_{a,e}$. Then

$$\Delta_\mu f(z) = \sum_{n=0}^\infty 4n(n+\mu)a_n z^{2n}, \quad z \in \mathbf{C}.$$

By using [5, Proposition 3] (see also [2, Proposition 4.5.3]), we have that

$$\begin{aligned}\liminf_{n\to\infty} \frac{-\log(4n(n+\mu)|a_n|)}{2n\log(2n)} &= \liminf_{n\to\infty} \frac{-\log(4n(n+\mu)) - \log|a_n|}{2n\log(2n)} \\ &\ge \liminf_{n\to\infty} \frac{-\log|a_n|}{2n\log(2n)} \\ &\ge \frac{1}{a}.\end{aligned}$$

Hence, $\Delta_\mu f \in Z_{a,e}$.

https://doi.org/10.4153/CMB-2005-014-2 Published online by Cambridge University Press

The continuity of Δ_μ follows from the closed graph theorem. Indeed, assume that $(f_n)_{n\in\mathbf{N}} \subset Z_{a,e}$ is such that $f_n \to f$ and $\Delta_\mu f_n \to g$, as $n \to \infty$, in $Z_{a,e}$, where $f, g \in Z_{a,e}$. Since the convergence in $Z_{a,e}$ implies the convergence in $\mathcal{H}_e$ and Δ_μ is continuous from $\mathcal{H}_e$ into itself, $\Delta_\mu f = g$. Thus we conclude that Δ_μ is continuous from $Z_{a,e}$ into itself. ∎

The operator Δ_μ is defined on $Z'_{a,e}$, as usual, by transposition.

According to [12, (6) and (7), Chapter 5] if δ denotes the Dirac functional we have that

$$h_\mu((-\Delta_\mu)^k\delta)(z) = z^{2k}, \quad z \in \mathbf{C} \text{ and } k \in \mathbf{N}.$$

Indeed, if $k \in \mathbf{N}$,

$$\begin{aligned} h_\mu((-\Delta_\mu)^k\delta)(z) &= \langle(-\Delta_\mu)^k\delta(t), 2^\mu\Gamma(\mu+1)(zt)^{-\mu}J_\mu(zt)\rangle \\ &= \langle\delta(t), 2^\mu\Gamma(\mu+1)(-1)^k\Delta^k_{\mu,t}((zt)^{-\mu}J_\mu(zt))\rangle \\ &= \langle\delta(t), 2^\mu\Gamma(\mu+1)z^{2k}(zt)^{-\mu}J_\mu(zt)\rangle \\ &= z^{2k}, \quad z \in \mathbf{C}. \end{aligned}$$

Proposition 2.3 *Suppose that $f(z) = \sum_{n=0}^\infty a_n z^{2n}$, $z \in \mathbf{C}$, is a function in $Q_{a',e}$. Then the series $\sum_{n=0}^\infty a_n(-\Delta_\mu)^k\delta$ converges in the weak * (equivalently in the strong) topology of $Z'_{a,e}$.*

Proof Assume that $g(z) = \sum_{n=0}^\infty b_n z^{2n}$, $z \in \mathbf{C}$, is in $Z_{a,e}$. We can write

$$\begin{aligned} \Big\langle \sum_{k=0}^n (-1)^k a_k \Delta_\mu^k\delta, g\Big\rangle &= \sum_{k=0}^n a_k(-1)^k(\Delta_\mu^k g)(0) \\ &= \sum_{k=1}^n (-1)^k a_k b_k 2^{2k} k!\,\Gamma(\mu+k+1), \quad n \in \mathbf{N}. \end{aligned}$$

Since $f \in Q_{a',e}$, there exists $\eta \in (0, a'-1)$ such that $f \in Z_{a'-\eta,e}$. By [5, Proposition 3] we can choose $\varepsilon_1, \varepsilon_2 > 0$ and $k_0 \in \mathbf{N}$ such that

$$\frac{-\log|a_k|}{2k\log(2k)} \geq \frac{1}{a'-\varepsilon_1} \quad\text{and}\quad \frac{-\log|b_k|}{2k\log(2k)} \geq \frac{1}{a+\varepsilon_2},$$

for every $k \in \mathbf{N}$, $k \geq k_0$, where $\frac{1}{a'-\varepsilon_1} + \frac{1}{a+\varepsilon_2} = 1$. Hence, for each $k \geq k_0$,

$$|a_k||b_k| \leq (2k)^{-2k(\frac{1}{a'-\varepsilon_1}+\frac{1}{a+\varepsilon_2})} = (2k)^{-2k}.$$

Then, by using the Stirling formula, we obtain

$$\begin{aligned} |a_k||b_k|2^{2k}k!\,\Gamma(\mu+k+1) &\leq C(2k)^{-2k}k^k e^{-k}\sqrt{2\pi k}\,2^{2k}(\mu+k)^{\mu+k}e^{-\mu-k}\sqrt{2\pi(\mu+k)} \\ &\leq Ce^{-2k}k^{\mu+1}, \quad k \geq k_0. \end{aligned}$$

Therefore the series $\sum_{k=0}^\infty (-1)^k a_k b_k 2^{2k} k!\,\Gamma(\mu+k+1)$ is convergent.

Thus, according to [7, 5.b, p. 242], the proof is completed. ∎

By combining Propositions 2.1 and 2.3 we can obtain the following result.

Proposition 2.4 *Let $a > 1$. The Hankel transform h_μ is a one-to-one mapping from $Z'_{a,e}$ onto $Q_{a',e}$.*

Proof We proved in Proposition 2.1 that h_μ is a one-to-one mapping from $Z'_{a,e}$ into $Q_{a',e}$.

Now let $f \in Q_{a',e}$. We define the functional T on $Z'_{a,e}$ through

$$\langle T, g\rangle = \sum_{n=0}^{\infty} a_n(-1)^n \langle \Delta_\mu^n \delta, g\rangle, \quad g \in Z_{a,e},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$, $z \in \mathbf{C}$. According to Proposition 2.3, $T \in Z'_{a,e}$ and we have that

$$h_\mu(T)(z) = \sum_{n=0}^{\infty} a_n(-1)^n \langle \Delta_\mu^n \delta(t), (zt)^{-\mu} J_\mu(zt)\rangle = f(z), \quad z \in \mathbf{C}.$$

Hence h_μ defines a mapping from $Z'_{a,e}$ onto $Q_{a',e}$. ■

According to [5, Proposition 1] $Z_{a,e}$ is a Montel and Schwartz space. Hence the strong dual $Z'_{a,e}$ of $Z_{a,e}$ is bornological [7, p. 257]. We consider on $Q_{a',e}$ the topology induced by $Z_{a,e}$ via the Hankel transform h_μ, and then $Q_{a',e}$ is denoted by $Q^\mu_{a',e}$. We will prove that $Q^\mu_{a',e} = Q^\nu_{a',e}$, provided that $\mu, \nu > -\frac{1}{2}$. We first describe the bounded sets of $Q^\mu_{a',e}$. Before making this we note that if f is an even and entire function, we can write

$$f(z) = \sum_{n=0}^{\infty} \frac{\Delta_\mu^k f(0)}{2^{2k}\Gamma(\mu+k+1)k!} z^{2k}, \quad z \in \mathbf{C},$$

where Δ_μ represents the Bessel operator $\Delta_\mu = z^{-2\mu-1} D z^{2\mu+1} D$. Then the following formula

$$\Delta_\mu^k f(0) = \frac{2^{2k}\Gamma(\mu+k+1)k!}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{w^{2k+1}}\, dw, \tag{2.2}$$

holds for every $k \in \mathbf{N}$, where, for every $r > 0$, Γ_r denotes the circular path $\Gamma : w(t) = re^{it}$, $t \in [0, 2\pi)$. This formula will be useful in the sequel.

Proposition 2.5 *Let $a > 1$ and let B be a subset of $Q^\mu_{a,e}$. Then B is a bounded subset of $Q^\mu_{a,e}$ if, and only if, for some $\varepsilon > 0$ there exists $M > 0$ for which*

$$|F(z)| \le M \exp(|z|^{a-\varepsilon}), \quad z \in \mathbf{C} \text{ and } F \in B. \tag{2.3}$$

https://doi.org/10.4153/CMB-2005-014-2 Published online by Cambridge University Press

Proof First suppose that B is a bounded subset of $Q_{a,e}$.

We define $W = h_\mu^{-1}(B)$. Thus W is a bounded set of $Z'_{a',e}$. Then there exists $C, \eta > 0$ for which

$$|\langle T, f\rangle| \leq C \sup_{z\in\mathbf{C}} \exp(-|z|^{a'+\eta})|f(z)|, \quad f \in Z_{a',e} \text{ and } T \in W.$$

By proceeding now as in the proof of Proposition 2.1, we can find $\varepsilon > 0$ such that for a certain $M > 0$ we have

$$|F(z)| \leq M \exp(|z|^{a-\varepsilon}), \quad z \in \mathbf{C} \text{ and } F \in B.$$

Assume now that for some $\varepsilon > 0$ such that $a - \varepsilon > 1$, there exists $M > 0$ for which (2.3) holds.

Our objective is to see that $W = h_\mu^{-1}(B)$ is a bounded subset of $Z'_{a',e}$. We choose $\eta > 0$ being $\frac{1}{a-\varepsilon} + \frac{1}{a'+\eta} = 1$. By [5, Proposition 3], if $f \in Z_{a',e}$,

$$\langle T, f\rangle = \sum_{k=0}^{\infty} a_k \langle T, z^{2k}\rangle, \quad T \in Z'_{a',e},$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^{2k}$, $z \in \mathbf{C}$. Hence we can write, for every $F \in Q_{a,e}$,

$$\begin{aligned}\langle h_\mu^{-1}(F), f\rangle &= \sum_{k=0}^{\infty} a_k \langle h_\mu^{-1}(F), z^{2k}\rangle \\ &= \sum_{k=0}^{\infty} (-1)^k a_k (\Delta_\mu^k F)(0),\end{aligned}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^{2k}$, $z \in \mathbf{C}$, is in $Z_{a',e}$. Let $F \in B$. By (2.2), we have

$$(\Delta_\mu^k F)(0) = \frac{2^{2k} k!\, \Gamma(\mu + k + 1)}{2\pi i} \int_{\Gamma_r} \frac{F(w)}{w^{2k+1}}\, dw, \quad k \in \mathbf{N},$$

where Γ_r denotes the circular path $\Gamma_r : w(t) = re^{it}$, $t \in [0, 2\pi)$. In particular, for every $k \in \mathbf{N}$, by taking

$$r = \left(\frac{2k}{a-\varepsilon}\right)^{1/(a-\varepsilon)},$$

we obtain

$$|(\Delta_\mu^k F)(0)| \leq M 2^{2k} k!\, \Gamma(\mu + k + 1) \exp\left(\frac{2k}{a-\varepsilon}\right) \left(\frac{2k}{a-\varepsilon}\right)^{-2k/(a-\varepsilon)}.$$

Moreover, [5, Proposition 3] implies that if $f(z) = \sum_{k=0}^{\infty} a_k z^{2k}$, $z \in \mathbf{C}$, is in $Z_{a',e}$,

$$|a_k| \leq C(2k)^{-2k/(a'+\eta)}, \quad k \in \mathbf{N}.$$

Hence, if $f(z) = \sum_{k=0}^{\infty} a_k z^{2k}$, $z \in \mathbf{C}$, is in $Z_{a',e}$, Stirling's formula leads to

$$\begin{aligned}
|\langle h_\mu^{-1}(F), f\rangle| &\leq \sum_{k=0}^{\infty} |a_k||(\Delta_\mu^k F)(0)| \\
&\leq C \sum_{k=0}^{\infty} 2^{2k}\Gamma(\mu + k + 1)k!\, \exp\Big(\frac{2k}{a-\varepsilon}\Big)\Big(\frac{2k}{a-\varepsilon}\Big)^{-2k/(a-\varepsilon)} \\
&\quad \times (2k)^{-2k/(a'+\eta)} \\
&\leq C \sum_{k=0}^{\infty} k^{\mu+1} \exp\Big(2k\Big(\frac{1}{a-\varepsilon} - 1\Big)\Big)(a-\varepsilon)^{2k/(a-\varepsilon)} \\
&\leq C \sum_{k=0}^{\infty} k^{\mu+1} \exp\Big(2k\Big(\frac{1+\log(a-\varepsilon)}{a-\varepsilon} - 1\Big)\Big) \leq C,
\end{aligned}$$

since the function $\alpha(x) = \frac{1+\log x}{x} < 1$, $x > 1$. Moreover the constant $C > 0$ is not depending on $F \in B$. Then we prove that W is a bounded set in $Z'_{a',e}$ when on $Z'_{a',e}$ we consider the weak $*$ topology. Then W is also a bounded set in the strong dual $Z'_{a',e}$ of $Z_{a',e}$ because $Z_{a',e}$ is reflexive [5, Proposition 1].

Thus the proof of Proposition is complete. ■

Proposition 2.6 *Let $\mu, \nu > -\frac{1}{2}$ and $a > 1$. Then $Q^\mu_{a,e} = Q^\nu_{a,e}$ where the equality is understood algebraically and topologically.*

Proof According to [5, Proposition 1], for every $\gamma > -\frac{1}{2}$, $Q^\gamma_{a,e}$ is a bornological space (see [7, p. 257]). Since, by virtue of Proposition 2.5, a subset of $Q_{a,e}$ is bounded in $Q^\nu_{a,e}$ if, and only if, it is bounded in $Q^\mu_{a,e}$, we conclude that the topology of $Q^\nu_{a,e}$ coincides with the one of $Q^\mu_{a,e}$. ■

In view of Proposition 2.6, to simplify the following we will write $Q_{a,e}$ to refer to $Q^\mu_{a,e}$, $\mu > -\frac{1}{2}$.

As mentioned in the introduction, if $f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$, $z \in \mathbf{C}$, then the Hankel translation ${}_\mu\tau_w f$, $w \in \mathbf{C}$, is given by

(2.4)

$$\begin{aligned}
{}_\mu\tau_w(f)(z) &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\mu+1)\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} w^{2(n-k)} z^{2k} \\
&= \sum_{k=0}^{\infty} z^{2k} \frac{\Gamma(\mu+1)}{\Gamma(\mu+k+1)} \sum_{n=k}^{\infty} \binom{n}{k} w^{2(n-k)} a_n \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)}, \quad w, z \in \mathbf{C},
\end{aligned}$$

where the convergence of the series is uniform in every compact subset of $\mathbf{C} \times \mathbf{C}$ (see [1, 4]). Note that ${}_\mu\tau_w f$ is an even and entire function, for every $w \in \mathbf{C}$.

Proposition 2.7 *Let $w \in \mathbf{C}$ and $a > 1$. The Hankel translation ${}_\mu\tau_w$ defines a continuous linear mapping from $Z_{a,e}$ into itself.*

https://doi.org/10.4153/CMB-2005-014-2 Published online by Cambridge University Press

Proof Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$, $z \in \mathbf{C}$ is a function in $Z_{a,e}$. According to [5, Proposition 2] ${}_\mu\tau_w(f) \in Z_{a,e}$ if, and only if, $g_w \in Z_{a,e}$, where

$$g_w(z) = \sum_{k=0}^{\infty} z^{2k} \frac{\Gamma(\mu+1)}{\Gamma(\mu+k+1)} \Big| \sum_{n=k}^{\infty} \binom{n}{k} w^{2(n-k)} a_n \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)} \Big|, \quad z \in \mathbf{C}.$$

Assume that $z \in \mathbf{C}$. We can write

$$\begin{aligned} |g_w(z)| &\le \sum_{k=0}^{\infty} |z|^{2k} \frac{\Gamma(\mu+1)}{\Gamma(\mu+k+1)} \sum_{n=k}^{\infty} \binom{n}{k} |w|^{2(n-k)} |a_n| \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)} \\ &= {}_\mu\tau_{|w|} \Big(\sum_{n=0}^{\infty} |a_n| t^{2n} \Big) (|z|) \\ &= \int_{||w|-|z||}^{|w|+|z|} D(|w|,|z|,t) \sum_{n=0}^{\infty} |a_n| t^{2n} \frac{t^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dt. \end{aligned}$$

By using again [5, Proposition 2], the function $\sum_{n=0}^{\infty} |a_n| t^{2n}$, $t \in \mathbf{C}$, is in $Z_{a,e}$. Hence from [9, (2), Section 2] we deduce that for every $\varepsilon > 0$,

$$\begin{aligned} |g_w(z)| &\le C \int_{||w|-|z||}^{|w|+|z|} D(|w|,|z|,t) \exp(t^{a+\varepsilon}) t^{2\mu+1}\, dt \\ &\le C \exp(||w|+|z||^{a+\varepsilon}) \int_{||w|-|z||}^{|w|+|z|} D(|w|,z,t) t^{2\mu+1}\, dt \\ &\le C \exp(2^{a+\varepsilon}(|w|^{a+\varepsilon} + |z|^{a+\varepsilon})). \end{aligned}$$

Hence $g_w \in Z_{a,e}$. Thus we prove that ${}_\mu\tau_w f \in Z_{a,e}$, $w \in \mathbf{C}$.

The continuity of the mapping $f \mapsto {}_\mu\tau_w f$, $w \in \mathbf{C}$, can be proved by using the closed graph theorem. Indeed, assume that $w \in (0,\infty)$. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence in $Z_{a,e}$ such that $f_n \to f$ and ${}_\mu\tau_w f_n \to g$ as $n \to \infty$ in $Z_{a,e}$ where $f, g \in Z_{a,e}$. Then since the convergence in $Z_{a,e}$ implies the convergence in $\mathcal{H}_e$, ${}_\mu\tau_w(f)(x) \to_\mu \tau_w(f)(x)$ as $n \to \infty$ for every $x \in (0,\infty)$. Hence ${}_\mu\tau_w(f)(x) = g(x)$, $x \in (0,\infty)$. Since ${}_\mu\tau_w f, g \in Z_{a,e}$, we conclude that ${}_\mu\tau_w(f) = g$. Thus we prove in this case the continuity of the mapping $f \mapsto {}_\mu\tau_w f$ from $Z_{a,e}$ into itself. On the other hand, if $w \in \mathbf{C}$, we have that for every $x \in (0,\infty)$,

$${}_\mu\tau_w(f_n)(x) = {}_\mu\tau_x(f_n)(w) \to {}_\mu\tau_x(f)(w) = {}_\mu\tau_w(f)(x), \text{ as } n \to \infty.$$

Then we obtain again that ${}_\mu\tau_w(f) = g$ and thus the proof is finished. ■

Proposition 2.7 allows us to define the Hankel convolution $T\#_\mu f$ of $T \in Z'_{a,e}$ and $f \in Z_{a,e}$ by

$$(T\#_\mu f)(z) = \langle T(t), {}_\mu\tau_z(f)(t) \rangle, \quad z \in \mathbf{C}.$$

To study the behaviour of Hankel convolution on the space $Z_{a,e}$ we need first to obtain a representation of the elements $Z'_{a,e}$.

https://doi.org/10.4153/CMB-2005-014-2 Published online by Cambridge University Press

Proposition 2.8 *Let T be a functional on $Z_{a,e}$. Then $T \in Z'_{a,e}$ if, and only if, there exist a complex regular Borel measure γ on $\mathbf{C}$ and $\varepsilon > 0$ such that*

$$\langle T, f\rangle = \int_{\mathbf{C}} f(z)\exp(-|z|^{a+\varepsilon})\, d\gamma(z), \quad f \in Z_{a,e}. \tag{2.5}$$

Proof Suppose that T admits the representation (2.5) for a certain complex regular Borel measure γ on $\mathbf{C}$ and an $\varepsilon > 0$. Then, we have

$$\begin{aligned}|\langle T, f\rangle| &\le \int_{\mathbf{C}} \exp(-|z|^{a+\varepsilon})|f(z)|\, d|\gamma|(z)\\ &\le C\sup_{z\in\mathbf{C}} \exp(-|z|^{a+\varepsilon})|f(z)|, \quad f \in Z_{a,e},\end{aligned}$$

Where, as usual, $|\gamma|$ denotes the total variation measure of γ. Hence $T \in Z'_{a,e}$.

Assume now that $T \in Z'_{a,e}$. Then there exists $C, \varepsilon > 0$ for which

$$|\langle T, f\rangle| \le C\sup_{z\in\mathbf{C}} |f(z)|\exp(-|z|^{a+\varepsilon}), \quad f \in Z_{a,e}. \tag{2.6}$$

We denote by $\mathcal{C}_0$ the space of continuous functions in $\mathbf{C}$ vanishing in infinity. If $f \in Z_{a,e}$, then $f(z)\exp(-|z|^{a+\varepsilon}) \in \mathcal{C}_0$. Indeed, if $0 < \eta < \varepsilon$, one has

$$\begin{aligned}|f(z)|\exp(-|z|^{a+\varepsilon}) &\le |f(z)|\exp(-|z|^{a+\eta})\exp(|z|^{a+\eta} - |z|^{a+\varepsilon})\\ &\le \exp(|z|^{a+\eta}(1 - |z|^{\varepsilon-\eta}))\sup_{w\in\mathbf{C}} |f(w)|\exp(-|w|^{a+\eta}) \to 0,\end{aligned}$$

as $|z| \to \infty$.

We define the mappings

$$J\colon Z_{a,e} \longrightarrow \mathcal{C}_0, \quad f \mapsto f(z)\exp(-|z|^{a+\varepsilon}),$$

and

$$L\colon J(Z_{a,e}) \subset \mathcal{C}_0 \longrightarrow \mathbf{C}, \quad f(z)\exp(-|z|^{a+\varepsilon}) \mapsto \langle T, f\rangle.$$

By (2.6) L is continuous when on $J(Z_{a,e})$ we consider the topology induced in it by the usual topology of $\mathcal{C}_0$. By using Hanh-Banach and Riesz representation theorems in a standard way, we can conclude that T admits a representation like (2.5) for a certain complex regular Borel measure γ on $\mathbf{C}$ and an $\varepsilon > 0$. ∎

Proposition 2.9 *Let $T \in Z'_{a,e}$, where $a > 1$. Then the mapping $f \mapsto T\#_\mu f$ is continuous from $Z_{a,e}$ into itself.*

Proof Assume, by Proposition 2.8, that γ is a complex regular Borel measure on $\mathbf{C}$ and $\varepsilon > 0$ such that

$$\langle T, f\rangle = \int_{\mathbf{C}} \exp(-|z|^{a+\varepsilon})f(z)\, d\gamma(z), \quad f \in Z_{a,e}.$$

https://doi.org/10.4153/CMB-2005-014-2 Published online by Cambridge University Press

In particular, for every $f \in Z_{a,e}$ and $z \in \mathbf{C}$,

$$(T\#_\mu f)(z) = \int_{\mathbf{C}} \exp(-|w|^{a+\varepsilon})_\mu \tau_z(f)(w)\, d\gamma(w).$$

Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^{2n}$, $z \in \mathbf{C}$, is in $Z_{a,e}$. Then, for every $z \in \mathbf{C}$,

$$(T\#_\mu f)(z) = \sum_{k=0}^{\infty} z^{2k} \frac{\Gamma(\mu+1)}{\Gamma(k+\mu+1)} \sum_{n=k}^{\infty} \binom{n}{n-k} \times a_n \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)} \int_{\mathbf{C}} w^{2(n-k)} exp(-|w|^{a+\varepsilon})\, d\gamma(w).$$

Indeed, let $z \in \mathbf{C}$. We can write

$${}_\mu\tau_z(f)(w) = \sum_{k=0}^{\infty} w^{2k} \frac{\Gamma(\mu+1)}{\Gamma(k+\mu+1)} \sum_{n=k}^{\infty} \binom{n}{k} a_n \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)} z^{2(n-k)}, \quad w \in \mathbf{C}.$$

Moreover the series converges in $Z_{a,e}$. Hence

$$\langle T, {}_\mu\tau_z f\rangle = \sum_{k=0}^{\infty} \langle T(w), w^{2k}\rangle \frac{\Gamma(\mu+1)}{\Gamma(k+\mu+1)} \sum_{n=k}^{\infty} \binom{n}{k} a_n \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)} z^{2(n-k)}. \tag{2.7}$$

The last series converges absolutely. In fact, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} |\langle T(w), w^{2k}\rangle| \frac{\Gamma(\mu+1)}{\Gamma(k+\mu+1)} \sum_{n=k}^{\infty} \binom{n}{k} |a_n| \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)} |z|^{2(n-k)} \\
&\quad \le \sum_{k=0}^{\infty} \int_{\mathbf{C}} |w|^{2k} \exp(-|w|^{a+\varepsilon})\, d|\gamma|(w) \frac{\Gamma(\mu+1)}{\Gamma(k+\mu+1)} \\
&\qquad \times \sum_{n=k}^{\infty} \binom{n}{k} |a_n| \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)} |z|^{2(n-k)} \\
&\quad \le \int_{\mathbf{C}} \sum_{k=0}^{\infty} |w|^{2k} \frac{\Gamma(\mu+1)}{\Gamma(k+\mu+1)} \\
&\qquad \times \sum_{n=k}^{\infty} \binom{n}{k} |a_n| \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)} |z|^{2(n-k)} \exp(-|w|^{a+\varepsilon})\, d|\gamma|(w) \\
&\quad \le C \int_{\mathbf{C}} \exp(2^{a+\varepsilon/2}(|z|^{a+\varepsilon/2} + |w|^{a+\varepsilon/2})) \exp(-|w|^{a+\varepsilon})\, d|\gamma|(w) < \infty.
\end{aligned}$$

https://doi.org/10.4153/CMB-2005-014-2 Published online by Cambridge University Press

Then, we can permute the order of summation in (2.7) obtaining

$$\langle T, {}_\mu\tau_z f\rangle = \sum_{k=0}^{\infty} z^{2k}\frac{\Gamma(\mu+1)}{\Gamma(k+\mu+1)}\sum_{n=k}^{\infty}\binom{n}{n-k} \times a_n \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)}\int_{\mathbf{C}} w^{2(n-k)} exp(-|w|^{a+\varepsilon})\,d\gamma(w).$$

Hence $T\#_\mu f$ is an even and entire function. Moreover, we have obtained that for every $\varepsilon > 0$,

$$|(T\#_\mu f)(z)| \le C\exp(|z|^{a+\varepsilon}), \quad z \in \mathbf{C}.$$

Thus we prove that $T\#_\mu f \in Z_{a,e}$.

Suppose now that $(f_n)_{n\in\mathbf{N}}$ is a sequence in $Z_{a,e}$ for which $f_n \to f$ and $T\#_\mu f_n \to g$, as $n \to \infty$, in $Z_{a,e}$, where $f, g \in Z_{a,e}$.

According to Proposition 2.7, for every $z \in \mathbf{C}$, ${}_\mu\tau_z f_n \to_\mu \tau_z f$, as $n \to \infty$, in $Z_{a,e}$. Then,

$$\langle T, {}_\mu\tau_z f_n\rangle = (T\#_\mu f_n)(z) \to \langle T,_\mu \tau_z f\rangle = (T\#_\mu f)(z), \quad \text{as } n \to \infty,$$

for every $z \in \mathbf{C}$. Hence, since the convergence in $Z_{a,e}$ implies the pointwise convergence, $T\#_\mu f = g$. The closed graph theorem allows us to conclude that the convolution operator defined by T is continuous from $Z_{a,e}$ into itself. ■

The Hankel convolution $T\#_\mu S$ of T and S in $Z'_{a,e}$ is defined as follows

$$\langle T\#_\mu S, f\rangle = \langle T, S\#_\mu f\rangle, \quad f \in Z_{a,e}.$$

Thus $T\#_\mu S \in Z'_{a,e}$.

The interchange formula of Hankel transform and Hankel convolution holds.

Proposition 2.10 *Let $T, S \in Z'_{a,e}$. Then*

$$h_\mu(T\#_\mu S) = h_\mu(T)h_\mu(S).$$

Proof Let $z \in \mathbf{C}$. According to [9, (1), Section 2], we have

$$\begin{aligned} h_\mu(T\#_\mu S)(z) &= \langle (T\#_\mu S)(t), 2^\mu\Gamma(\mu+1)(zt)^{-\mu}J_\mu(zt)\rangle \\ &= \langle T(w), \langle S(t), {}_\mu\tau_w(2^\mu\Gamma(\mu+1)(zu)^{-\mu}J_\mu(zu))(t)\rangle\rangle \\ &= \langle T(w), 2^\mu\Gamma(\mu+1)(zw)^{-\mu}J_\mu(zw)\rangle\langle S(t), 2^\mu\Gamma(\mu+1)(zt)^{-\mu}J_\mu(zt)\rangle \\ &= h_\mu(T)(z)h_\mu(S)(z). \end{aligned}$$

■

By using Proposition 2.10 and the uniqueness of the Hankel transform (Proposition 2.1), we can establish the following algebraic properties of the Hankel convolution.

Proposition 2.11 *Let $T, R, S \in Z'_{a,e}$. Then*

(a) $T\#_\mu R = R\#_\mu T$.
(b) $(T\#_\mu R)\#_\mu S = T\#_\mu(R\#_\mu S)$.
(c) $T\#_\mu\delta = T$, *where T denotes the Dirac functional.*
(d) $\Delta_\mu(T\#_\mu R) = (\Delta_\mu T)\#_\mu R$.

We now show the surjectivity of the convolution operators defined on $Z_{a,e}$ by the elements of $Z'_{a,e}$.

Proposition 2.12 *Let $T \in Z'_{a,e}$. If $T \neq 0$, the Hankel convolution operator generated by T from $Z_{a,e}$ into itself is surjective.*

Proof Assume that $T \neq 0$. To see that T defines a surjective Hankel convolution operator F_T on $Z_{a,e}$ into itself by $F_T(f) = T\#_\mu f$, $f \in Z_{a,e}$, we will use the surjectivity criterion in Meise and Vogt [10, 26.2]. To show that F_T is surjective we have to prove that if B is a subset of $Z'_{a,e}$ then, B is bounded in $Z'_{a,e}$ provided that $T\#_\mu B$ is a bounded set in $Z'_{a,e}$.

Let B be a subset of $Z'_{a,e}$ such that $T\#_\mu B$ is bounded in $Z'_{a,e}$. Then, by Proposition 2.10, $h_\mu(T\#_\mu B) = h_\mu(T)h_\mu(B)$ is a bounded set in $Q_{a',e}$. Moreover, since $h_\mu(T) \neq 0$, [5, Theorem 12] implies that $h_\mu(B)$ is a bounded set in $Q_{a',e}$ and, then B is a bounded set in $Z'_{a,e}$.

Thus we conclude that the Hankel convolution operator F_T generated by T is surjective from $Z_{a,e}$ onto itself. ■

References

[1] M. Belhadj and J. J. Betancor, *Hankel convolution operators on entire functions and distributions.* J. Math Anal. Appl. **276**(2002), 40–63.
[2] C. A. Berenstein and R. Gay, *Complex variables, an introduction.* Graduate Texts in Mathematics, 125, Springer-Verlag, New York, 1991.
[3] J. J. Betancor and A. Bonilla, *On the universality property of certain integral operators.* J. Math. Anal. Appl. **250**(2000), 162–180.
[4] F. M. Cholewinski and D. T. Haimo, *The Weierstrass-Hankel convolution.* J. Analyse Math. **17**(1966), 1–58.
[5] L. Ehrenpreis, *Solution of some problems of division, III.* Amer. J. Math. **78**(1956), 685–715.
[6] S. J. L. van Eijndhoven and M. J. Kerkhof, *The Hankel transformation and spaces of type W.* Reports on Appl. and Numer. Analysis, 10, Dept. of Maths. and Comp. Sci., Eindhoven University of Technology, 1988.
[7] H. G. Garnir, M. De Wilde, J. Schmets, *Analyse fonctionnelle, Tome I.* Birkhäuser Verlag, Basel, 1968.
[8] D. T. Haimo, *Integral equations associated with Hankel convolutions.* Trans. Amer. Math. Soc. **116**(1965), 330–375.
[9] I. I. Hirschman, Jr., *Variation diminishing Hankel transforms.* J. Analyse Math. **8**(1960/61), 307–336.
[10] R. Meise and D. Vogt, *Einführung in die Funktionalanalysis.* Vieweg, Braunschweig, 1992.

[11] G. N. Watson, *A treatise on the theory of Bessel functions.* Cambridge University Press, Cambridge, 1995.
[12] A. H. Zemanian, *Generalized integral transformations.* Interscience Publishers, New York, 1968.

Departamento de Análisis Matemático
Universidad de La Laguna
38271 La Laguna.
Tenerife
Islas Canarias
Spain
e-mail: jbetanco@ull.es