ON EASTON SUPPORT ITERATION OF PRIKRY-TYPE FORCING NOTIONS

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Abstract. We consider of constructing normal ultrafilters in extensions are here Easton support iterations of Prikry-type forcing notions. New ways presented. It turns out that, in contrast with other supports, seemingly unrelated measures or extenders can be involved here.

§1. Introduction.

We continue here the study of the structure of normal ultrafilters in generic extensions by iterated Prikry-type forcings.

In [1, 7, 9], nonstationary support and full support iterations were considered. When iterating Prikry forcings below a measurable limit of measurables κ, all the normal measures it carries in the extension are characterized in terms of normal measures in the ground model; furthermore, for every normal measure on κ in the generic extension, the restrictions of its ultrapower to the ground model is an iteration of it by normal measures only.

Here we concentrate on Easton support iteration of arbitrary Prikry-type forcings. The situation turned out to be radically different. Namely, we show the following:

Theorem 1.1. *Let κ be a measurable cardinal with $2^\kappa = \kappa^+$. Let $\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ be an Easton support iteration of Prikry-type forcing notions.*

Assume that $\Delta \subseteq \kappa$ is unbounded, such that for every $\alpha < \kappa$, $\underset{\sim}{Q}_\alpha$ is forced to be trivial if and only if $\alpha \notin \Delta$. Let $U \in V$ be a normal measure on κ with $\Delta \notin U$, and let $i\colon V \to N$ is an elementary embedding, definable in V, such that the following properties hold[1]:

1. $crit(i) = \kappa$.
2. ${}^\kappa N \subseteq N$.
3. $\kappa \notin i(\Delta)$.
4. $U = \{X \subseteq \kappa : \kappa \in i(X)\}$.
5. $|i(\kappa)| = \kappa^+$.
6. $\{i(f)(\kappa) : f \in V,\ f : \kappa \to \kappa\}$ *is unbounded in* $i(\kappa)$.

Assume also that every element of N has the form $i(f)(\beta_1, \dots, \beta_l)$ for some $f \in V$ and $\beta_1 < \dots < \beta_l < i(\kappa)$.

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[1] A typical example of such N is an ultrapower of V by its κ-closed extender, and $i\colon V \to N$ is its embedding.

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Then there exists a measure $W \in V[G]$ extending U, such that, denoting $Ult(V[G], W) \simeq M_W\left[j_W(G)\right]$, there exists $k\colon N \to M_W$ with $crit(k) > \kappa$ such that $j_W \restriction V = k \circ i$.

Furthermore, under mild assumptions on the forcings participating in the iteration, there are $(2^\kappa)^+ = \kappa^{++}$ normal measures W as above extending U (see Theorem 2.19). This generalizes the Kunen–Paris theorem on the number of normal measures [10].

In Sections 3 and 4 we analyze the properties of the ultrapower embedding $j_W\colon V[G] \to M_W\left[j_W(G)\right]$ for an arbitrary measure $W \in V[G]$.

Assume that $V = \mathcal{K}$ is the core model. By a well-known series of results in inner model theory, $j_W \restriction V$ is an iterated ultrapower of V, provided that the variety of large cardinals in the universe is limited. For instance, by Mitchell [11], assuming that there is no inner model with a cardinal α with $o(\alpha) = \alpha^{++}$ and $V = \mathcal{K}$ is the core model, $j_W \restriction \mathcal{K}$ is an iteration of $\mathcal{K}$ by normal measures. By a result of Schindler [12], assuming that there is no inner model with a Woodin cardinal, $j_W \restriction \mathcal{K}$ is an iteration of $\mathcal{K}$ by its extenders.

Theorem 1.1 shows that $j_W \restriction V$ decomposes to the form $k \circ i$. In particular, $j_W(\kappa) \geq i(\kappa) \geq j_U(\kappa)$. In Section 3, we analyze the requirements needed to ensure strict inequality, namely $j_W(\kappa) > j_U(\kappa)$, by concentrating on the context where $V = \mathcal{K}$ is the core model and $j_W \restriction \mathcal{K}$ is an iteration of $\mathcal{K}$ by measures or extenders.

In Section 4 we focus on the question of what can be said about the embedding $k\colon N \to M_W$. In particular, whether it is an iteration of N by measures or extenders (without assuming that $V = \mathcal{K}$ is the core model). We will prove in Theorem 4.7 that this is the case where $P = P_\kappa$ is an iteration of Prikry forcings (under some restrictions on the normal measures used; see Section 4.1). Furthermore, in this case, k is an iterated ultrapower with normal measures only.

§2. The general framework

Definition 2.1. An iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta\colon \alpha \leq \kappa\ ,\ \beta < \kappa\rangle$ is called an Easton support iteration of Prikry-type forcings if and only if,

1. For every $\alpha < \kappa$, the weakest condition in P_α forces that $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_{Q_\alpha}, \underset{\sim}{\leq}^*_{Q_\alpha}\rangle$ is a Prikry-type forcing notion.
2. For every $\alpha \leq \kappa$ and $p \in P_\alpha$,
    (a) p is a function with domain α such that for every $\beta < \alpha$, $p \restriction \beta \in P_\beta$, and $p \restriction \beta \Vdash p(\beta) \in \underset{\sim}{Q}_\beta$.
    (b) If $\alpha \leq \kappa$ is inaccessible, then $\text{supp}(p) \cap \alpha$ is bounded in α ($\text{supp}(p) \subseteq \alpha$ is the set of indices γ on which $p(\gamma)$ is forced to be non-trivial).

Suppose that $p, q \in P_\alpha$. Then $p \geq q$, which means that p extends q, holds if and only if:

1. $\text{supp}(q) \subseteq \text{supp}(p)$.
2. For every $\beta \in \text{supp}(q)$, $p \restriction \beta \Vdash p(\beta) \geq_\beta q(\beta)$ (where $\geq_\beta$ is the order of Q_β).
3. There is a finite subset $b \subseteq \text{supp}(q)$, such that for every $\beta \in \text{supp}(q) \setminus b$, $p \restriction \beta \Vdash p(\beta) \geq^*_\beta q(\beta)$ (where $\geq^*_\beta$ is the direct extension order of Q_β).

If $b = \emptyset$, we say that p is a direct extension of q, and denote it by $p \geq^* q$.

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Assume that κ is measurable, and let U be a normal ultrafilter over κ. Let $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ be an Easton support iteration of a Prikry-type forcing notions. Suppose that the following hold:

1. There exists an unbounded subset $\Delta \subseteq \kappa$, $\Delta \notin U$, such that, for every $\alpha < \kappa$,
   (a) $\alpha \in \Delta \longrightarrow \Vdash_{P_\alpha} \underset{\sim}{Q}_\alpha$ is nontrivial.
   (b) $\alpha \notin \Delta \longrightarrow \Vdash_{P_\alpha} \underset{\sim}{Q}_\alpha$ is trivial.
2. For every $\alpha < \kappa$, $\Vdash_{P_\alpha} \langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is α-closed.
3. For every $\alpha \in \Delta$, $\Vdash_{P_\alpha} |\underset{\sim}{Q}_\alpha| < \min(\Delta \setminus \alpha + 1)$.

The following properties are standard (see, e.g., [5]):

Lemma 2.2. *For every $\lambda \leq \kappa$, P_λ satisfies the Prikry property.*

Lemma 2.3. *For every $\lambda \leq \kappa$ which is Mahlo, P_λ has the $\lambda - c.c.$*

Let G be a generic subset of $P = P_\kappa$. We would like to analyze the normal measures on κ in $V[G]$ extending U. The standard way to do so appears in [5], we present it here for the sake of completeness.

Lemma 2.4. *There exists a normal measure $U^* \in V[G]$ on κ which extends U.*

Proof. Let $\langle \underset{\sim}{A}_\alpha : \alpha < \kappa^+\rangle$ be an enumeration, in V, of $P = P_\kappa$-names, such that every $X \in \left(\mathcal{P}(\kappa)\right)^{V[G]}$ has the form $(\underset{\sim}{A}_\alpha)_G$ for some $\alpha < \kappa^+$. Such a list of names exists since $P = P_\kappa$ is $\kappa - c.c.$ Now, construct, in $V[G]$, a $\leq^*$-increasing sequence of conditions $\langle q_\alpha : \alpha < \kappa^+\rangle$, such that, over $N[G]$, $q_\alpha \parallel \kappa \in j_U(\underset{\sim}{A}_\alpha)$. Such a sequence exists since $V[G] \models$ "$\langle j_U(P) \setminus \kappa, \leq^*\rangle$ is κ^+-*closed*."

Let $\langle \underset{\sim}{q}_\alpha : \alpha < \kappa^+\rangle$ be a P-name for the above sequence. Now, define $U^* \supseteq U$ as follows: For every $\alpha < \kappa^+$, $(\underset{\sim}{A}_\alpha)_G \in U^*$ if and only if there exist $p \in G$ and $\alpha < \kappa^+$ such that

$$p^\frown \underset{\sim}{q}_\alpha \Vdash \kappa \in i(\underset{\sim}{A}_\alpha).$$

We argue that U^* defined above is a normal measure which extends U.

Assume that $\delta < \kappa$ and $\langle \underset{\sim}{X}_\alpha : \alpha < \delta\rangle$ is a P_κ-name for a partition of κ in $V[G]$. For every $\alpha < \delta$, define

$$Y_\alpha = \{\beta < \kappa^+ : \exists p \in P_\kappa,\ p \Vdash \underset{\sim}{X}_\alpha = \underset{\sim}{A}_\beta\}.$$

Since P is $\kappa - c.c.$, $|Y_\alpha| < \kappa$. Denote

$$Y = \bigcup_{\alpha<\delta} Y_\alpha.$$

Then $Y \subseteq \kappa^+$ is a bounded subset. Pick $\alpha^* < \kappa^+$ high enough which bounds Y. Let us argue that there exists $p \in G$ and a unique $\beta < \delta$ such that

$$p^\frown \underset{\sim}{q}_{\alpha^*} \Vdash \kappa \in j_U\left(\underset{\sim}{A}_\beta\right),$$

and thus $\left(\underset{\sim}{A}_\beta\right)_G \in U^*$.

Work in $N[G]$. Note that $\langle A_\beta : \beta \in Y\rangle$ covers the sequence $\langle X_\alpha : \alpha < \delta\rangle$. Since q_{α^*} is $\leq^*$ above any q_β for $\beta \in Y$,

$$\forall \xi < \alpha, q_{\alpha^*} \parallel \kappa \in i(\underset{\sim}{X}_\xi).$$

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Since $\langle i(X_\xi) : \xi < \delta\rangle$ is a partition of $i(\kappa)$, there exists a unique $\xi^* < \delta$ such that $q_{\alpha^*} \Vdash \kappa \in i(\underset{\sim}{A}_{\xi^*})$. Let $p \in G$ be a condition forcing this. Then $p^\frown \underset{\sim}{q}_{\alpha^*} \Vdash \kappa \in i(\underset{\sim}{X}_{\xi^*})$, as desired.

A similar argument shows that U^* is normal. Indeed, given a P_κ-name for a regressive function $\underset{\sim}{f} : \kappa \to \kappa$, define, for every $\alpha < \kappa$,

$$X_\alpha = \{\xi < \kappa \colon f(\xi) = \alpha\}$$

and proceed as before to find a unique $\alpha < \kappa$ such that $X_\alpha \in U^*$. ⊣

In particular, U can be extended to a normal measure $U^* \in V[G]$, such that the ultrapower embedding $j_{U^*} \colon V[G] \to M\left[j_{U^*}(G)\right]$ satisfies that $j_{U^*} \restriction V = k \circ j_U$, for an embedding $k \colon M_U \to M$ which satisfies $\text{crit}(k) > \kappa$. Indeed, define $k([f]_U) = [f]_{U^*}$ for every $f : \kappa \to V$ in V.

A natural question here is whether this is the only way to generate a normal ultrafilter on κ in $V[G]$. In [6, 7] it was established that this is the case when considering the nonstationary support iteration. However, this is not true anymore once full support iterations are considered: in [1] and later in [9], iterations of the standard Prikry forcing were considered. It was proved that every normal measure $U \in V$ on κ with $\Delta \notin U$ can be extended to a normal measure $U^* \in V[G]$ similarly as above, but not every normal measure extending U is generated this way; nevertheless, all the normal measures on κ in $V[G]$ were characterized, either as extensions U^* of measures $U \in V$ with $\Delta \notin U$, or as the projections to normal measure of extensions U^* of a normal ultrafilter $U \in V$ with $\Delta \in U$.

It turns out that the picture in the Easton support iteration of Prikry-type forcing notions (and even of the standard Prikry forcings) is radically different. Given an elementary embedding $i \colon V \to N$ with critical point κ, definable in V, the normal measure derived from it, $U = \{X \subseteq \kappa \colon \kappa \in i(X)\}$, can be extended to a normal measure $W \in V[G]$ such that $j_W \restriction V = k \circ i$, for some $k \colon N \to M$ with $\text{crit}(k) > \kappa$. In the case of iterations of the standard Prikry forcing, k is an iterated ultrapower of N by normal measures only (see Section 4), while $i \colon V \to N$ can be an embedding derived from an extender (as in the formulation of Theorem 1.1).

Let us demonstrate that, in the Easton support iteration, there are many more possibilities to get normal measures $W \in V[G]$. We show that an arbitrary embedding $i \colon V \to N$ can be used to extend the normal measure U derived from it.

LEMMA 2.5. *Assume that $i \colon V \to N$ is an elementary embedding definable in V, with $\text{crit}(i) = \kappa$, such that $|i(\kappa)| = \kappa^+$, $\kappa \notin i(\Delta)$, $N \subseteq V$, and ${}^\kappa N \subseteq N$. Denote*

$$U = \{X \subseteq \kappa \colon \kappa \in i(X)\}.$$

Then G is $i(P) \restriction \kappa = P$-generic over N, and:

1. *For every $q \in i(P) \setminus \kappa$, there is $H \in V[G]$ with $q \in H$, which is $\langle i(P) \setminus \kappa, \leq^* \rangle$-generic over $N[G]$.*
2. *Given such $H \in V[G]$, define*

$$U_H = \{(\underset{\sim}{A})_G : \underset{\sim}{A} \text{ is a } P\text{-name for a subset of } \kappa, \text{ and there exists } p \in G * H \text{ such that } p \Vdash \kappa \in i(\underset{\sim}{A})\}.$$

Then U_H is a normal, κ-complete ultrafilter on κ which extends U.

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PROOF.

1. We can enumerate, in $V[G]$, all the maximal antichains in $\langle i(P)\setminus\kappa, \leq^*\rangle$ with order type κ^+, by $i(\kappa)$-c.c. of the forcing, and since $V[G]\vDash |i(\kappa)|=\kappa^+$. Note that $\kappa\notin i(\Delta)$, so in the sense of $N[G]$, the forcing $\langle i(P)\setminus\kappa, \leq^*\rangle$ is more than κ-closed. Moreover, since $V\vDash {}^{\kappa}N\subseteq N$, and $P=P_\kappa$ is κ–c.c., $V[G]\vDash {}^{<\kappa}N[G]\subseteq N[G]$. Therefore, every sequence of length κ of conditions in $i(P)\setminus\kappa$ which belongs to $V[G]$ belongs to $N[G]$ as well. Thus, in the sense of $V[G]$, the forcing $\langle i(P)\setminus\kappa, \leq^*\rangle$ is κ^+-closed.

   Starting from any condition in $i(P)\setminus\kappa$, we can construct (in $V[G]$) a sequence of direct extensions of it, meeting every maximal antichain. This sequence generates a $\leq^*$-generic over $N[G]$ for $i(P)\setminus\kappa$, which belongs to $V[G]$.
2. First, we prove that $W=U_H$ is a normal, κ-complete ultrafilter on κ which extends U. It is not hard to verify that W is a filter. We prove that W is a κ-complete ultrafilter. Assume that $\langle X_\alpha\colon\alpha<\delta\rangle$ is a partition of κ, for some $\delta<\kappa$. Work in $N[G]$. Let $D\subseteq i(P)\setminus\kappa$ be the $\leq^*$-dense open set of conditions which decide the unique $\alpha<\delta$ for which $\kappa\in i(\underset{\sim}{X}_\alpha)$. Then such a statement is forced by some $r\in H$. Let $p\in G$ be a condition which forces that r has this property, and also decides the value of α. Then $p^\frown r\Vdash\kappa\in i(X_\alpha)$ and thus $X_\alpha\in W$. Normality of W follows by a similar argument, using the dense set of conditions deciding the value of $i(\underset{\sim}{f})(\kappa)$ for a given regressive function $f\colon\kappa\to\kappa$. The argument works since we don't force over κ in N. ⊣

REMARK 2.6. M. Magidor pointed out the following: Assuming that $N\subseteq V$ and $i\colon V\to N$ is definable in $V[G]$, it follows that N is already a class of V. Indeed, pick a formula φ and a parameter $a\in V[G]$ such that for every x,y in V, $\varphi(x,y,a)$ holds in $V[G]$ if and only if $i(x)=y$. For every ordinal α pick a condition $p_\alpha\in G$ which decides the value of the set $\left(V_{i(\alpha)}\right)^N$, which is the set y for which $\varphi(V_\alpha,y,a)$ holds. Since P is a set forcing, there exists $p^*\in G$ such that, for unboundedly many ordinals α, $p_\alpha=p^*$. Then N can be defined as a class of V using p^*, $N=\bigcup\{y\colon\exists\alpha\in\mathrm{ON},\ p^*\Vdash\varphi(V_\alpha,y,\underset{\sim}{a})\}$.

In general, the settings of Lemma 2.5 are not enough to ensure that $j_{U_H}\restriction V=k\circ i$ for some k with $\mathrm{crit}(k)>\kappa$. For instance, given a normal measure U on κ in V with $\Delta\notin U$, the embedding $i=j_{U^2}$ satisfies the settings of Lemma 2.5, but cannot be used to extend U to a measure U_H for which $j_{U_H}=k\circ i$ for some embedding k with $\mathrm{crit}(k)>\kappa$. This follows since i fails to satisfy clause 3 in the next claim:

PROPOSITION 2.7. *Assume that* $2^\kappa=\kappa^+$, $U\in V$ *is a normal measure on* κ, $W\in V[G]$ *is a normal measure which extends* U, $i\colon V\to N$ *is an elementary embedding and* $j_W\restriction V=k\circ i$ *for some* $k\colon N\to M$ *with* $\mathrm{crit}(k)>\kappa$. *Then*:

1. $\{X\subseteq\kappa\colon\kappa\in i(X)\}=U$.
2. $|i(\kappa)|=\kappa^+$.
3. $\{i(f)(\kappa)\colon f\in V,\ f\colon\kappa\to\kappa\}$ *is unbounded in* $i(\kappa)$.

PROOF.

1. $\{X\subseteq\kappa\colon\kappa\in i(X)\}=U$: Indeed, let $X\subseteq\kappa$ in V with $\kappa\in i(X)$. By applying $k\colon N\to M$ it follows that $\kappa\in j_W(X)$ and hence $X\in W$. Since $X\in V$ and $U=W\cap V$, it follows that $X\in U$.

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2. $|i(\kappa)| = \kappa^+$: This holds since, in $V[G]$, $|j_W(\kappa)| = 2^\kappa = \kappa^+$ (since, in V, $2^\kappa = \kappa^+$), and $i(\kappa) \leq j_W(\kappa)$.
3. $\{i(f)(\kappa) | f : \kappa \to \kappa\}$ is unbounded in $i(\kappa)$: Given $\beta < i(\kappa)$, let $f \in V[G]$ be a function such that $[f]_W = k(\beta)$. Since $k(\beta) < k(i(\kappa)) = j_W(\kappa)$, we can assume that $f : \kappa \to \kappa$. The Easton support ensures that there exists $g : \kappa \to \kappa$ in V which dominates f. Thus $i(g)(\kappa) \geq \beta$ (indeed, by applying $k : N \to M$ on both sides, this is equivalent to $j_W(g)(\kappa) \geq k(\beta) = [f]_W$, which holds, since g dominates f. Note that, when applying k, we used the fact that $\text{crit}(k) > \kappa$). ⊣

Theorem 1.1 will be proved by a sequence of lemmata, concluded in Lemma 2.15. The main idea in the proof of Theorem 1.1 is to add representing functions for all the generators of i above κ. This is needed since $j_W \restriction V$ has a single generator κ.

Definition 2.8. An ordinal β is called a generator of $i : V \to N$ if there are no $n < \omega$, ordinals $\beta_1, \dots, \beta_n$ below β and a function $f \in V$ such that $\beta = i(f)(\beta_1, \dots, \beta_n)$.

In the next lemma we construct a function $\alpha \mapsto \theta_\alpha$ in $V[G]$, which will be utilized, alongside functions in V, to represent the generators of i in $\text{Ult}(V[G], W)$.

Lemma 2.9. *There exists a P_κ-name for a sequence of ordinals, $\langle \theta_\alpha : \alpha < \kappa \rangle$, such that the following properties hold*:

1. *For every $\beta < \kappa$ and $p \in P_\kappa$, there is $\alpha_0 < \kappa$ such that for every $\alpha \geq \alpha_0$ there exists $p^* \geq^* p$ such that $p^* \Vdash \theta_\alpha = \beta$.*
2. *For every $\alpha < \kappa$ and condition $p \in P_\kappa$, there exists a condition $p^* \geq^* p$ which decides the value of $\underset{\sim}{\theta}_\alpha$.*

Remark 2.10. When iterating Prikry forcings, the natural candidate for the function $\alpha \mapsto \theta_\alpha$ is the function which maps every $\alpha \in \Delta$ to $d(\alpha)$, which is the first element in its Prikry sequence (this function does not have domain κ, but this can be fixed by defining the function on elements outside of Δ as follows: for every $\alpha < \kappa$ outside of Δ, let $\theta_\alpha = d(\alpha')$, where α' is the least element above α in Δ). The main problem with such a function is that it fails to satisfy clause 2 of the lemma (from density, every $\leq^*$-generic set has some $\alpha \in \Delta$ for which it does not decide $d(\alpha)$).

In the proof below we work under more general settings, and do not assume that we iterate Prikry forcings.

Proof. For every $\alpha < \kappa$, let $\tau_\alpha < \kappa$ be the least ordinal such that $P \restriction (\alpha, \tau_\alpha)$ is not $\alpha - c.c.$ We will argue below that such $\tau_\alpha < \kappa$ exists, but first, let us show that this suffices: Pick an unbounded subset $X \subseteq \kappa$, such that, for every $\alpha, \alpha' \in X$,

$$\alpha < \alpha' \implies \tau_\alpha < \tau_{\alpha'}$$

(for instance, let X be the club of closure points of the function $\alpha \mapsto \tau_\alpha$). Enumerate $X = \langle x_\alpha : \alpha \in \Delta \rangle$. For every $\alpha \in \Delta$, let $\langle q_{x_\alpha, \xi} : \xi < x_\alpha \rangle$ be an antichain in $P_{(x_\alpha, \tau_{x_\alpha})}$ of cardinality x_α. Define θ_α to be the unique ordinal $\xi < x_\alpha$ for which $q_{x_\alpha, \xi} \in G \restriction (x_\alpha, \tau_{x_\alpha})$ (if there is no such ξ, which is possible since the antichain is not necessarily maximal, set $\theta_\alpha = 0$).

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Now, given $\beta < \kappa$ and a condition $p \in P_\kappa$, pick first $\alpha \in \Delta$ for which x_α bounds the support of p. Direct extend p to p^* such that $p^* \restriction (x_\alpha, \tau_{x_\alpha}) = q_{x_\alpha,\beta}$. Then[2] by our definition, p^* forces that $\theta_\alpha = \beta$.

Let us prove now that for every $\alpha < \kappa$ and condition $p \in P_\kappa$, there exists $p^* \geq^* p$ which decides the value of θ_α.

We will direct extend p in the interval $(x_\alpha, \tau_{x_\alpha})$, x_α-many times, to decide whether $q_{x_\alpha,\xi} \in G \restriction (x_\alpha, \tau_{x_\alpha})$, for every $\xi < x_\alpha$. Note that this is possible since $\langle P \restriction (x_\alpha, \tau_{x_\alpha}), \leq^* \rangle$ is more than x_α-closed. Let $p^* \geq^* p$ be the obtained condition. Then either there exists $\xi < x_\alpha$ such that p^* forces that $q_{x_\alpha,\xi}$ is in the generic, and then $p^* \Vdash \underset{\sim}{\theta}_\alpha = \xi$; or, there is no such ξ, and then $p^* \Vdash \underset{\sim}{\theta}_\alpha = 0$.

Let us argue now that indeed, for every $\alpha < \kappa$ there exists $\tau_\alpha < \kappa$ such that $P \restriction (\alpha, \tau_\alpha)$ is not $\alpha - c.c.$: Pick τ_α such that there are α-many elements of Δ in the interval (α, τ_α). Let $\langle \tau_{\alpha,\xi} \colon \xi < \alpha \rangle$ be an enumeration of the first α-many elements in $(\alpha, \tau_\alpha) \cap \Delta$. For every $\xi < \alpha$, let $\underset{\sim}{x}_\xi$, $\underset{\sim}{y}_\xi$ be $P_{\tau_{\alpha,\xi}}$-names, forced by $0_{P_{\tau_{\alpha,\xi}}}$ to be a pair of incompatible elements of $\underset{\sim}{Q}_{\tau_{\alpha,\xi}}$. Such a pair exists since $\underset{\sim}{Q}_{\tau_{\alpha,\xi}}$ is nontrivial.

Now, for every $\sigma \in 2^\alpha$, let $p_\sigma \in P \restriction (\alpha, \tau_\alpha)$ be the condition which satisfies, for every $\xi < \alpha$, that

$$p_\sigma \restriction \xi \Vdash p_\sigma(\xi) = \begin{cases} \underset{\sim}{x}_\xi, & \text{if } \sigma(\xi) = 0, \\ \underset{\sim}{y}_\xi, & \text{if } \sigma(\xi) = 1. \end{cases}$$

Note that τ_α is the limit of the first α many elements above α in Δ, and thus τ_α is singular, so the support of a condition in $P = P_\kappa$ may be unbounded in τ_α.

Then $\langle p_\sigma \colon \sigma \in 2^{<\alpha} \rangle$ is an antichain in $P \restriction (\alpha, \tau_\alpha)$ of cardinality at least α. ⊣

Remark 2.11. Given a function $\alpha \mapsto \underset{\sim}{\theta}_\alpha$ as in Lemma 2.9, we slightly abuse the notation and denote $i(\alpha \mapsto \underset{\sim}{\theta}_\alpha)$ by $\langle \underset{\sim}{\theta}_\alpha \colon \alpha < i(\kappa) \rangle$.

Lemma 2.12. *Under the assumptions of Theorem 1.1, there exists $H \in V[G]$ which is $\langle i(P) \setminus \kappa, \leq^* \rangle$-generic over $N[G]$, with the following property:*

(∗) *For every generator $\beta \in i(\kappa) \setminus (\kappa + 1)$ of i, there exists a function $f = f_\beta \in V$,*
$$f \colon \kappa \to \kappa \text{ and a condition } q \in H \text{ such that } q \Vdash \beta = i\left(\alpha \mapsto \underset{\sim}{\theta}_{f(\alpha)}\right)(\kappa),$$
where $\langle \underset{\sim}{\theta}_\alpha \colon \alpha < i(\kappa) \rangle$ is as in Remark 2.11.

Proof. In $V[G]$, let $\langle A_\xi \mid \xi < \kappa^+ \rangle$ be an enumeration of maximal antichains in $i(P)$. Let $\langle \beta_\xi \mid \xi < \kappa^+ \rangle$ be an enumeration of all the generators of i below $i(\kappa)$. Define in $V[G]$ a $\leq^*$-increasing sequence $\langle r_\xi \mid \xi < \kappa^+ \rangle$. Assume that $\langle r_\xi \colon \xi < \xi^* \rangle$ has been constructed for some $\xi^* < \kappa^+$. Pick a condition r which $\leq^*$ extends all the conditions $\langle r_\xi \colon \xi < \xi^* \rangle$ constructed so far, and, by extending it, assume that r extends a condition in A_{ξ^*}. Finally, let $\alpha_0 < i(\kappa)$ be such that for every $\alpha \geq \alpha_0$ there exists $r^* \geq^* r$ which forces that $i(\xi \mapsto \theta_\xi)(\alpha) = \beta_{\xi^*}$. Pick any $\alpha \geq \alpha_0$ below $i(\kappa)$ which has the form $i(f)(\kappa)$ for some $f = f_{\beta_{\xi^*}} \in V$, and let $r_{\xi^*} \geq^* r$ be a condition which forces that $i(\xi \mapsto \theta_\xi)(\alpha) = \beta_{\xi^*}$.

Finally, let H be the $\leq^*$-generic generated from $\langle r_\xi \colon \xi < \xi^* \rangle$. ⊣

[2]It is crucial here that the Easton support is used.

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REMARK 2.13. Repeating the above argument, we can construct 2^{κ^+}-many distinct generic sets H satisfying property $(*)$, by constructing a binary tree $\langle r_\sigma \colon \sigma \in 2^{<\kappa^+}\rangle$ of conditions, which are $\leq^*$-increasing in each branch, and for each $\sigma \in 2^{<\kappa^+}$, $r_{\sigma^\frown\langle 0\rangle}$ and $r_{\sigma^\frown\langle 1\rangle}$ are $\leq^*$-incompatible. Assuming $2^{\kappa^+} = \kappa^{++}$, this provides the maximal number of generic sets H in $V[G]$ for $\langle i(P)\setminus\kappa, \leq^*\rangle$ over $N[G]$.

Below we will define for every such H a measure $U_H \in V[G]$ on κ which extends U; under mild assumptions on the forcing notions $\underset{\sim}{Q}_\alpha$, we will prove that for $H \neq H'$ satisfying property $(*)$, $U_H \neq U_{H'}$ (see Theorem 2.19). Assuming GCH, this produces the maximal number κ^{++} of normal measures on κ, generalizing the well-known result of Kunen and Paris [10].

REMARK 2.14. Not every generic set $H \in V[G]$ for $\langle i(P)\setminus\kappa, \leq^*\rangle$ satisfies property $(*)$.

Indeed, assume that Δ consists only of inaccessibles and $i\colon V \to N$ has a nonempty set of generators in $(\kappa, i(\kappa))$ which is bounded by some ordinal $\eta = i(f)(\kappa)$ below $\min(i(\Delta)\setminus\kappa)$, for some $f \in V$. This holds in the typical case where Δ consists of measurables below κ and i is a (κ,κ^+)-extender (the length of i is κ^+ since i has to satisfy the requirement $|i(\kappa)| = \kappa^+$ of Theorem 1.1). Let $\sigma\colon M_U \to N$ be the embedding which maps each element $[g]_U$ of M_U to $i(g)(\kappa)$ (here $g \in V$ is any function with domain κ). σ has critical point strictly above κ^+, since $(\kappa^+)^N = \kappa^+$.

In $V[G]$, let $H_U \subseteq j_U(P)\setminus\kappa$ be $\leq^*$-generic over $M_U[G]$. Let $H \subseteq i(P)\setminus\kappa$ be the generic set generated from $\sigma'' H_U$. We argue that H is indeed $\leq^*$-generic over N. Let $D \in N[G]$ be a $\leq^*$-dense open subset of $i(P)\setminus\kappa$. Write $\underset{\sim}{D} = i(F)(\kappa,\beta_1,\dots,\beta_l)$ for some function $F \in V$, $l < \omega$ and generators $\beta_1,\dots,\beta_l < i(f)(\kappa)$ of i. We can assume that for every $\xi,\eta_1,\dots,\eta_l < f(\xi)$, $F(\xi,\eta_1,\dots,\eta_l) \subseteq P\setminus\xi$ is forced to be $\leq^*$-dense open subset of $P\setminus\xi$. Define, in M_U,

$$D_U = \bigcap_{\gamma_1,\dots,\gamma_l < j_U(f)(\kappa)} j_U(F)(\kappa,\gamma_1,\dots,\gamma_l)$$

and note that, since the amount of sequences $\gamma_1,\dots,\gamma_l < j_U(f)(\kappa)$ in M_U is below $\min(\Delta\setminus\kappa)$, and $\langle j_U(P)\setminus\kappa, \leq^*\rangle$ is more than $\min(j_U(\Delta)\setminus\kappa)$-closed, D_U is $\leq^*$-dense open subset of $j_U(P)\setminus\kappa$. Pick any $q \in H_U \cap D_U$. Then $\sigma(q) \in D \cap H$, since $\sigma(D_U) \subseteq D$.

Since $\sigma'' G * H_U \subseteq G * H$, the embedding $\sigma\colon M_U \to N$ can be lifted to an embedding $\sigma^*\colon M_U[G * H_U] \to N[G * H]$.

Pick now any generator β of i in the interval $(\kappa, i(\kappa))$. We argue that there is no $f \in V$ such that $H \Vdash \beta = i\left(\alpha \mapsto \underset{\sim}{\theta}_{f(\alpha)}\right)(\kappa)$. Indeed, otherwise, by elementarity of σ^*, there exists $\beta^* < j_U(\kappa)$ such that

$$H_U \Vdash \beta^* = j_U\left(\alpha \mapsto \underset{\sim}{\theta}_{f(\alpha)}\right)(\kappa).$$

Let $g \in V$ be a function such that $\beta^* = j_U(g)(\kappa)$. Then

$$\beta = \sigma^*(\beta^*) = i(g)(\kappa)$$

contradicting the fact that β is a generator of i.

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Given i, N, U as in Theorem 1.1 and a generic set $H \in V[G]$ for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$, define

$$U_H = \{(\underset{\sim}{A})_G : \underset{\sim}{A} \text{ is a } P\text{-name for a subset of } \kappa, \text{ and there exists } p \in G * H \text{ such that } p \Vdash \kappa \in i(\underset{\sim}{A})\}.$$

Then U_H is a normal, κ-complete ultrafilter which extends U. This follows by repeating the argument of Lemma 2.5.

The model $M_{U_H} \simeq \text{Ult}(V[G], U_H)$ is of the form $M[G^*]$, where M is the image of V and $G^* = j_{U_H}(G)$ is $j_{U_H}(P)$-generic over M in the sense of M_{U_H}. We conclude the proof of Theorem 1.1 by defining an elementary embedding $k \colon N \to M$ and proving that $\text{crit}(k) > \kappa$.

In the next lemma we continue the abuse of notation as in Remark 2.11, and denote

$$j_{U_H}(\langle \theta_\xi \colon \xi \in \Delta \rangle) = \langle \theta_\xi \colon \xi \in j_{U_H}(\Delta) \rangle.$$

Lemma 2.15. *Assume the settings of Theorem 1.1. Suppose that H is a generic set for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$ with the property $(*)$. Define then $k : N \to M$ as follows*:

$$k\left(i(f)(\kappa, \beta_1, \dots, \beta_l)\right) = j_{U_H}(f)\left(\kappa, \theta_{\left[f_{\beta_1}\right]_{U_H}}, \dots, \theta_{\left[f_{\beta_l}\right]_{U_H}}\right)$$

for every $l < \omega$, $\beta_1, \dots, \beta_l < i(\kappa)$ generators of i and $f \in V$ (the functions f_{β_i}, $1 \leq i \leq l$, are as in Lemma 2.12).

Then $k \colon N \to M$ is elementary, crit$(k) > \kappa$ and $j_{U_H} \restriction V = k \circ i$.

Proof. Denote $W = U_H$. Let us prove that the embedding k defined above is elementary. Let us prove, for example, that for every $x, y \in N$, $x \in y$ if and only if $k(x) \in k(y)$. Fix such x, y and let $f, g \in V$, $\beta_1, \dots, \beta_l$ and $\alpha_1 < \dots < \alpha_k < \alpha$ be such that

$$x = i(f)(\kappa, \beta_1, \dots, \beta_l), \; y = i(g)(\kappa, \beta_1, \dots, \beta_l).$$

Assume now that $k(x) = k(y)$, namely

$$j_W(f)\left(\kappa, \theta_{j_W\left(f_{\beta_1}\right)(\kappa)}, \dots, \theta_{j_W\left(f_{\beta_l}\right)(\kappa)}\right) \in j_W(g)\left(\kappa, \theta_{j_W\left(f_{\beta_1}\right)(\kappa)}, \dots, \theta_{j_W\left(f_{\beta_l}\right)(\kappa)}\right).$$

Then

$$\{\xi < \kappa \colon f\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}\right) \in g\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}\right)\} \in W$$

and by the definition of W, there exists $p \in G$ and $r \in H$ such that

$$p^\frown r \Vdash \kappa \in i\left(\{\xi < \kappa \colon f\left(\xi, \underset{\sim}{\theta}_{f_{\beta_1}(\xi)}, \dots, \underset{\sim}{\theta}_{f_{\beta_l}(\xi)}\right) \in g\left(\xi, \underset{\sim}{\theta}_{f_{\beta_1}(\xi)}, \dots, \underset{\sim}{\theta}_{f_{\beta_l}(\xi)}\right)\}\right).$$

By extending $r \in H$ finitely many times, $p^\frown r \Vdash \underset{\sim}{\theta}_{\left(i(f_{\beta_m})(\kappa)\right)} = \beta_m$ holds for every $1 \leq m \leq k$. Thus, the last equation can be replaced with

$$p^\frown r \Vdash i(f)(\kappa, \beta_1, \dots, \beta_l) \in i(g)(\kappa, \beta_1, \dots, \beta_l)$$

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but the forced statement above is entirely in N, and since a condition forces it, it is true in N. Thus

$$i(f)(\kappa, \beta_1, \dots, \beta_l) \in i(g)(\kappa, \beta_1, \dots, \beta_l),$$

as desired. The implication in the other direction can be proved similarly.

Clearly $\mathrm{crit}(k) > \kappa$. We finish the proof by showing that $j_W \restriction V = k \circ i$. Let $x \in V$ and let $c_x : \kappa \to V$ be the constant function with value x. Then

$$k\left(i(x)\right) = k\left(i\left(c_x\right)(\kappa)\right) = j_W\left(c_x\right)(\kappa) = j_W(x),$$

as desired. ⊣

Let us now study the properties of the embedding $k : N \to M$. We assume the settings of Theorem 1.1.

LEMMA 2.16. *If $\leq = \leq^*$, or at least $\leq_\alpha = \leq^*_\alpha$, for a final segment of $\alpha < \kappa$, then k is the identity and $M = N$.*

PROOF. Fix an ordinal η, and let $f \in V[G]$ be a function such that $\eta = [f]_W$. We will prove that $\eta \in \mathrm{Im}(k)$. Indeed, consider the set

$$\{p \in i(P)/G \mid \exists \tau (p \Vdash i(\underset{\sim}{f})(\kappa) = \tau)\}.$$

It is $\leq$-dense in $N[G]$. So, if $\leq = \leq^*$, then H meets it. Thus, there exists a condition $q \in H$, a function $g \in V$ and generators $\beta_1, \dots, \beta_l$ of i, such that $q \Vdash i(\underset{\sim}{f})(\kappa) = i(g)(\kappa, \beta_1, \dots, \beta_l)$. Thus, by the definition of W,

$$\{\xi < \kappa : f(\xi) = g\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}\right)\} \in W,$$

and thus $\eta = [f]_W = k\left(i(g)(\kappa, \beta_1, \dots, \beta_l)\right)$. ⊣

In general, M should not be equal to N. Thus, for example, they will differ if the Prikry forcing was used unboundedly often below κ. Indeed, if $M = N$ then k is the identity, and $j_W \restriction V = i$, which implies that $j_W(G)$ is generic over N for $i(P)$. Pick a measurable of N, $\mu \in (\kappa, i(\kappa))$ for which the Prikry forcing at stage μ is used in the iteration $i(P)$. Such a measurable exists since the Prikry forcing was used cofinally in P over V. Then, in $N\left[j_W(G)\right]$, μ changes cofinality to ω. Thus, in $V[G]$, μ has cofinality ω, and hence, in V, $\mathrm{cf}(\mu) \leq \kappa$. By closure of N under κ-sequences which belong to V, $\mathrm{cf}^N(\mu) \leq \kappa$, contradicting the fact that $\mu > \kappa$ is measurable in N.

We do not know whether the assumption of 2.16 is necessary.

QUESTION 2.17. *Suppose that for unboundedly many $\alpha < \kappa$, $\leq_\alpha \neq \leq^*_\alpha$. Is then $M \neq N$?*

We do not know what are the requirements on the forcings $\underset{\sim}{Q}_\alpha$ for $\alpha \in \Delta$ which imply $M = N$. We conjecture that the requirement should be that there is $\delta < \kappa$, such that every set of ordinals x of $V[G]$ can be covered by a set $y \in V$ of cardinality $\leq |x| + \delta$.

LEMMA 2.18. $k''H \subseteq G^* \setminus \kappa$.

PROOF. Let q be in H, and let $p \in G$ be a condition such that $p \Vdash \underset{\sim}{q} \in \underset{\sim}{H}$ (recall that $H \in V[G]$). Clearly,

$$p^\frown \underset{\sim}{q} \Vdash \underset{\sim}{q} \in \Gamma \setminus \kappa,$$

where Γ is the canonical $i(P)$-name for the generic set for $i(P)$ over V.

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Pick $f : [\kappa]^n \to \kappa, f \in V$ and $\beta_1 < \cdots < \beta_n < i(\kappa)$ such that $q = i(f)(\beta_1, \dots, \beta_n)$. For every $m, 1 \leq m \leq n$, there are $f_m : \kappa \to \kappa, f_m \in V$ such that $q_{i(f_m)(\kappa), \beta_m} \in H$, namely, $\beta_m = \theta_{i(f_m)(\kappa)}$.

Let us argue that the set

$$A_q = \{\nu < \kappa \mid f(\theta_{f_1(\nu)}, \dots, \theta_{f_n(\nu)}) \in G \setminus \nu\}$$

is in W. Pick any $q \leq^* q^* \in H$ which $\leq^*$ which forces that $\beta_m = \theta_{i(f_{\beta_m})(\kappa)}$, for every $1 \leq m \leq n$. Recall that

$$q = i(f)(\beta_1, \dots, \beta_n) = i(f)(\theta_{i(f_1)(\kappa)}, \dots, \theta_{i(f_n)(\kappa)}),$$

and thus $p^\frown \underset{\sim}{q}^* \Vdash \kappa \in i(\underset{\sim}{A}_q)$. ⊣

The next lemma generalizes a Kunen–Paris result (see Remark 2.13).

THEOREM 2.19. *Let $H, H' \in V[G]$ be generic sets for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$. Suppose that H and H' satisfy $(*)$. Assume that for every $\beta < \kappa$, if $q, q' \in Q_\beta$ are incompatible according to the order $\leq^*$, then*

$$D_\beta(q) = \{r \in Q_\beta \mid r \text{ is } \leq\text{-incompatible with } q\}$$

is $\leq^*$-dense above q', or

$$D_\beta(q') = \{r \in Q_\beta \mid r \text{ is } \leq\text{-incompatible with } q'\}$$

is $\leq^*$-dense above q.[3]

Suppose that $H \neq H'$, then $U_H \neq U_{H'}$.

REMARK 2.20. Note that if the Q_β's are taken to be Prikry forcings, then the above property holds. Indeed, assume that $q = \langle t, A \rangle$ and $q' = \langle t', A' \rangle$ are $\leq^*$-incompatible. Then $t \neq t'$. Assume without loss of generality that t is an end extension of t'. Then $D(q) = \{r : r, q \text{ are } \leq^* \text{-incompatible}\}$ is $\leq^*$-dense open above q'. Indeed, pick a condition $\langle t', B \rangle \geq^* \langle t', A \rangle$. Shrink B to the set $B^* = B \setminus (\max(t) + 1)$. Then $\langle t', B^* \rangle \geq^* \langle t', B \rangle$ and is incompatible with $q = \langle t, A \rangle$.

PROOF. Suppose otherwise, i.e., $H \neq H'$, but $U_H = U_{H'} := W$.

Let $k : N \to M$ be the elementary embedding defined from H and $k' : N \to M$ from H'. ⊣

CLAIM 1. $k \neq k'$.

PROOF. Assume for contradiction that $k = k'$. Thus, by Lemma 2.18, every pair of elements from H, H' are $\leq$-compatible. We will argue that this implies that $H = H'$. It suffices to prove that every pair of conditions $q \in H, q' \in H'$ are $\leq^*$-compatible.

Assume otherwise. Let $\alpha < \kappa$ be the least ordinal such that there are pair of conditions $q \in H, q' \in H'$ for which $q \restriction \alpha, q' \restriction \alpha$ are $\leq^*$-incompatible. α cannot be limit, since $\leq^*$-compatibility of all the initial segments of q, q' below α implies that $q \restriction \alpha$ and $q' \restriction \alpha$ are $\leq^*$-compatible themselves (if α is inaccessible, this is clear since the support of q, q' is bounded in α; if the supports of q, q' are unbounded

[3]This type of condition usually holds. For example, if we iterate Prikry forcings, then just shrinking sets of measure one will produce such type of incomparability.

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in α, direct extend both q, q' coordinate by coordinate to find a common direct extension). Thus $\alpha = \beta + 1$ is successor, and $q(\beta)$, $q'(\beta)$ are $\leq^*$-incompatible. By the property of the forcing Q_β, without loss of generality, $D_\beta(q)$ is $\leq^*$-dense open above $q'(\beta)$. Since $q'(\beta) \in H'(\beta)$, q' can be extended to a condition $r \in H'$, such that $r(\beta) \in D_\beta(q)$. In particular, $q \in H$, $r \in H'$ are $\leq$-incompatible, which is a contradiction. □ of Claim 1. ⊣

Since $k \neq k'$, there exists a generator β of i such that $k(\beta) \neq k'(\beta)$. Pick the least such generator β.

Claim 2. *For every generator $\beta' < \beta$ of i, there exists a function $f_{\beta'} \in V$ such that each generic H, H' has a condition which forces that $\beta' = \underset{\sim}{\theta}_{i\left(f_{\beta'}\right)(\kappa)}$.*

Proof. Let f, f' be functions such that some condition in H forces that $\beta' = \underset{\sim}{\theta}_{i(f)(\kappa)}$, and some condition in H' forces that $\beta' = \underset{\sim}{\theta}_{i(f')(\kappa)}$. Let $q \in H$ be a condition which decides the statement $\underset{\sim}{\theta}_{i(f)(\kappa)} = \underset{\sim}{\theta}_{i(f')(\kappa)}$ and assume for contradiction that it is decided negatively. By applying k, $k(q) \in j_W(G)$ forces that

$$\underset{\sim}{\theta}_{[f]_W} \neq \underset{\sim}{\theta}_{[f']_W},$$

namely $k(\beta') \neq k'(\beta')$, contradicting the minimality of β. □ of Claim 2. ⊣

Recall now that $k(\beta) \neq k'(\beta)$. Thus, there are two distinct functions f, f' in V such that:

1. Some condition in H forces that $\beta = \underset{\sim}{\theta}_{i(f)(\kappa)}$.
2. Some condition in H' forces that $\beta = \underset{\sim}{\theta}_{i(f')(\kappa)}$.
3. Without loss of generality, $\{\xi < \kappa \colon \theta_{f'(\xi)} < \theta_{f(\xi)}\} \in W$.

By property (2) of the names $\langle \underset{\sim}{\theta}_\alpha \colon \alpha < \kappa \rangle$, presented in Lemma 2.9, there exists an ordinal β' such that some condition in H forces that $\underset{\sim}{\theta}_{i(f')(\kappa)} = \beta'$. By the above assumptions, $\beta' < \beta$.

We argue that β' is a generator of i as well. This will finish the proof: once we prove that β' is a generator of i, it follows from Claim 2 that $\theta_{i(f')(\kappa)}$ represents β' in the sense of both generics, H, H'. However, in the sense of H', it represents β, which is a contradiction.

Assume for contradiction that β' is not a generator of i. Then there is a function $g \in V$ and $\beta_1, \ldots, \beta_l$ below β', such that $\beta' = i(h)(\kappa, \beta_1, \ldots, \beta_l)$. Since H forces that $\beta' = \underset{\sim}{\theta}_{i(f')(\kappa)}$, it follows that

$$\{\xi < \kappa \colon g\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}\right) = \theta_{f'(\xi)}\} \in U_H = W.$$

Thus the same set belongs to $U_{H'}$. Therefore, H' forces that

$$\beta = \theta_{i(f')(\kappa)} = i(g)(\kappa, \beta_1, \ldots, \beta_l)$$

contradicting the fact that β is a generator of i (note that we used Claim 2 when arguing that the generators β_i, $1 \leq i \leq l$, are represented the same way in the sense of H, H').

Definition 2.21. A measure $W \in V[G]$ is called simply generated if $W = U_H$ for some $U \in V$, where H is generic for $\langle j_U(P) \setminus \kappa, \leq^* \rangle$ over $M_U[G]$.

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REMARK 2.22. Given a simply generated normal measure $W \in V[G]$ as above, with $\Delta \notin W$, the parameters U and H are uniquely defined from it[4]. Indeed, we will prove in the next lemma that $U = W \cap V$ belongs to V, and is a normal measure there with $\Delta \notin U$, Now, assume that there are H, H', generic over $M_U[G]$ for $\langle j_U(P) \setminus \kappa, \leq^* \rangle$, with $W = U_H = U_{H'}$. Then H, H' satisfy the conditions of Lemma 2.19 (since j_U has no generators other than κ). Thus, by the theorem, $H = H'$.

Given $W \in V[G]$ normal on κ (which is not necessarily simply generated), we can say the following:

LEMMA 2.23. *Assume that $2^\kappa = \kappa^+$. Then every normal measure $W \in V[G]$ on κ extends a measure $U = W \cap V \in V$.*

PROOF. By Proposition 2.1 in [6], it suffices to prove that there are no new fresh unbounded subsets of cardinals in the interval $\left[\kappa, (2^\kappa)^V\right] = \left[\kappa, \kappa^+\right]$. Thus, it suffices to prove the following pair of claims: ⊣

CLAIM 3. *$P = P_\kappa$ does not add fresh unbounded subsets to κ.*

PROOF. The fact that there are no fresh unbounded subsets of κ follows essentially from the facts that $2^\kappa = \kappa^+$, and that there exists a normal measure on κ in $V[G]$: Given a normal measure $U \in V$ with $\Delta \notin U$, take any $U^* \in V[G]$ which extends it. Given a fresh unbounded $A \subseteq \kappa$, $A = j_{U^*}(A) \cap \kappa$ and thus, by elementarity, A belongs to the ground model M of $\mathrm{Ult}(V[G], U^*)$. Now set $k_U \colon M_U \to M$ to be the function which maps $[f]_U$ to $[f]_{U^*}$. Then k_U is a well-defined elementary embedding since $U \subseteq U^*$, and $\mathrm{crit}(k_U) > \kappa$ by normality of U^*. Since $2^\kappa = \kappa^+$ holds in M, k_U maps the sequence of subsets of κ to itself, and thus every subset of κ which belongs to M, already belongs to V. So the above set A belongs to V, which is a contradiction. ⊣

CLAIM 4. *For every measurable (in V) $\lambda \leq \kappa$, P_λ doesn't add fresh unbounded subsets of λ^+. In particular, P_κ does not add fresh subsets to λ^+.*

PROOF. Let $f \in V[G]$ be the characteristic function of a fresh unbounded subset of λ^+. Let $\underset{\sim}{f}$ be a P_λ-name and assume that $p \in P$ forces that $\underset{\sim}{f}$ is fresh.

Let $G \subseteq P_\lambda$ be generic over V. For every $\xi < \lambda^+$, let $p_\xi \in G$ be a condition which decides $\underset{\sim}{f} \restriction \xi$. For every $\xi < \lambda^+$ there exists $\alpha_\xi < \lambda$ such that the support of p_ξ is bounded by α_ξ. Let $A \subseteq \lambda^+$ and $\alpha^* < \lambda$ be such that $|A| = \lambda^+$ and $\alpha_\xi = \alpha^*$ for every $\xi \in A$.

By shrinking $A \subseteq \lambda^+$ even further (to a set of cardinality λ^+), we can assume that there exists $q^* \in P_\lambda$ such that, for every $\xi \in A$, $p_\xi \restriction \alpha^* = q^* \restriction \alpha^*$, and $q^* \restriction [\alpha^*, \lambda)$ is trivial.

Let $h = \bigcup\{g \colon \exists \xi < \lambda \ q^* \Vdash \underset{\sim}{f} \restriction \xi = g\}$. Clearly, $h \colon \lambda^+ \to 2$ is a function and $q^* \Vdash \underset{\sim}{f} = \check{h}$.

□ of Lemma 2.23. ⊣

[4]Note that when iterating Prikry forcings, $\Delta \notin W$ holds for every normal measure $W \in V[G]$ on κ, since every such W concentrates on regulars.

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§3. On $j_W(\kappa) > j_U(\kappa)$ and the existence of N. Assume in this section that $P = P_\kappa$ is an iteration of Prikry forcings. Let $W, U = W \cap V, i \colon V \to N$ be as in the previous section.

Clearly, $j_W(\kappa) \geq j_U(\kappa)$. Our interest here will be in situations where a strict inequality holds.

Note that such a phenomenon is impossible with the nonstationary support, where, for every normal measure $W \in V[G]$ on κ, $j_W(\kappa) = j_U(\kappa)$ (see [7]).

On the other hand, in the full support iteration, it is possible that $j_W(\kappa) > j_U(\kappa)$ starting with $o(\kappa) \geq 2$. Indeed, under the assumption that there are normal measures $U_0 \lhd U_1$ on κ in V, take $W = (U_1)^\times$ (in the notations of [9]). Assume that $\xi \mapsto U_0(\xi)$ is a function in V which represents U_0 in $\mathrm{Ult}(V, U_1)$, and, for every $\xi \in \Delta$, $U_0(\xi)$ is the normal measure used to singularize ξ at stage ξ in the iteration $P = P_\kappa$. Then W extends U_0, but $j_W(\kappa) = j_{U_1}(\kappa) > j_{U_0}(\kappa)$. The other direction is also true: if $j_W(\kappa) > j_U(\kappa)$, let U' be a measure on κ in V such that $W = (U')^\times$. Then $U \lhd U'$ and thus $o(\kappa) \geq 2$.

Let us discuss this situation in the context of the Easton support iteration.

3.1. On $j_W(\kappa) > j_U(\kappa)$. Start with the following simple observation:

PROPOSITION 3.1. *The set*

$$\{j_W(f)(\kappa) \mid f : \kappa \to \kappa, f \in V\}$$

is unbounded in $j_W(\kappa)$.

Hence, $k'' j_U(\kappa)$ is unbounded in $j_W(\kappa)$, where $k([f]_U) = [f]_W$ is the embedding defined in Lemma 2.15.

PROOF. P satisfies κ-c.c. Hence for every $g : \kappa \to \kappa$ in $V[G]$ there is $f : \kappa \to \kappa$ in V which dominates it, i.e., for every $\nu < \kappa$, $g(\nu) < f(\nu)$. ⊣

Let us present a first example of a situation where $j_W(\kappa) > j_U(\kappa)$.

DEFINITION 3.2 (W. Mitchell). A cardinal κ is called *μ-measurable* iff there exists an extender E over κ such that $E_\kappa \in M_E$, where $E_\kappa = \{A \subseteq \kappa \mid \kappa \in j_E(A)\}$.

Note that we can use a witnessing extender E with two generators only—κ and the ordinal $\eta < 2^{2^\kappa}$ which codes E_κ. The ultrapower by such an extender is closed under κ-sequences.

The next lemma is obvious:

LEMMA 3.3. *Suppose that κ is μ-measurable and E is an extender witnessing this. Then $j_{E_\kappa}(\kappa) < j_E(\kappa)$.*

PROPOSITION 3.4. *Suppose that κ is μ-measurable and E is an extender witnessing this which ultrapower is closed under κ-sequences. Let $U = E_\kappa$ and $\Delta \subseteq \kappa$ be a set of measurable cardinals which is not in U. Force with an Easton support iteration P of the Prikry forcings over Δ. Let $G \subseteq P$ be a generic.*

Then, in $V[G]$, there is a normal ultrafilter W which extends U such that $j_W(\kappa) > j_U(\kappa)$.

PROOF. Construct W as in Theorem 1.1 using E, i.e., $i = j_E$ and $N = M_E$. Then $j_U(\kappa) < j_E(\kappa) = i(\kappa) = j_W(\kappa)$. ⊣

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Let us observe now that we need a μ-measurable in order to have $j_W(\kappa) > j_U(\kappa)$, provided $V = \mathcal{K}$, where $\mathcal{K}$ denotes the core model.

PROPOSITION 3.5. *Assume $\neg 0^{\P}$. Suppose that $V = \mathcal{K}$. Let U be a normal ultrafilter over κ and $\Delta \subseteq \kappa$ be a set of measurable cardinals which is not in U. Force with an Easton support iteration P of the Prikry forcings over Δ. Let $G \subseteq P$ be a generic.*

Suppose that, in $V[G]$, there is a normal ultrafilter W which extends U such that $j_W(\kappa) > j_U(\kappa)$.

Then κ is a μ-measurable in V. Moreover, U is a normal measure of a witnessing extender.

PROOF. Suppose otherwise.

Consider $j_W \restriction V$. Since $V = \mathcal{K}$ is the core model, $j_W \restriction V$ is a normal[5] iterated ultrapower of $\mathcal{K} = V$ by its measures and extenders (see [13]).

Recall that $W \cap V = U$, and so, $U = \{A \subseteq \kappa \mid A \in V, \kappa \in j_W(A)\}$. The assumption that U is not a normal measure of an extender which witnesses a μ-measurability of κ implies then that U must be used first in this iterated ultrapower.

Apply now the arguments of [4, 8] in $\mathcal{K}_U$ the core model of M_U. For every measurable $\alpha, \kappa \leq \alpha < j_U(\kappa)$, there will be a bound η_α (which depends on $o(\alpha)$) on the number of possible applications of measures and extenders over α with their images, and, by the assumption that there are no strong cardinals, $\eta_\alpha < j_U(\kappa)$. Therefore, for every such α, there exists an upper bound $\mu^*_\alpha < j_U(\kappa)$ on the image of α in the iterated ultrapower by the measures or extenders taken on α or its images. $\mu^*_\alpha < j_U(\kappa)$ since none of the extenders participating has length $j_U(\kappa)$ or above (by $\neg 0^{\P}$), and $\eta_\alpha < j_U(\kappa)$.

This implies that the rest of the iteration from j_U to $j_W \restriction \mathcal{K}$ cannot move $j_U(\kappa)$: otherwise, $j_U(\kappa)$ participates as a critical point in the iteration (it cannot be moved by an extender with a critical point below $j_U(\kappa)$, as explained above; it surely cannot moved by an extender with a critical point above $j_U(\kappa)$; thus, it moves since an extender on it participates in the iteration). Decompose $j_W \restriction V = k_1 \circ i_1$, where i_1 is the iteration with all the extenders below $j_U(\kappa)$, and k_1 is an iteration with critical point $j_U(\kappa)$. Then

$$\{j_W(f)(\kappa) \colon f : \kappa \to \kappa,\ f \in V\}$$

is bounded in $j_W(\kappa)$, since, for every $f : \kappa \to \kappa$ in V,

$$j_W(f)(\kappa) = k_1 \circ \big(i_1(f)(\kappa)\big) = i_1(f)(\kappa) < j_U(\kappa),$$

where we used the fact that $i_1(f)(\kappa)$ does not move in k_1 since it is strictly below $j_U(\kappa)$. This, contradicts Proposition 3.1. ⊣

The situation changes if we do not assume $V = \mathcal{K}$. Let us argue now that the consistency strength of $j_W(\kappa) > j_U(\kappa)$ is just a measurable which is a limit of measurable cardinals.

PROPOSITION 3.6. *Let V_0 be a model of GCH with a measurable cardinal κ which is a limit of measurable cardinals.*

[5]Extenders with smaller indexes are used first.

Then there is a cardinal preserving generic extension V of V_0 which satisfies the following:

Let Δ be an unbounded subset of κ consisting of measurable cardinals. Force with an Easton support iteration P of the Prikry forcings over Δ. Let $G \subseteq P$ be a generic.

There exists a normal ultrafilter U over κ in V and a normal ultrafilter W in $V[G]$ which extends U such that $j_W(\kappa) > j_U(\kappa)$.

PROOF. The idea is as follows. Let U_0 be a normal ultrafilter over κ in V_0 which concentrates on non-measurable cardinals. Consider $U_0^2 = U_0 \times U_0$ and $U_0^3 = U_0 \times U_0 \times U_0$.

Let $j_1 = j_{U_0}, j_2 = j_{U_0^2}, j_3 = j_{U_0^3}, M_1 = M_{U_0}, M_2 = M_{U_0^2}, M_3 = M_{U_0^3}, \kappa_1 = j(\kappa)$, $\kappa_2 = j_2(\kappa), \kappa_3 = j_3(\kappa)$. We have natural commuting embeddings $j_{12} : M_1 \to M_2, j_{23} : M_2 \to M_3$ and $j_{13} : M_1 \to M_3$. Namely, $j_{12}(j_1(f)(\kappa)) = j_2(f)(\kappa)$, $j_{23}(j_2(g)(\kappa, \kappa_1)) = j_3(g)(\kappa, \kappa_1)$, etc. Note that the critical point of j_{12}, j_{13} is κ_1 and of j_{23} is κ_2. However there is an additional way to embed M_2 into M_3. Define $\sigma : M_2 \to M_3$ by setting $\sigma(j_2(f)(\kappa, \kappa_1)) = j_3(f)(\kappa, \kappa_2)$. Clearly, σ is elementary and its critical point is κ_1 and it is moved to κ_2.

The idea will be to force in order to extend U_0 to a normal ultrafilter U such that:

1. M_U is a generic extension of M_2,
2. U_0^3 extends to a κ-complete ultrafilter E with M_E a generic extension of M_3,
3. U is the normal ultrafilter which is strictly below E with the corresponding embedding extending σ.

Now, $\kappa_2 < \kappa_3$ will imply $j_U(\kappa) < j_E(\kappa)$, since $j_U(\kappa) = \kappa_2$ and $j_E(\kappa) = \kappa_3$.

Such construction was used in [3]. We refer to this paper for details. Let us only sketch the argument.

We force a Cohen function $f_\alpha : \alpha \to \alpha$ for every inaccessible $\alpha \leq \kappa$ using the iteration with an Easton support.

Denote a generic object which produces such $\langle f_\alpha \mid \alpha \leq \kappa, \alpha \text{ is an inaccessible } \rangle$ by G_0.

Let $V = V_0[G_0]$.

It is possible to extend all the embeddings, $j_1, j_2, j_3, j_{12}, j_{13}, j_{23}, \sigma$. We change one value of f_{κ_3} at κ by setting it to κ_2. Let G_3 be such generic over M_3 Then, $j_3 : V_0 \to M_3$ extends to $j_3^* : V_0[G_0] \to M_3[G_3]$. Derive now U and E from j_3^*, in $V = V_0[G_0]$, by setting $U = \{A \subseteq \kappa \mid \kappa \in j_3^*(A)\}$ and $E = \{B \subseteq \kappa^3 \mid \langle \kappa, \kappa_1, \kappa_2 \rangle \in j_3^*(B)\}$.

Finally, we apply the construction of Section 2 to U and E to produce an extension W of U in $V[G]$. ⊣

Note that U produced in 3.6 can be picked to be the minimal in the Mitchell order, which is not true about one of 3.4, where $V = \mathcal{K}$. Let us argue that under rather strong assumptions it is possible to find such U in $\mathcal{K}$.

PROPOSITION 3.7. *Let U be a normal ultrafilter over κ. Suppose that the set*

$$\{\alpha < \kappa \mid \alpha \text{ is } \kappa\text{-strong} \}$$

is unbounded in κ. Force with P as above. Let $G \subseteq P$ be a generic. Then, in $V[G]$, there is a normal ultrafilter W over κ such that:

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1. $U \subseteq W$,
2. $j_U(\kappa) < j_W(\kappa)$,
   moreover, $j_W \restriction V = k \circ i$, *where*
   - $i : V \to N$,
   - $j_U(\kappa) < i(\kappa)$,
   - i, N *satisfy the conditions of Theorem* 1.1.

PROOF. Work in M_U. Pick some $\alpha, \kappa < \alpha < j_U(\kappa)$ which is $j_U(\kappa)$-strong. Let $E \in M_U$ be an $(\alpha, j_U(\kappa))$-extender witnessing this. Set N to be the ultrapower of M_U by E and let $i = j_E \circ j_U$. We have

$$j_U(\kappa) \leq j_E(\alpha) < j_E(j_U(\kappa)) = i(\kappa).$$

Note that the embedding i satisfies the assumptions of Theorem 1.1. Indeed, the only nontrivial properties of i that require verification are:

1. $\{i(f)(\kappa) \colon f : \kappa \to \kappa\}$ is unbounded in $i(\kappa)$: In M_U, denote $\lambda = j_U(\kappa)$. Then λ is regular, and thus the (α, λ)-extender embedding j_E is continuous at λ. Thus, for every $\beta < i(\kappa) = j_E(\lambda)$, there exists $\beta' < \lambda$ such that $j_E(\beta') > \beta$. Now, find $f \in V$, $f : \kappa \to \kappa$, such that $\beta' = j_U(f)(\kappa)$. Then $i(f)(\kappa) = j_E(\beta') > \beta$, as desired. $|j_E(\kappa)| = \left(j_U(\kappa)\right)^+$.
2. $|i(\kappa)| = \kappa^+$: Using the above notations, $M_U \Vdash |j_E(\lambda)| = \lambda^+$. But $V \Vdash |\lambda^+| = \kappa^+$ since $2^\kappa = \kappa^+$.

Now apply Theorem 1.1 to construct the desired measure W. ⊣

We do not know whether the assumption of 3.7 is really necessary. However, it is possible to show the following.

PROPOSITION 3.8. *Suppose* $\neg 0^{\P}$.
Assume $V = \mathcal{K}$.
Let U *be a normal ultrafilter over* κ *which is minimal in the Mitchell order.*
Let P *be an Easton support iteration of Prikry-type forcing notions up to* κ *and* $G \subseteq P$ *be a generic.*
Suppose that W *is a normal ultrafilter in* $V[G]$ *which extends* U.
Then $j_U(\kappa) = j_W(\kappa)$.

PROOF. Since $V = \mathcal{K}$ is the core model, $j_W \restriction \mathcal{K}$ is a normal iterated ultrapower of $\mathcal{K}$ by its measures and extenders (see [13]).

The minimality of U implies that it must be used first in this iteration.

Apply now the arguments of [4, 8] in $\mathcal{K}_U$ the core model of M_U. For every measurable $\alpha, \kappa \leq \alpha < j_U(\kappa)$, there will be a bound η_α (which depends on $o(\alpha)$) on number of possible applications of measures and extenders over α with their images, and, by the assumption that there is no strong cardinal, $\eta_\alpha < j_U(\kappa)$. Now complete the argument as in Proposition 3.5 by showing that, if $j_U(\kappa) < j_W(\kappa)$ then an extender with critical point $j_U(\kappa)$ participates in the iteration, and thus the set $\{j_W(f)(\kappa) \colon f : \kappa \to \kappa,\ f \in V\}$ is bounded by $j_U(\kappa)$; this contradicts Proposition 3.1. ⊣

We conjecture that the needed strength (for 3.8) is exactly

$$\{\alpha < \kappa \mid \alpha \text{ is } \kappa\text{-strong}\} \text{ is unbounded in } \kappa.$$

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Thus, Schindler [12] extension of the Mitchell result can be used to argue that $j_W \restriction \mathcal{K}$ is a normal iterated ultrapower of $\mathcal{K}$ by its measures and extenders. A missing part is an extension of [4] beyond strong which is likely to hold.

3.2. On existence of N. As before, let $W \in V[G]$ be a normal measure on κ, and $U = W \cap V \in V$. In Section 4 we will prove that if W is constructed as in Theorem 1.1 then $j_W \restriction V = k \circ i$, where k is an iteration of N by normal measures only. A natural question in view of this result is whether for every $W \in V[G]$ there exists $N, {}^{\kappa}N \subseteq N$ such that M is obtained from it by iterating normal measures only. We do not know the answer in general. However, it turns out to be an affirmative provided some anti large cardinal assumptions and $V = \mathcal{K}$.

Proposition 3.9. *Assume $\neg 0^{\P}$ and $V = \mathcal{K}$.*

Let U be a normal ultrafilter over κ and $\Delta \subseteq \kappa$ be a set of measurable cardinals which is not in U. Force with an Easton support iteration P of the Prikry forcings over Δ. Let $G \subseteq P$ be a generic.

Suppose that, in $V[G]$, there is a normal ultrafilter W which extends U.

Then there are $N, i : V \to N$ which satisfy the conditions of Theorem 1.1 such that $j_W \restriction V = k \circ i$ and k is formed by iterating normal measures only, starting from N.

Proof. As in Section 3.1, we analyze $j := j_W \restriction \mathcal{K}$.

By elementarity, $j : \mathcal{K} \to (\mathcal{K})^{M_W}$ and M_W is a generic extension of $(\mathcal{K})^{M_W}$ by an Easton support iteration of Prikry forcings with normal measures in $j(\Delta)$.

Since $V = \mathcal{K}$ is the core model, j is an iterated ultrapower of $\mathcal{K}$ by its measures and extenders (see [13]). Recall that $W \cap \mathcal{K} = U$, and so, $U = \{A \subseteq \kappa \mid A \in V, \kappa \in j_W(A)\}$. So, this iterated ultrapower starts either with U or with an extender F such that $U = \{X \subseteq \kappa : \kappa \in j_F(X)\}$.

Note that M_F must be closed under κ-sequences. Otherwise, there will be a set of ordinals a, $|a| < \kappa$ which consists of generators and which is not in M_F. The further Easton support iteration of Prikry forcings will not be able to add such a. Thus, by our assumption, the length of F must be below the first measurable cardinal above κ in M_F. The iteration of Prikry forcings above κ does not add new bounded subsets below the first measurable $> \kappa$.

By the same reason, extenders used to continue the iteration must be κ-closed.

None of them can be used infinitely many times (or infinitely many extenders cannot be used), since otherwise, ω-sequences which cannot be added by an Easton support iteration of Prikry forcings, will be produced. It follows from the strong Prikry condition of the forcing, which can be shown for the relevant parts as in Ben Neria [2].

This leaves us with a finite iteration by κ-closed extenders (measures).

It is the first part of the iteration.

The rest consists of an iteration of normal measures, each of them is applied ω-many times.

Take N to be the first part of the iteration and $i : \mathcal{K} \to N$ be the corresponding embedding. ⊣

§4. Properties of k. We continue and use the notations of Theorem 1.1. We first state the following lemma.

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LEMMA 4.1. *Assume that $2^\kappa = \kappa^+$. Let $P = P_\kappa$ be an Easton support iteration of Prikry-type forcings, and $i : V \to N$, $\Delta \subseteq \kappa$, $U \in V$, $W \in V[G]$ and $k : N \to M$ be as in Section 2.*

Assume that there are no elements in $(\kappa, crit(k)) \cap i(\Delta)$. Then $crit(k) \in i(\Delta)$, namely, it is the least element above κ in $i(\Delta)$.

REMARK 4.2. The assumption $(\kappa, \mathrm{crit}(k)) \cap i(\Delta) = \emptyset$ holds in the typical case where $P = P_\kappa$ is an iteration of Prikry forcings. Indeed, denote $\lambda = \mathrm{crit}(k)$, and assume, by contradiction, that there exists $\mu \in (\kappa, \lambda) \cap i(\Delta)$. Then $\mu = k(\mu)$, and thus in $M[j_W(G)]$, μ changes cofinality to ω. Therefore, in $V[G]$, $\mathrm{cf}(\mu) = \omega$. Since P preserves cofinalities above κ, it follows that in V, $\mathrm{cf}(\mu) \le \kappa$. The sequence witnessing this belongs to $V \cap ({}^\kappa N)$ and thus, by our assumption on N, belongs already to N. This contradicts the measurability of μ in N.

PROOF. Denote $\lambda = \mathrm{crit}(k)$. Then for some $h \in V$ and $\kappa = \beta_0 < \beta_1 < \cdots < \beta_k$,

$$\lambda = i(h)(\kappa, \beta_1, \dots, \beta_k).$$

By the definition of k, $\lambda > \kappa$.

We first prove that $\lambda \in i(\Delta)$. Assume otherwise. We can assume without loss of generality that for every $\xi, \nu_1, \dots, \nu_k$ below κ, $h(\xi, \nu_1, \dots, \nu_k) > \xi$ does not belongs to Δ: this can be assumed by replacing the function h with the function $h' : [\kappa]^{k+1} \to \kappa$ defined as follows: For every $\xi, \eta_1, \dots, \eta_k$, $h'(\xi, \eta_1, \dots, \eta_k)$ equals $h(\xi, \eta_1, \dots, \eta_k)$ if $h(\xi, \eta_1, \dots, \eta_k) > \xi$ is not measurable in V; and else, $h'(\xi, \eta_1, \dots, \eta_k)$ is an arbitrary non-measurable above ξ. By our assumption,

$$i(h)(\kappa, \beta_1, \dots, \beta_k) = i(h')(\kappa, \beta_1, \dots, \beta_k)$$

so we can replace h with h'. Since λ is regular (as a critical point of an elementary embedding), we can assume, using a similar argument, that each $h(\xi, \nu_1, \dots, \nu_k)$ is regular.

We can assume that for every $\xi, \mu_1, \dots, \nu_k$, there are no elements of Δ in the interval $(\xi, h(\xi, \nu_1, \dots, \nu_k))$.

Let $f \in V[G]$ be a function such that $[f]_W = \lambda$. Then

$$[f]_W = \lambda < k(\lambda) = j_W(h)\left(\kappa, \theta_{\left[f_{\beta_1}\right]_W}, \dots, \theta_{\left[f_{\beta_k}\right]_W}\right).$$

By the definition of W, there exist $p \in G$ and $r \in H$ such that

$$p^\frown r \Vdash i(\underset{\sim}{f})(\kappa) < i(h)\big(\kappa, \theta_{i\left(f_{\beta_1}\right)(\kappa)}, \dots, \theta_{i\left(f_{\beta_k}\right)(\kappa)}\big).$$

Recall that, for every $1 \le i \le k$, there exists a condition in H forcing that $\theta_{i\left(f_{\beta_i}\right)(\kappa)} = \beta_i$. Thus by extending r inside H,

$$p^\frown r \Vdash i(\underset{\sim}{f})(\kappa) < i(h)(\kappa, \beta_1, \dots, \beta_k).$$

Since there are no measurables of N in the interval $\big(\kappa, i(h)(\kappa, \beta_0, \dots, \beta_k)\big]$, we can find $r' \ge^* r$ inside H such that

$$p \Vdash \exists \alpha < i(h)(\kappa, \beta_1, \dots, \beta_k),\ r' \Vdash i(f)(\kappa) < \alpha,$$

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and since $P = P_\kappa$ is κ-c.c. and $i(h)(\kappa, \beta_1, \dots, \beta_k)$ is regular, there exists $\alpha < i(h)(\kappa, \beta_1, \dots, \beta_k)$ such that

$$p^\frown r' \Vdash i(\underset{\sim}{f})(\kappa) < \alpha.$$

Now apply k on both sides. By Lemma 2.18,

$$M\left[j_W(G)\right] \vDash \lambda = [f]_W < k(\alpha)$$

but $\alpha < i(h)(\kappa, \beta_1, \dots, \beta_k) = \lambda$ and thus $\lambda < k(\alpha) = \alpha < \lambda$, which is a contradiction. ⊣

REMARK 4.3. Assume that $P = P_\kappa$ is an iteration of the one-point Prikry forcings. A one-point Prikry forcing on a measurable α is a forcing, which depends on a normal measure U on α, and is defined as follows: Conditions are of the form A where $A \in U$ or ξ for some ordinal $\xi < \alpha$. The latter kind of condition cannot be extended. A condition of the form A for $A \in U$ can be extended in two ways: A direct extension is a condition B where $B \in U$ and $B \subseteq A$; a non-direct extension is of the form ξ where $\xi \in A$ is an ordinal.

We argue that in this case, the question of whether $\left(\kappa, \mathrm{crit}(k)\right) \cap i(\Delta) \neq \emptyset$, and, as a result, the value of $\mathrm{crit}(k)$, depend on the choice of H:

1. Denote by μ the first element above κ in $i(\Delta)$. Assume first that H is chosen such that the condition on coordinate μ is a measure one set. In this case, $\mu = \mathrm{crit}(k)$. Indeed, $\mathrm{crit}(k) < \mu$ cannot hold, since then $\left(\kappa, \mathrm{crit}(k)\right) \cap i(\Delta) = \emptyset$ which implies, by the last lemma, that $\mu = \mathrm{crit}(k)$. And $\mu < \mathrm{crit}(k)$ cannot hold since then $k(\mu) = \mu$; in this case, denote by $\mu_0 < \mu$ the one point added below μ in $j_W(G)$. Then H at coordinate μ has a condition which is incompatible with μ_0 (by shrinking the large set and applying a density argument), which is a contradiction. Thus $\mu = \mathrm{crit}(k)$.
2. Denote now by μ the least element in $i(\Delta)$, for which H does not specify the one-point element added to it. We argue that $\mathrm{crit}(k) = \mu$, even though μ doesn't have to be the least element above κ in $i(\Delta)$.

   Repeat the proof of the last lemma, and note that the $\leq^*$ forcing in the interval (κ, μ) is trivial, since no condition in this interval can be non-trivially extended. This replaces the assumption that there are no elements of $i(\Delta)$ in the interval $\left(\kappa, i(h)(\kappa, \beta_1, \dots, \beta_k)\right)$. Therefore, $\mu = \mathrm{crit}(k)$.

Let us deal here with an Easton support iteration P of the Prikry forcings over a set Δ of a measurable length κ. Let U be a normal ultrafilter over κ in V with $\Delta \notin U$. Let $G \subseteq P$ be a generic and W be a normal ultrafilter in $V[G]$ which extends U.

Let $i\colon V \to N$ be an elementary embedding as in Theorem 1.1, and assume that $W = U_H$ and $k\colon N \to M$ are as in Lemma 2.15.

In the setting of iteration of Prikry forcings, much more can be said about the embedding $k\colon N \to M$. From Remark 4.2, it follows that $\mathrm{crit}(k)$ is the least element in $i(\Delta)$ above κ. In particular, by elementarity, $k(\mu) \in j_W(\Delta)$ in M, and thus a Prikry sequence is added to $k(\mu)$ in $j_W(G)$.

LEMMA 4.4. *Denote $\mu = crit(k)$. Then μ appears in the Prikry sequence of $k(\mu)$.*

REMARK 4.5. μ is not necessarily the first element in the Prikry sequence of $k(\mu)$. The initial segment of this Prikry sequence below μ depends on the choice of H.

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For every finite sequence $t \in [\mu]^{<\omega}$, we can choose $H \subseteq i(P) \setminus \kappa$ such that t is an initial segment of the Prikry sequence of μ. This way, in $M\left[j_W(G)\right]$, t will be an initial segment of the Prikry sequence of $k(\mu)$ below μ.

Proof. Let t be the finite initial segment of the Prikry sequence of $k(\mu)$ below μ, and assume that $\langle \xi, \eta_1, \dots, \eta_l \rangle \mapsto t(\xi, \eta_1, \dots, \eta_l)$ is a function in V, such that

$$t = i\left(\langle \xi, \eta_1, \dots, \eta_l \rangle \mapsto t(\xi, \eta_1, \dots, \eta_l)\right)(\kappa, \beta_1, \dots, \beta_l)$$

for some generators $\beta_1, \dots, \beta_l$ of i. For every $\xi < \kappa$, let $s(\xi) = \min\{\Delta \setminus (\xi + 1)\}$, so $\left[\xi \mapsto s(\xi)\right]_W = \mu$. In $V[G]$, define, for every $\xi < \kappa$,

$$\mu(\xi) = \text{the first element above } t\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_k}(\xi)}\right) \text{ in the Prikry sequence of } s(\xi)$$

and, if $t\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_k}(\xi)}\right)$ is not an initial segment of the Prikry sequence of $s(\xi)$, set $\mu(\xi) = 0$.

It suffices to prove that $\left[\xi \mapsto \mu(\xi)\right]_W = \mu$.

Assume first that $\eta < \mu$. Work in $N[G]$. Since H is $\leq^*$-generic, it meets an element $q \in i(P) \setminus \kappa$, for which $A^q_\mu \subseteq \mu \setminus (\eta + 1)$. Since $q \in H$, we can assume that t^q_μ is an initial segment of t: Indeed, t, t^q_μ are compatible sequences, since, for any $p \in G$ which forces that $q \in H$ and decides the value of t^q_μ, the condition $k\left(p^\frown q\right) = p^\frown k(q)$ belongs to $j_W(G)$, and decides an initial segment, below μ, of the Prikry sequence of $k(\mu)$. By our assumption, this initial segment is contained in t, and $p^\frown k(q)$ forces that every possible extension of it is above η. Thus, in $M\left[j_W(G)\right]$, each element in the Prikry sequence of $k(\mu)$ after t is strictly above η.

The argument given in the previous paragraph also shows that for every $q \in H$, t^q_μ is either empty or equals to t: As mentioned, it must be an initial segment of t. Let us argue that if it is proper, then it is empty. Apply the above paragraph for $\eta = \max(t)$. Then by direct extending q inside H, it forces that the element after t^q_μ in the Prikry sequence of μ is strictly above η. By applying $k\colon N \to M$, there exists a condition in $j_W(G)$ which forces that the Prikry sequence of $k(\mu)$ has an initial segment t^q_μ, followed only by elements above η. So t^q_μ cannot be a proper initial segment of t.

Assume now that $\eta < \left[\xi \mapsto \mu(\xi)\right]_W$. Write $\eta = [f]_W$ and assume that for every $\xi < \kappa$,

$$f(\xi) < \mu(\xi) < s(\xi).$$

Let $p \in G$ be a condition which forces this. Work in $N[G]$. Take $q \in H$ such that $t^q_\mu = t$. Then $i(p)^\frown q = p^\frown q$ forces that $i(\underset{\sim}{f})(\kappa)$ is below the first element above t in the Prikry sequence of μ. Thus, its value can be decided by taking a direct extension. So, by direct extending q inside H we can assume that

$$p \Vdash \exists \alpha < \mu,\ q \Vdash i\left(\underset{\sim}{f}\right)(\kappa) < \alpha,$$

and thus there exists $\alpha < \mu$ in V, such that

$$p^\frown q \Vdash i(f)(\kappa) < \alpha.$$

Thus, in $M\left[j_W(G)\right]$, $\eta = j_W(f)(\kappa) < k(\alpha) = \alpha < \mu$, as desired. ⊣

In the next subsection we will decompose the embedding k to an iterated ultrapower of N. We now demonstrate the first step in the iteration:

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LEMMA 4.6. *Let $\mu = crit(k)$ and let $U_\mu = \{X \subseteq \mu \colon \mu \in k(X)\} \cap N$. Then $U_\mu \in N$.*

PROOF. For every $\xi < \kappa$, denote by W_ξ the measure in $V\left[G_\xi\right]$ used to singularize ξ in the Prikry forcing at stage ξ in the iteration. Let $U_\xi = W_\xi \cap V$. We first argue that there exists a set $\mathcal{F} \in N$ of measures on μ, with $|\mathcal{F}| < \mu$, such that, for some $p \in G$ and $q \in H$,

$$p^\frown q \Vdash i\left(\xi \mapsto \underset{\sim}{U}_\xi\right)(\mu) \in \mathcal{F}. \tag{1}$$

Indeed, let $\underset{\sim}{\alpha}$ be a $j_U(P)$-name for the index of $i\left(\xi \mapsto \underset{\sim}{U}_\xi\right)(\mu)$ in a prescribed well order of the normal measures μ carries in N. Work in $N[G]$. For some $q \in H$, there exists an ordinal β such that $q \Vdash \underset{\sim}{\alpha} = \beta$. Thus, by $\kappa - c.c.$ of the forcing $i(P)_\mu = P_\kappa$, there exist $p \in G$ and a set $S \subseteq 2^{2^\mu}$ of ordinals with $|S| < \mu$, such that $p^\frown q \Vdash \underset{\sim}{\alpha} \in S$. In particular, $p^\frown q$ forces that $i\left(\xi \mapsto \underset{\sim}{U}_\xi\right)(\mu)$ belongs to $\mathcal{F}$, where $\mathcal{F}$ is the set of measures on μ indexed in S.

Now apply k on equation (1), and work in $M\left[j_W(G)\right]$. Since $|\mathcal{F}| < \mu$, it follows that there exists a measure $F \in \mathcal{F}$ such that

$$j_W\left(\xi \mapsto U_\xi\right)(k(\mu)) = k(F)$$

so it suffices to argue that $F = \{X \subseteq \mu \colon \mu \in k(X)\} \cap N$. Fix $X \in F$. Write $X = i(g)(\kappa, \beta_0, \dots, \beta_k)$. Then

$$j_W(g)\left(\kappa, \theta_{\left[f_{\beta_1}\right]_W}, \dots, \theta_{\left[f_{\beta_k}\right]_W}\right) \in j_W\left(\xi \mapsto U_\xi\right)\left(k(\mu)\right).$$

Recall the function $\xi \mapsto s(\xi) = \min\left(\Delta \setminus (\xi + 1)\right)$, for which $\left[\xi \mapsto s(\xi)\right]_W = k(\mu)$. We can assume that for every $\xi < \kappa$,

$$g\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_k}(\xi)}\right) \in U_{s(\xi)}$$

and let $p \in G$ be a condition which forces this. Then for strong enough $q \in H$,

$$p^\frown q \Vdash i(g)(\kappa, \beta_1, \dots, \beta_k) \in i\left(\xi \mapsto \underset{\sim}{U}_\xi\right)(\mu),$$

and thus by direct extending q further, we can assume that q forces that the first element after t in the Prikry sequence of μ belongs to $i(g)(\kappa, \beta_1, \dots, \beta_k) = X$. Thus $k(q) \in j_W(G)$ forces that the first element after t in the Prikry sequence of $k(\mu)$ belongs to $k(X)$. By the previous lemma, it follows that $\mu \in k(X)$, as desired. ⊣

4.1. Description of $j_W \restriction V$. We now generalize the previous subsection, in order to completely decompose $j_W \restriction V$. We continue to assume the hypotheses of Theorem 1.1, and also that $P = P_\kappa$ is an iteration of Prikry forcings. For technical reasons, we will assume that the measures used in the iteration $P = P_\kappa$ to singularize the measurables in Δ are all simply generated; this is needed only in the proof of Claim 6 which will be presented in the next subsection.

At each stage $\alpha \in \Delta$, let $\underset{\sim}{Q}_\alpha$ be the P_α-name for the Prikry forcing on α, using a simply generated normal measure $\underset{\sim}{W}_\alpha$ on α. Denote $\underset{\sim}{U}_\alpha = \underset{\sim}{W}_\alpha \cap V \in V$. Let $\underset{\sim}{H}_\alpha \subseteq (j_{\underset{\sim}{U}_\alpha}(P_\alpha) \setminus \alpha, \leq^*)$, $\underset{\sim}{H}_\alpha \in V[G_\alpha]$, be $\leq^*$-generic over $M_{\underset{\sim}{U}_\alpha}[G_\alpha]$, such that $\underset{\sim}{W}_\alpha = (\underset{\sim}{U}_\alpha)_{\underset{\sim}{H}_\alpha}$.

Let $G \subseteq P_\kappa$ be generic over V.

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Our goal is to prove the following theorem:

THEOREM 4.7. *Assume the hypotheses of Theorem 1.1. Let $H \in V[G]$ be a generic set for $\langle i(P) \setminus \kappa, \leq^* \rangle$ which satisfies $(*)$. Let $W = U_H$ be the corresponding normal measure on κ extending U, and denote its ultrapower embedding $j_W \colon V[G] \to M\left[j_W(G)\right] \simeq Ult(V[G], W)$ for some model M. Factor $j_W \restriction V$ to the form $j_W \restriction V = k \circ i$ for some elementary $k \colon N \to M$, as in Theorem 1.1.*

Assume that P is an Easton support iteration, where at each step $\beta \in \Delta$, $\underset{\sim}{Q}_\beta$ is forced to be Prikry forcing with a simply generated normal measure on β.

Then k is an iterated ultrapower of N by normal measures and $j_W(\kappa) = i(\kappa)$.

Furthermore, if W itself is simply generated, than $j_W \restriction V = k \circ j_U$ is an iterated ultrapower with normal measures only, and $j_W(\kappa) = j_U(\kappa)$.

This, in contrast to full-support and nonstationary-support iterations of Prikry forcings, where, assuming $\mathrm{GCH}_{\leq\kappa}$, $j_W \restriction V$ is an iteration of V by normal measures only, for every normal measure $W \in V[G]$.

Throughout this section, we fix the notation mentioned in the formulation of the theorem. Let $H \in V[G]$ be a generic for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$ with the property $(*)$. In the case where $i = j_U$ and $N = M_U$, any generic for $\langle i(P) \setminus \kappa \rangle, \leq^* \rangle$ is such. Let $W = U_H \in V[G]$ be the corresponding normal measure on κ. Let $j_W \colon V[G] \to M\left[j_W(G)\right]$ be the corresponding ultrapower embedding.

Denote by $B \subseteq \left(\kappa, i(\kappa)\right)$ the set of generators of i. By property $(*)$ of H, for every $\beta \in B$, there exists a function f_β in V such that H forces that $\beta = \theta_{i(f)(\kappa)}$. The mapping $\beta \mapsto f_\beta$ is available in $V[G]$.

Recall the embedding $k \colon N \to M$ defined in Lemma 2.15:

$$k\left(i(f)(\kappa, \beta_1, \ldots, \beta_k)\right) = j_W(f)\left(\kappa, \theta_{\left[f_{\beta_1}\right]_W}, \ldots, \theta_{\left[f_{\beta_k}\right]_W}\right)$$

for every $f \in V$ and $\beta_1, \ldots, \beta_k \in B$. Then k is elementary, $\mathrm{crit}(k) > \kappa$ and $j_W \restriction V = k \circ i$.

Denote $\kappa^* = i(\kappa)$. Define by induction a linearly directed system $\langle\langle M_\alpha \colon \alpha \leq \kappa^* \rangle, \langle j_{\alpha,\beta} \colon \alpha < \beta \leq \kappa^* \rangle\rangle$ such that:

1. $M_0 = N$, $j_0 = i$.
2. *Successor Step:* Assume that $\alpha < \kappa^*$ and M_α has been defined. We will define an elementary embedding $k_\alpha \colon M_\alpha \to M$, such that $j_W \restriction V = k_\alpha \circ j_\alpha$. We denote $\mu_\alpha = \mathrm{crit}\,(k_\alpha)$ and define

$$U_{\mu_\alpha} = \{X \subseteq \mu_\alpha \colon \mu_\alpha \in k_\alpha(X)\} \cap M_\alpha.$$

   We will prove that $U_{\mu_\alpha} \in M_\alpha$ and take $M_{\alpha+1} \simeq \mathrm{Ult}\left(M_\alpha, U_{\mu_\alpha}\right)$. We also take $j_{\alpha,\alpha+1} \colon M_\alpha \to M_{\alpha+1}$ to be the ultrapower embedding $j^{M_\alpha}_{U_{\mu_\alpha}}$, and $j_{\alpha+1} = j_{\alpha,\alpha+1} \circ j_\alpha$.
3. *Limit Step:* For every limit $\alpha \leq \kappa^*$, the system $\langle M_\beta \colon \beta < \alpha \rangle, \langle j_{\beta,\gamma} \colon \beta < \gamma < \alpha \rangle$ is linearly directed, and we take direct limit to form the model M_α and the embedding $j_\alpha \colon V \to M_\alpha$.

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For every $\alpha < \kappa^*$, define $k_\alpha \colon M_\alpha \to M$ as follows:

$$\begin{aligned} k_\alpha \left(j_\alpha (f) \left(\kappa, j_{0,\alpha}(\beta_1), \dots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_k} \right) \right) \\ = j_W (f) \left(\kappa, \theta_{\left[f_{\beta_1} \right]_W}, \dots, \theta_{\left[f_{\beta_l} \right]_W}, \mu_{\alpha_1}, \dots, \mu_{\alpha_k} \right) \end{aligned}$$

for every $f \in V$, $\beta_1, \dots, \beta_l$ generators of i and $\alpha_1 < \cdots < \alpha_k < \alpha$.

Our goal is to prove by induction on $\alpha < \kappa^*$ the following properties:

(A) $k_\alpha \colon M_\alpha \to M$ is an elementary embedding, and $j_W \restriction V = k_\alpha \circ j_\alpha$.

(B) μ_α is measurable in M_α. Moreover, it is the least measurable in $j_\alpha(\Delta)$, which is greater or equal to $\sup\{\mu_\beta \colon \beta < \alpha\}$, and whose cofinality is above κ in V.

(C) μ_{μ_α} appears in the Prikry sequence of $k_\alpha(\mu_\alpha)$.

(D) Let U_{μ_α} be defined in $V[G]$ as above. Then $U_{\mu_\alpha} \in M_\alpha$ is a normal measure which concentrates on $\mu_\alpha \setminus j_\alpha(\Delta)$. Moreover,

$$k_\alpha \left(U_{\mu_\alpha} \right) = j_W (\delta \mapsto U_\delta) \left(k_\alpha (\mu_\alpha) \right),$$

where, for every $\delta \in \Delta$, $U_\delta = W_\delta \cap V$, for W_δ which is the measure used in the Prikry forcing at stage δ in the iteration P.

After that, we will prove in Lemma 4.20, that $k_{\kappa^*} \colon M_{\kappa^*} \to M$ is the identity, and thus $j_W \restriction V = j_{\kappa^*}$. This will conclude the proof of Theorem 4.7.

REMARK 4.8. We remark that k_α is well-defined in the sense that there is no $\alpha' < \alpha$ and generator β of i, for which $j_{0,\alpha}(\beta) = \mu_{\alpha'}$. Indeed, assume otherwise. Note that $\mu_{\alpha'} = j_{0,\alpha}(\beta) \geq j_{0,\alpha'}(\beta)$. Strict inequality is not possible here, since if $j_{0,\alpha'}(\beta) < \mu_{\alpha'}$ then $j_{0,\alpha'}(\beta) = j_{0,\alpha}(\beta) = \mu_{\alpha'}$, which is a contradiction. Thus, $j_{0,\alpha'}(\beta) = \mu_{\alpha'}$ (which is, by itself, possible for $\alpha' < \alpha$ - see Remark 4.9), but then, applying $j_{\alpha',\alpha}$ on both sides, we get

$$j_{0,\alpha}(\beta) = j_{\alpha',\alpha}(\mu_{\alpha'}) > \mu_{\alpha'},$$

where the last inequality follows since $\mu_{\alpha'} = \mathrm{crit}(j_{\alpha',\alpha})$.

REMARK 4.9. It is possible that a generator β of i is measurable in N and belongs to $i(\Delta)$. In this case, there exists $\alpha < \kappa^*$ such that $\mu_\alpha = \beta = j_{0,\alpha}(\beta)$. Such β will appear as an element in the Prikry sequence of $k_\alpha(\beta) \in j_W(\Delta)$, which also has the form $\theta_{[f_\beta]_W}$.

Properties (A) – (D) of k_α, presented above, will be proved by induction on $\alpha < \kappa^*$. The proof of the inductive step at stage $\alpha < \kappa^*$ will be carried out in Section 4.3, using the tools presented in [7] and [9]. Fixing $\alpha < \kappa^*$, we can assume by induction that $k_{\alpha'} \colon M_{\alpha'} \to M$ and $\mu_{\alpha'}, U_{\mu_{\alpha'}}$, for $\alpha' < \alpha$, satisfy properties (A) – (D). Denote by $t_{\alpha'}$ the initial segment of the Prikry sequence of $k_{\alpha'}(\mu_{\alpha'})$ below $\mu_{\alpha'}$.

DEFINITION 4.10. Fix $\alpha < \kappa^*$ and a sequence of generators $\langle \beta_1, \dots, \beta_l \rangle$ for i. An increasing sequence $\langle \alpha_1, \dots, \alpha_k \rangle$ below α is called a $\langle \beta_1, \dots, \beta_l \rangle$-nice sequence if there are functions $g_1, \dots, g_k, t_1, \dots, t_k$ in V, such that

$$\begin{aligned} \mu_{\alpha_1} &= j_{\alpha_1}(g_1) \left(\kappa, j_{0,\alpha_1}(\beta_1), \dots, j_{0,\alpha_1}(\beta_l) \right), \\ t_{\alpha_1} &= j_{\alpha_1}(t_{\alpha_1}) \left(\kappa, j_{0,\alpha_1}(\beta_1), \dots, j_{0,\alpha_1}(\beta_l) \right), \\ U_{\mu_{\alpha_1}} &= j_{\alpha_1}(F_1) \left(\kappa, j_{0,\alpha_1}(\beta_1), \dots, j_{0,\alpha_1}(\beta_l) \right), \end{aligned}$$

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and, for every $1 \leq i < k$,

$$\begin{aligned}
\mu_{\alpha_{i+1}} &= j_{\alpha_{i+1}}\left(g_{i+1}\right)\left(\kappa, j_{0,\alpha_1}\left(\beta_1\right), \dots, j_{0,\alpha_1}\left(\beta_l\right), \mu_{\alpha_1}, \dots, \mu_{\alpha_i}\right), \\
t_{\alpha_{i+1}} &= j_{\alpha_{i+1}}\left(t_{i+1}\right)\left(\kappa, j_{0,\alpha_1}\left(\beta_1\right), \dots, j_{0,\alpha_1}\left(\beta_l\right), \mu_{\alpha_1}, \dots, \mu_{\alpha_i}\right), \\
U_{\mu_{\alpha_{i+1}}} &= j_{\alpha_{i+1}}\left(F_{i+1}\right)\left(\kappa, j_{0,\alpha_1}\left(\beta_1\right), \dots, j_{0,\alpha_1}\left(\beta_l\right), \mu_{\alpha_1}, \dots, \mu_{\alpha_i}\right).
\end{aligned}$$

Fix now $\alpha < \kappa^*$. Assume by induction that properties (A) – (D) above hold for every $\alpha' < \alpha$. Fix also a sequence of generators $\langle \beta_1, \dots, \beta_l \rangle$ for i, and a $\langle \beta_1, \dots, \beta_l \rangle$-nice sequence $\langle \alpha_1, \dots, \alpha_k \rangle$ below α. We define, in $V[G]$, functions which can be used to represent μ_{α_i}, t_{α_i}, U_{α_i}. Assume that μ_{α_i} is the n_ith element in the Prikry sequence of $k_{\alpha_i}\left(\mu_{\alpha_i}\right)$.

First, set

$$\mu_{\alpha_1}(\xi) = \text{the } n_1\text{th element in the Prikry sequence of } g_1(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}).$$

By induction, define, for every $i < k$,

$$\begin{aligned}
\mu_{\alpha_{i+1}}(\xi) =& \text{the } (n_{i+1})\text{th element in the Prikry sequence of} \\
& g_{i+1}(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi))
\end{aligned}$$

and $U_{\mu_{\alpha_i}}(\xi) = W_{\mu_{\alpha_i}(\xi)} \cap V$. Here, given $\delta \in \Delta$, W_δ is the measure on δ used in the Prikry forcing which was applied at stage δ in the iteration.

Claim 5. $\left[\xi \mapsto \mu_{\alpha_i}(\xi)\right]_W = \mu_{\alpha_i}$ *and* $\left[\xi \mapsto U_{\mu_{\alpha_i}(\xi)}\right]_W = k_{\alpha_i}\left(U_{\mu_{\alpha_i}}\right)$.

Proof. We begin by proving that $\left[\xi \mapsto \mu_{\alpha_i}(\xi)\right]_W = \mu_{\alpha_i}$. We present the argument for $i = 1$. Higher values of $i \leq k$ are proved similarly, using induction. Recall that

$$\mu_{\alpha_1} = j_{\alpha_1}\left(g_1\right)\left(\kappa, j_{0,\alpha_1}\left(\beta_1\right), \dots, j_{0,\alpha_1}\left(\beta_l\right)\right)$$

and by applying k_{α_1} on both sides,

$$k_{\alpha_1}\left(\mu_{\alpha_1}\right) = j_W\left(g_1\right)\left(\kappa, \theta_{\left[f_{\beta_1}(\xi)\right]_W}, \dots, \theta_{\left[f_{\beta_l}(\xi)\right]_W}\right).$$

By induction, μ_{α_1} is the n_1th element in the Prikry sequence of $k_{\alpha_1}\left(\mu_{\alpha_1}\right)$, and thus it is represented as the n_1th element in the Prikry sequence of $g_1\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}\right)$.

As for $\left[\xi \mapsto U_{\mu_{\alpha_i}(\xi)}\right]_W = k_{\alpha_i}\left(U_{\mu_{\alpha_i}}\right)$, this follows since, by induction,

$$k_{\alpha_i}\left(U_{\mu_{\alpha_i}}\right) = j_W\left(\delta \mapsto U_\delta\right)\left(k_{\alpha_i}\left(\mu_{\alpha_i}\right)\right). \qquad \dashv$$

Let us argue that $k_\alpha \colon M_\alpha \to M$ is elementary.

Lemma 4.11. *$k_\alpha \colon M_\alpha \to M$ is elementary.*

Proof. Assume that $x, y \in M_\alpha$, and let us prove, for example, that $x \in y$ if and only if $k(x) \in k(y)$. Let $f, g \in V$, $\beta_1, \dots, \beta_l$ and $\alpha_1 < \cdots < \alpha_k < \alpha$ be such that

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$$x = j_\alpha(f)\left(\kappa, j_{0,\alpha}(\beta_1), \dots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right),$$
$$y = j_\alpha(g)\left(\kappa, j_{0,\alpha}(\beta_1), \dots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right).$$

Assume that $\alpha = \alpha' + 1$ is successor (the limit case is simpler). For simplicity, we assume also that $\alpha_k = \alpha'$. Then $x \in y$ if and only if

$$\mu_{\alpha'} \in j_{\alpha',\alpha}\left(\{\xi < \mu_{\alpha'} : j_{\alpha'}(f)\left(\kappa, j_{0,\alpha'}(\beta_1), \dots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_{k-1}}, \xi\right) \in j_{\alpha'}(g)\left(\kappa, j_{0,\alpha'}(\beta_1), \dots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_{k-1}}, \xi\right)\}\right),$$

which is equivalent to

$$\{\xi < \mu_{\alpha'} : j_{\alpha'}(f)\left(\kappa, j_{0,\alpha'}(\beta_1), \dots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_{k-1}}, \xi\right) \in j_{\alpha'}(g)\left(\kappa, j_{0,\alpha'}(\beta_1), \dots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_{k-1}}, \xi\right)\} \in U_{\mu_{\alpha'}}$$

which, by the definition of $U_{\mu_{\alpha'}}$, is equivalent to

$$\mu_{\alpha'} \in k_{\alpha'}\left(\{\xi < \mu_{\alpha'} : j_{\alpha'}(f)\left(\kappa, j_{0,\alpha'}(\beta_1), \dots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_{k-1}}, \xi\right) \in j_{\alpha'}(g)\left(\kappa, j_{0,\alpha'}(\beta_1), \dots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_{k-1}}, \xi\right)\}\right),$$

namely $k_\alpha(x) \in k_\alpha(y)$. ⊣

Let us describe now the main ideas behind the proof that $\mu_\alpha = \operatorname{crit}(k_\alpha)$ is measurable in M_α. Note that this is not trivial since $k_\alpha : M_\alpha \to M$ is not definable in M_α. The full argument will be presented in Lemma 4.17, but will require a technical theorem (Theorem 4.12). Mainly we would like to follow the methods developed in [7] and [9], which deal with nonstationary and full support iterations of Prikry forcings, respectively.

We consider the function $f \in V[G]$, for which $\mu_\alpha = [f]_W$. We will prove that if μ_α is not measurable in M_α, then $\mu_\alpha = [f]_W \in \operatorname{Im}(k_\alpha)$, contradicting the fact that $\mu_\alpha = \operatorname{crit}(k_\alpha)$. For that, we first fix a function $h \in V$ such that, for some sequence $\beta_1, \dots, \beta_l$ of generators of i, and for some nice sequence $\langle \alpha_1, \dots, \alpha_k \rangle$ below α,

$$\mu_\alpha = j_\alpha(h)\left(\kappa, j_{0,\alpha}(\beta_1), \dots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)$$

since $\mu_\alpha = \operatorname{crit}(k_\alpha)$, we can assume that for every $\xi < \kappa$,

$$f(\xi) < h\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right).$$

Pick a condition $p \in G$ which forces this. For every $\xi < \kappa$, $\vec{\eta} = \langle \eta_1, \dots, \eta_l \rangle$ and $\vec{\nu} = \langle \nu_1, \dots, \nu_k \rangle$, denote

$$e(\xi, \vec{\eta}, \vec{\nu}) = \{r \in P \setminus \nu_k : \text{ there exists a bounded subset } A \subseteq h(\xi, \vec{\eta}, \vec{\nu}) \text{ such that } r \Vdash \underset{\sim}{f}(\xi) \in A\}.$$

This set is $\leq^*$-dense open above conditions which extend p and force that

$$\langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi) \rangle = \langle \vec{\eta}, \vec{\nu} \rangle. \tag{2}$$

We would like to follow [7] and [9], and construct a condition $p^* \in G$ above p, such that, very roughly[6], for every $\xi, \vec{\eta}, \vec{\nu}$ as above, and for every extension r of p^* which

[6]We omitted some of the details in the version described here, for sake of simplicity.

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forces (2),

$$r \restriction \nu_k \Vdash r\setminus_{\nu_k} \in e(\xi, \vec{\eta}, \vec{\nu}).$$

Essentially, such p^* will have the following property: every extension r of it which forces that equation (2) holds, forces also that $f(\xi)$ belongs to a bounded subset $A(\xi, \vec{\eta}, \vec{\nu}) \subseteq h(\xi, \vec{\eta}, \vec{\nu})$ (which depends only on p^* and $\langle \xi, \vec{\eta}, \vec{\nu}\rangle$, and not on the choice of the extension of p^* which forces (2)). In [7] and [9] the construction of such p^* was done by a Fusion argument which allows, in a sense, to absorb a lot of data into a single direct extension p^* of p. Such a method is not available in the Easton support iteration. We bypass this problem by constructing, for every sequence $\langle \xi, \eta_1, \dots, \eta_l\rangle$, a system of non-direct extensions of p,

$$\langle p(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k) : \nu_1 < \cdots < \nu_k < \kappa\rangle$$

and sets

$$\langle A(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k) : \nu_1 < \cdots < \nu_k < \kappa\rangle$$

such that the following properties hold:

1. If $p(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k)$ forces (2), then it also forces that $\underset{\sim}{f}(\xi) \in A(\xi, \vec{\eta}, \vec{\nu})$, which is a bounded subset of $h(\xi, \vec{\eta}, \vec{\nu})$.
2. For a set of ξ-s in W, $p\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)$ belongs to G.

This suffices, since, by combining the above properties,

$$V[G] \vDash \{\xi < \kappa : f(\xi) \in A\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\} \in W,$$

and thus, in $M\left[j_W(G)\right]$,

$$\begin{aligned}\mu_\alpha = [f]_W &\in \left[\xi \mapsto A\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\right]_W \\ &= k_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})\right)(\kappa, \beta_1, \dots, \beta_l, \mu_1, \dots, \mu_k)\right) \subseteq \operatorname{Im}(k_\alpha),\end{aligned}$$

where the last inclusion follows since $j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})\right)(\kappa, \beta_1, \dots, \beta_l, \mu_1, \dots, \mu_k)$ is a bounded subset of $\mu_\alpha = j_\alpha(h)\left(\kappa, \beta_1, \dots, \beta_l, \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)$.

We will complete the missing details in the proof in Lemma 4.17. Before that, we present the proof of Theorem 4.12.

4.2. Theorem 4.12 and its proof. We devote this subsection to the proof of the following theorem:

THEOREM 4.12. *Let $p \in G$ be a condition. Assume that for every increasing sequence $\langle \xi, \nu_1, \dots, \nu_k\rangle$, and for every $\vec{\eta} = \langle \eta_1, \dots, \eta_l\rangle$ above ξ, the set*

$$e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k) \subseteq P \setminus \nu_k$$

is $\leq^*$-dense open above conditions in $P \setminus \nu_k$ which force that

$$\langle \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k\rangle = \langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\rangle.$$

Then there are $s < \omega$, a new sequence of generators $\beta'_l, \dots, \beta'_s$ of i which contains $\beta_1, \dots, \beta_l$, and a system of extensions of p,

$$\langle p(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k) : \eta_1, \dots, \eta_s < \kappa, \nu_1 < \cdots < \nu_k < \kappa\rangle$$

with the following properties:

1. *There exists a set of ξ-s in W for which*

$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \restriction \mu_{\alpha_k}(\xi) \Vdash$$
$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \setminus \mu_{\alpha_k}(\xi) \in$$
$$e\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right).$$

2. *There exists a set of ξ-s in W for which*

$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \in G.$$

(Intuitively, for the majority of values of $\langle \xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k \rangle$, the condition $p(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k)$ which we will construct, forces that

$$\langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi) \rangle = \langle \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k \rangle$$

and its final segment belongs to $e(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k)$.)

REMARK 4.13. When we extend a sequence of generators $\langle \beta_1, \dots, \beta_l \rangle$ to a sequence $\langle \beta_1', \dots, \beta_s' \rangle$ we will naturally identify the set $e(\xi, \eta_1, \dots, \eta_l)$, with

$$e'(\xi, \eta_1, \dots, \eta_s) = e\left(\xi, \eta_{i_1}, \dots, \eta_{i_l}\right),$$

where i_j is the index for which $\beta_{i_j}' = \beta_j$, for every $1 \le j \le l$.

Similarly, whenever a function $g \in V$ is given, whose variables are $\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k$, we abuse the notation and denote $g(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k)$ to mean $g\left(\xi, \eta_{i_1}, \dots, \eta_{i_l}, \nu_1, \dots, \nu_k\right)$.

The proof of Theorem 4.12 goes by generalizing the given sets $e\big(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k\big)$:

DEFINITION 4.14. For every $\eta_1, \dots, \eta_l < \kappa$, $1 \le i \le k$ and an increasing sequence $\langle \xi, \nu_1, \dots, \nu_i \rangle$, we define a set $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i) \subseteq P \setminus \nu_i$.

We provide the definition by inverse induction, namely, first define the above set for $i = k$; then, given $i < k$ and a sequence $\langle \xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i \rangle$, and, under the assumption that $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i, \nu)$ is defined for every $\nu < g_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$, we provide the definition of $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$.

For $i = k$, the set $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k)$ is given in the formulation of the theorem.

Assume that $1 \le i < k$. Assume that for every $\nu < g_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$, the set $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i, \nu)$ is defined. Denote $g_{i+1} = g_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$. Let us define the set $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$, as follows: A condition $q \in P \setminus \nu_i$ belongs to $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$ if and only if the following properties hold:

1. (A technical requirement) $q \restriction g_{i+1}$ decides the statements

$$F_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i) = \underset{\sim}{W}_{g_{i+1}} \cap V, \; t^q_{g_{i+1}} = t_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i).$$

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Also, if $q \restriction g_{i+1}$ decides that $t^q_{g_{i+1}} \neq t_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$, it also decides whether one of the sequences is an initial segment of the other, and if so, which one it is. Finally, if it forces that $t^q_{g_{i+1}}$ is a strict initial segment of $t_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$, it also forces that $A^q_{g_{i+1}} \subseteq g_{i+1} \setminus \max(t_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i))$.

2. (The essential requirement) If both statements in the technical requirement are decided positively, there exists a sequence

$$\langle\, q(\nu) : \nu < g_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)\,\rangle$$

such that, for every $\nu < g_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$ above ν_i, $q(\nu) \in P \setminus \nu$ extends $q \setminus \nu$, and

$$q \Vdash \text{if } \underset{\sim}{\mu}_{\alpha_{i+1}}(\xi) = \nu, \text{ then } q(\nu) \in G \setminus \nu \text{ and } q(\nu) \in e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i, \nu).$$

For sake of clearness, we explicitly define $e(\xi, \eta_1, \dots, \eta_l)$, assuming that $e(\xi, \eta_1, \dots, \eta_l, \nu)$ is already defined for every $\nu < g_1(\xi, \eta_1, \dots, \eta_l)$. Let $e(\xi, \eta_1, \dots, \eta_l)$ be the set of conditions $q \in P \setminus \xi$ which decide whether $F_1(\xi, \eta_1, \dots, \eta_l) = W_{g_1(\xi, \eta_1, \dots, \eta_l)} \cap V$, $t_1(\xi, \eta_1, \dots, \eta_l) = t^q_{g_1(\xi, \eta_1, \dots, \eta_l)}$, and, assuming that it is decided positively, have a system of extensions

$$\langle q(\nu) : \nu < g_1(\xi, \eta_1, \dots, \eta_l)\rangle$$

such that, for every $\nu < g_1(\xi, \eta_1, \dots, \eta_l)$, $q(\nu) \in P \setminus \nu$, and

$$q \Vdash \text{if } \underset{\sim}{\mu}_{\alpha_1}(\xi) = \nu \text{ then } q(\nu) \in G \setminus \nu \text{ and } q(\nu) \in e(\xi, \eta_1, \dots, \eta_l, \nu).$$

If it is decided negatively, then $q \restriction g_1$ knows how to compare $t^q_{g_1}$ and $t_1(\xi, \eta_1, \dots, \eta_l)$ as in the second point above.

By induction, we will argue that for every $i \leq k$ and $\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i$, the set $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i) \subseteq P \setminus \nu_i$ is $\leq^*$-dense open above conditions $q \in P \setminus \nu_i$ for which

$$q \Vdash \langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \underset{\sim}{\mu}_{\alpha_1}(\xi), \dots, \underset{\sim}{\mu}_{\alpha_i}(\xi)\rangle = \langle \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i\rangle, \text{ and for every } 1 \leq j \leq i,\ F_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = \underset{\sim}{W}_{g_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j)} \text{ and } t_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = t^q_{g_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)}.$$

The induction will be inverse: The basis, for $i = k$, is true, as it is known that the set $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k) \subseteq P \setminus \nu_k$ is $\leq^*$-dense open above conditions $q \in P \setminus \nu_k$ which force that

$$\langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \underset{\sim}{\mu}_{\alpha_1}(\xi), \dots, \underset{\sim}{\mu}_{\alpha_k}(\xi)\rangle = \langle \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k\rangle.$$

The inductive step is given in the following lemma:

LEMMA 4.15. *Fix* $\eta_1, \dots, \eta_l < \kappa$, $1 \leq i < k$ *and an increasing sequence* $\langle \xi, \nu_1, \dots, \nu_i\rangle$. *Denote* $g_{i+1} = g_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$. *Assume that for every* $\nu \in (\nu_i, g_{i+1})$, *the set*

$$e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i, \nu) \subseteq P \setminus \nu$$

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is $\leq^*$-dense open above conditions $q \in P \setminus \nu$ for which

$$q \Vdash \langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \underset{\sim}{\mu}_{\alpha_1}(\xi), \dots, \underset{\sim}{\mu}_{\alpha_i}(\xi), \underset{\sim}{\mu}_{\alpha_{i+1}}(\xi) \rangle = \langle \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i, \nu \rangle, \text{ and}$$
$$\text{for every } 1 \leq j \leq i+1,\ F_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = \underset{\sim}{W}_{g_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j)}$$
$$\text{and } t_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = t^q_{g_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)}.$$

Then $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$ is $\leq^*$-dense open above conditions $q \in P \setminus \nu_i$ for which

$$q \Vdash \langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \underset{\sim}{\mu}_{\alpha_1}(\xi), \dots, \underset{\sim}{\mu}_{\alpha_i}(\xi) \rangle = \langle \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i \rangle, \text{ and for}$$
$$\text{every } 1 \leq j \leq i,\ F_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = \underset{\sim}{W}_{g_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j)} \text{ and}$$
$$t_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = t^q_{g_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)}.$$

PROOF. Let $q \in P \setminus \nu_i$ be a condition which forces that

$$\theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \underset{\sim}{\mu}_1(\xi), \dots, \underset{\sim}{\mu}_i(\xi) \rangle = \langle \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i \rangle$$
$$\text{and for every } 1 \leq j \leq i,\ F_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = \underset{\sim}{W}_{g_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j)}$$
$$\text{and } t_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = t^q_{g_{j+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)}.$$

Denote

$$g = g_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i),$$

$$U_g = F_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i),$$

$$t = t_{i+1}(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i).$$

Assume that $q \restriction g$ forces that

$$\underset{\sim}{W}_g \cap V = U_g,\ t = \underset{\sim}{t}^q_g$$

(if not, we are done since $q \in e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$). Denote $n = \mathrm{lh}(t)$. We will now apply the following claim:

CLAIM 6. *Assume that $p \in G$ is a condition, $n < \omega$ and $g \in \Delta$ is measurable in V. Assume that U_g is a normal measure on g in V, t is a finite sequence below g of length n, and*

$$p \Vdash \underset{\sim}{t}^q_g = t,\ \underset{\sim}{W}_g \cap V = U_g.$$

For every $\nu < g$, assume that $e(\nu) \subseteq P \setminus \nu$ is a P_ν-name for a subset of $P \setminus \nu$, which is $\leq^*$-dense open above conditions which force that ν is the $(n+1)$th element in the Prikry sequence of g. Then there exists a direct extension $p^* \geq^* p$ and a sequence $\langle p(\nu) : \nu < g \rangle$, such that, for every $\nu < g$,

$$p^* \Vdash \text{if } \nu \text{ appears after } t \text{ in the Prikry sequence of } g,\ \text{then } p(\nu) \in (G \setminus \nu) \cap e(\nu)$$
$$\text{and } p^* \restriction \nu \Vdash p(\nu) \geq^* p^* \restriction [\nu, g)^\frown \langle t^\frown \langle \nu \rangle, \underset{\sim}{A}^{p^*}_g \setminus \nu \rangle^\frown p^* \setminus (g+1).$$

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PROOF. For every $\nu < g$, consider the set

$$d(\nu) = \{r \in P \restriction [\nu, g) : r \parallel \nu \in \underset{\sim}{A}^p_g, \text{ and if } r \Vdash \nu \in \underset{\sim}{A}^p_g \text{ then } r \Vdash \exists s \geq^* \langle t^\frown \langle \nu \rangle, \underset{\sim}{A}^p_g \setminus \nu \rangle^\frown p \setminus (g+1),\ r^\frown s \in e(\nu)\}.$$

Then $d(\nu) \subseteq P \restriction [\nu, g)$ is $\leq^*$-dense open above $p \restriction [\nu, g)$. Let H_g be the P_g-name, forced by $p \restriction g$, to be the $\leq^*$-generic subset of $j_{U_g}(P_g) \setminus g$, for which

$$\underset{\sim}{W}_g = (U_g)_{\underset{\sim}{H}_g}$$

(such a generic exists since W_g is simply generated). Let $\underset{\sim}{q} \in \text{Ult}(V, U_g)$ be a P_g-name, forced by p to be a condition in $\left[\nu \mapsto d(\nu)\right]_{U_g} \cap \underset{\sim}{H}_g$. Let $\nu \mapsto \underset{\sim}{q}(\nu) \in P \restriction [\nu, g)$ be a function in V such that $[\nu \mapsto \underset{\sim}{q}(\nu)]_{U_g} = \underset{\sim}{q}$. Then we can assume that for a set of ν-s in U_g,

$$p \restriction \nu \Vdash \underset{\sim}{q}(\nu) \in d(\nu) \tag{3}$$

and, by Lemma 2.18, $p \restriction g$ forces that there exists a set $\underset{\sim}{C} \in W_g$, such that for every $\nu \in C$,

$$p \restriction \nu^\frown \underset{\sim}{q}(\nu) \in \underset{\sim}{G} \restriction g.$$

By shrinking C if necessary, we can assume that every $\nu \in C$ also satisfies equation (3). Now let us define the extension $p^* \geq^* p$, and, for every $\nu < g$, the condition $p(\nu) \in P \setminus \nu$. First, set

$$p^* \restriction g = p \restriction g$$

and, in $V^{P \restriction \nu}$, set

$$p(\nu) \restriction g = \underset{\sim}{q}(\nu).$$

Work in an arbitrary generic extension for $P \restriction g$, where $p^* \restriction g$ belongs. For every $\nu \in C \cap A^p_g$ (which thus satisfies $p \restriction \nu^\frown \underset{\sim}{q}(\nu) \in G \restriction g$), there exists $s(\nu) \in P \setminus g$, $s(\nu) \geq^* \langle t^\frown \langle \nu \rangle, \underset{\sim}{A}^p_g \setminus \nu \rangle^\frown q \setminus (g+1)$, such that $p(\nu) \restriction g^\frown s(\nu) \in e(\nu)$. Set

$$p^*(g) = \langle \underset{\sim}{t}^p_g, \underset{\sim}{A}^p_g \cap C \cap \left(\triangle_{\nu < g,\ \nu \in C \cap A^p_g} \underset{\sim}{A}^{s(\nu)}_g \right) \rangle$$

(the definition above is carried in $V[G \restriction g]$, so $\underset{\sim}{C}$ is available there).

Let $p^* \setminus (g+1) = s(\underset{\sim}{\nu})$, where $\underset{\sim}{\nu}$ is the $(n+1)$th element in the Prikry sequence of g. Finally, let

$$p(\nu) \setminus g = \langle t^\frown \langle \nu \rangle, A^{p^*}_g \setminus \nu \rangle^\frown p^* \setminus (g+1),$$

where the above definition is possible if $p \restriction \nu^\frown p(\nu) \restriction g \Vdash \nu \in \underset{\sim}{A}^{p^*}_g$; if not, let $p(\nu) \setminus g$ be arbitrary.

This completes the definition of $p^* \geq^* p$ and $\langle p(\nu) : \nu < g \rangle$. Let us prove that for every $\nu < g$,

$$p^* \Vdash \text{if } \nu \text{ appears after } t \text{ in the Prikry sequence of } g, \text{ then } p(\nu) \in (G \setminus \nu) \cap e(\nu) \text{ and } p^* \restriction \nu \Vdash p(\nu) \geq^* p^* \restriction [\nu, g)^\frown \langle t^\frown \langle \nu \rangle, \underset{\sim}{A}^{p^*}_g \setminus \nu \rangle^\frown p^* \setminus (g+1).$$

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Fix $\nu < g$ and let G be a generic set for P which includes p^*, such that, in $V[G]$, ν appears after t in the Prikry sequence of g. In particular, $\nu \in C$ and thus $q(\nu) \in G \restriction [\nu, g)$. By the definition of $p(\nu)$, and since $p^* \in G, q(\nu) \in G \restriction [\nu, g)$, it follows that $p(\nu) \in G \setminus \nu$, as desired. $\dashv$

of Claim 6.

Apply Claim 6 with respect to the set $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i, \nu) \subseteq P \setminus \nu$ (recall that $\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i$ are fixed), and direct extend q further, to a condition $q^* \geq^* q$, which has a system of extensions

$$\langle q(\nu) : \nu < g \rangle$$

as in the statement of the lemma.

It follows that, for every $\nu < g$,

$$q^* \Vdash \text{if } \underset{\sim}{\mu}_{\alpha_{i+1}}(\xi) = \nu \text{ then } q(\nu) \in G \setminus \nu_i \text{ and } q(\nu) \setminus \nu \in e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i, \nu).$$

Therefore $\langle q(\nu) \colon \nu < g \rangle$ witnesses the fact that $q^* \in e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k)$. $\dashv$

of Lemma 4.15.

We now proceed towards the proof of Theorem 4.12. We use the same notations as in the formulation of the theorem.

By induction, the following holds: For every $\xi, \eta_1, \dots, \eta_l$, the set $e(\xi, \eta_1, \dots, \eta_l) \subseteq P \setminus \xi$ is $\leq^*$-dense open above conditions $q \in P \setminus \xi$ which force that

$$\langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)} \rangle = \langle \eta_1, \dots, \eta_l \rangle$$

and that

$$F_1(\xi, \eta_1, \dots, \eta_l) = \underset{\sim}{W}_{g_1(\xi, \eta_1, \dots, \eta_l)} \text{ and } t_1(\xi, \eta_1, \dots, \eta_l) = t^q_{g_1(\xi, \eta_1, \dots, \eta_l)}.$$

We would like to perform another step, and move from conditions in $P \setminus \xi$ to conditions in P. This might require extending the sequence generators $\beta_1, \dots, \beta_l$. We do this in the following lemma, which concludes the proof of Theorem 4.12.

Lemma 4.16. *There exists $s < \omega$, a sequence of generators $\langle \beta'_1, \dots, \beta'_s \rangle$ of i which extends $\langle \beta_1, \dots, \beta_l \rangle$, and a system of conditions*

$$\langle p(\xi, \eta'_1, \dots, \eta'_s, \nu_1, \dots, \nu_k) : \eta'_1, \dots, \eta'_s < \kappa, \ \xi < \nu_1 < \dots < \nu_k \rangle$$

(*all of them extend the condition $p \in G$ given in the statement of Theorem 4.12*), *such that,*

$$\begin{aligned}\{\xi < \kappa \colon & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \restriction \mu_{\alpha_k}(\xi) \Vdash \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \setminus \mu_{\alpha_k}(\xi) \in \\ & e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \text{ and} \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \in G\} \in W.\end{aligned}$$

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Proof. Recall that $W = U_H$ is generated from the elementary embedding $i\colon V \to N$. Let us consider the set

$$i\left(\langle \xi, \eta_1, \dots, \eta_l\rangle \mapsto e\left(\xi, \eta_1, \dots, \eta_l\right)\right)\left(\kappa, \beta_1, \dots, \beta_l\right) \subseteq i(P) \setminus \kappa,$$

it is $\leq^*$-dense open in $i(P) \setminus \kappa$, and thus meets a condition $r \in H$. Since $r \in N$, it can be represented using a sequence of generators $\langle \beta'_1, \dots, \beta'_s\rangle$, on which we can assume that it contains $\langle \beta_1, \dots, \beta_l\rangle$. Let

$$\langle \xi, \eta'_1, \dots, \eta'_s\rangle \mapsto r\left(\xi, \eta'_1, \dots, \eta'_s\right) \in P \setminus \xi$$

be a function in V, such that

$$r = i\left(\langle \xi, \eta'_1, \dots, \eta'_s\rangle \mapsto r\left(\xi, \eta'_1, \dots, \eta'_s\right)\right)\left(\kappa, \beta'_1, \dots, \beta'_s\right).$$

Now, for every $\langle \xi, \eta'_1, \dots, \eta'_s, \nu_1, \dots, \nu_k\rangle$, let us define the condition $p\big(\xi, \eta'_1, \dots, \eta'_s, \nu_1, \dots, \nu_k\big) \in P$. We do this recursively, and define, for every $1 \leq i \leq k$, a condition $p\left(\xi, \eta'_1, \dots, \eta'_s, \nu_1, \dots, \nu_i\right) \in P$. Simultaneously, we prove that

$$\begin{aligned}\{\xi < \kappa\colon\, & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) \restriction \mu_{\alpha_i}(\xi) \Vdash \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) \setminus \mu_{\alpha_i}(\xi) \in \\ & e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) \text{ and} \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) \in G\} \in W.\end{aligned}$$

This will complete the proof of the lemma, and thus, the proof of Theorem 4.12.

- First, fix $\xi, \eta_1, \dots, \eta_s$, and let us define $p\left(\xi, \eta_1, \dots, \eta_s\right)$. If $p \restriction \xi \Vdash r\left(\xi, \eta_1, \dots, \eta_s\right) \in e\left(\xi, \eta_1, \dots, \eta_l\right)$, set $p\left(\xi, \eta_1, \dots, \eta_s\right) = p \restriction \xi ^\frown r\left(\xi, \eta_1, \dots, \eta_s\right)$. Else, let $p\left(\xi, \eta_1, \dots, \eta_s\right)$ be an arbitrary condition above p. We argue that

  $$\begin{aligned}\{\xi < \kappa\colon\, & p \restriction \xi \Vdash r\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \in e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \text{ and} \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \in G\} \in W.\end{aligned}$$

  Recall that $r \in H$ was defined such that

  $$p \Vdash r \in i\left(\langle \xi, \eta_1, \dots, \eta_l\rangle \mapsto e\left(\xi, \eta_1, \dots, \eta_l\right)\right)\left(\kappa, \beta_1, \dots, \beta_l\right)$$

  applying the embedding $k\colon N \to M$ and reflecting down modulo W gives

  $$\{\xi < \kappa\colon p \restriction \xi \Vdash r\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \in e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right)\} \in W.$$

  Finally, $p \Vdash r \in H$ and thus $p \Vdash k(r) \in j_W(G)$, by Lemma 2.18. Reflecting this down gives

  $$\{\xi < \kappa\colon p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \in G\} \in W.$$

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- Fix $\xi, \eta'_1, \dots, \eta'_s, \nu_1$ and let us define $p\left(\xi, \eta'_1, \dots, \eta'_s, \nu_1\right)$. Denote $g_1 = g_1\left(\xi, \eta'_1, \dots, \eta'_s\right)$.If $p\left(\xi, \eta'_1, \dots, \eta'_s\right) \restriction \xi \Vdash p\left(\xi, \eta'_1, \dots, \eta'_s\right) \setminus \xi \in e\left(\xi, \eta'_1, \dots, \eta'_s\right)$, then $p\left(\xi, \eta'_1, \dots, \eta'_s\right) \restriction \xi = p \restriction \xi$ decides the statements

$$F_1\left(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i\right) = \underset{\sim}{W}_{g_1} \cap V\ ,\ t^q_{g_1} = t_1\left(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i\right)$$

and, if it decides them positively, it forces that there exists a sequence $\langle q(\nu) \colon \nu < g_1\rangle$ witnessing this. Define

$$p\left(\xi, \eta'_1, \dots, \eta'_s, \nu_1\right) = p\left(\xi, \eta'_1, \dots, \eta'_s\right) \restriction \nu_1 {}^\frown q(\nu_1).$$

If $p\left(\xi, \eta'_1, \dots, \eta'_s\right) \restriction \xi \nVdash p\left(\xi, \eta'_1, \dots, \eta'_s\right) \setminus \xi \in e\left(\xi, \eta'_1, \dots, \eta'_s\right)$, or $p\left(\xi, \eta'_1, \dots, \eta'_s\right) \restriction \xi \Vdash p\left(\xi, \eta'_1, \dots, \eta'_s\right) \setminus \xi \in e\left(\xi, \eta'_1, \dots, \eta'_s\right)$ but the statements

$$F_1\left(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i\right) = \underset{\sim}{W}_{g_1} \cap V\ ,\ t^q_{g_1} = t_1\left(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i\right)$$

are decided negatively, let $p\left(\xi, \eta'_1, \dots, \eta'_s, \nu_1\right)$ be an arbitrary condition above $p\left(\xi, \eta'_1, \dots, \eta'_s\right)$.We argue that

$$\begin{aligned}\{\xi < \kappa \colon p&\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi)\right) \restriction \mu_{\alpha_1}(\xi) \Vdash \\ &p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi)\right) \setminus \mu_{\alpha_1}(\xi) \in \\ &e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi)\right) \text{ and} \\ &p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi)\right) \in G\} \in W.\end{aligned}$$

First, by the previous point,

$$\begin{aligned}\{\xi < \kappa \colon p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \restriction \xi \Vdash p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \setminus \xi \in \\ e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right)\} \in W.\end{aligned}$$

By the properties of the set $e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right)$, the condition

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \restriction \xi$$

decides the statements

$$F_1\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) = \underset{\sim}{W}_{g_1} \cap V$$

and

$$t_{g_1}^{p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right)} = t_1\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right).$$

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CLAIM 7. *For a set of ξ-s in W, the above statements are decided in a positive way.Before the proof of the claim, let us proceed with our argument. By the claim and Definition 4.14,*

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)=p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction\mu_{\alpha_1}(\xi)^\frown q\left(\mu_{\alpha_1}(\xi)\right)$$

and, by the properties of the set $e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)$, *the condition*

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)$$

forces that

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)\\ =p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction\mu_{\alpha_1}(\xi)^\frown q\left(\mu_{\alpha_1}(\xi)\right)\in \underset{\sim}{G}$$

and

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)\setminus\mu_{\alpha_1}(\xi)=q\left(\mu_{\alpha_1}(\xi)\right)\in\\ e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right).$$

Thus,

$$\{\xi<\kappa\colon p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction\xi\Vdash p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\setminus\xi\in\\ e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\}\in W$$

which finishes the second step. Thus, it remains to prove Claim 7.

PROOF. Let us prove first that

$$\left\{\xi<\kappa\colon p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction\xi\Vdash F_1\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)=\underset{\sim}{W}_{g_1}\cap V\right\}.$$

Assume otherwise. Then in $M\left[j_W(G)\right]$,

$$j_W\left(\langle\xi,\eta_1,\dots,\eta_s\rangle\mapsto F_1\left(\langle\xi,\eta_1,\dots,\eta_s\rangle\right)\right)\left(\kappa,j_{0,\alpha}\left(\beta_1'\right),\dots,j_{0,\alpha}\left(\beta_s'\right)\right)\neq\\ \left[\xi\mapsto W_{g_1\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)}\cap V\right]_W$$

but both sides are equal to $k_1\left(U_{\mu_{\alpha_1}}\right)$, contradicting property (D) of the embedding k_{α_1}.Now let us prove that

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$$\{\xi<\kappa\colon p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction\xi\Vdash$$
$$t_{g_1}^{p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)}=t_1\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\}.$$

Assume otherwise. Then the condition $s=j_W\left(\xi\mapsto p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\dots,\theta_{f_{\beta_s'}(\xi)}\right)\right)$ (κ) forces that

$$t^s_{k_{\alpha_1}(\mu_{\alpha_1})}\neq k_{\alpha_1}(t_{\alpha_1})=t_{\alpha_1}.$$

Note that $s\in j_W(G)\restriction k_{\alpha_1}(\mu_{\alpha_1})$ and t_{α_1} is the initial segment of the Prikry sequence of $k_{\alpha_1}(\mu_{\alpha_1})$ below μ_{α_1} in $M\left[j_W(G)\right]$. Thus, one of the sequences $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}$ and t_{α_1} is a strict initial segment of the other. By the second requirement in Definition 4.14, $s\restriction k_{\alpha_1}(\mu_{\alpha_1})$ decides which one is an initial segment of the other. Now this yields a contradiction:

1. If t_{α_1} is a strict initial segment of $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}$: Recall that $s=k_{\alpha_1}(s')$, where

$$s'=j_{\alpha_1}\left(\langle\xi,\eta_1,\dots,\eta_s\rangle\mapsto p\left(\xi,\eta_1,\dots,\eta_s\right)\right)\left(\kappa,j_{0,\alpha_1}(\beta_1'),\dots,j_{0,\alpha_1}\left(\beta_s'\right)\right).$$

   Then $s'\restriction\mu_{\alpha_1}$ forces that t_{α_1} is a strict initial segment of $t^{s'}_{\mu_{\alpha_1}}$. Work over M_{α_1}. Let $\gamma<\mu_{\alpha_1}$ be an ordinal, forced by $s'\restriction\mu_{\alpha_1}$ to be a bound on the first ordinal in $t^{s'}_{\mu_{\alpha_1}}\setminus t_{\alpha_1}$ (such a bound exists since the forcing $j_{\alpha_1}(P)\restriction\mu_{\alpha_1}$ is μ_{α_1}-c.c. in M_{α_1}). Applying $k_{\alpha_1}\colon M_{\alpha_1}\to M$, $\gamma<\mu_{\alpha_1}$ is an upper bound on the first ordinal in $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}\setminus t_{\alpha_1}$. However, in $M\left[j_W(G)\right]$, this element is μ_{α_1} itself, which is strictly above γ. A contradiction.
2. Else, $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}$ is a strict initial segment of t_{α_1}: Denote $\gamma=\max(t_{\alpha_1})$. Then, by Definition 4.14, s forces that the initial segment of the Prikry sequence of $k_{\alpha_1}(\mu_{\alpha_1})$ is $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}$, followed by an element strictly above γ; in particular, t_{α_1} is not an initial segment of the Prikry sequence of $k_{\alpha_1}(\mu_{\alpha_1})$ in $M\left[j_W(G)\right]$, which is a contradiction. ⊣

□ of Claim 7.

- Assume now that $1\leq i<k$ is arbitrary, and for every $\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i$, a condition $p\left(\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i\right)$ is defined. Denote $g_{i+1}=g_{i+1}\big(\xi,\eta_1',\dots,\eta_s',$ $\nu_1,\dots,\nu_i\big)$. For every $\nu_{i+1}<g_{i+1}$, let us define the condition $p\big(\xi,\eta_1',\dots,\eta_s',\nu_1,$ $\dots,\nu_i,\nu_{i+1}\big)$. If $p\left(\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i\right)\restriction\nu_i\Vdash p\left(\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i\right)\setminus\nu_i\in$ $e\left(\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i\right)$ and $p\left(\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i\right)\restriction\nu_i$ forces the statements

$$F_{i+1}\left(\xi,\eta_1,\dots,\eta_l,\nu_1,\dots,\nu_i\right)=\underset{\sim}{W}_{g_{i+1}}\cap V\ ,\ t^q_{g_{i+1}}=t_{i+1}\left(\xi,\eta_1,\dots,\eta_l,\nu_1,\dots,\nu_i\right)$$

then $p\left(\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i\right)\restriction\nu_i$ forces that there exists a sequence $\langle q(\nu)\colon\nu<$ $g_{i+1}\rangle$ witnessing this. In this case, define

$$p\left(\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i,\nu_{i+1}\right)=p\left(\xi,\eta_1',\dots,\eta_s',\nu_1,\dots,\nu_i\right)\restriction\nu_{i+1}{}^\frown q(\nu_{i+1}).$$

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Else, let $p\left(\xi, \eta_1', \dots, \eta_s', v_1, \dots, v_i, v_{i+1}\right)$ be an arbitrary condition which extends the condition $p\left(\xi, \eta_1', \dots, \eta_s', v_1, \dots, v_i\right)$.Let us argue now that

$$\begin{aligned}\{\xi < \kappa \colon\ & p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \restriction \mu_{\alpha_{i+1}}(\xi) \Vdash \\ & p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \setminus \mu_{\alpha_{i+1}}(\xi) \in \\ & e\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \text{ and} \\ & p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \in G\} \in W.\end{aligned}$$

We do this as in the previous point. First,

$$\begin{aligned}\{\xi < \kappa \colon\ & p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) \restriction \mu_{\alpha_i}(\xi) \Vdash \\ & p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) \setminus \mu_{\alpha_i}(\xi) \in \\ & e\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right)\} \in W.\end{aligned}$$

Thus, for a set of ξ-s in W, the condition

$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) \restriction \mu_{\alpha_i}(\xi)$$

decides the statements

$$\begin{aligned}& F_{i+1}\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) = \\ & \underset{\sim}{W}_{g_{i+1}\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right)} \cap V\end{aligned}$$

and

$$t^{p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right)}_{g_{i+1}\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right)} = t_{i+1}\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right).$$

Arguing as in Claim 7, both statements are decided positively for a set of ξ-s in W. Thus,

$$\begin{aligned}& p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) = \\ & p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right) \restriction \mu_{\alpha_{i+1}}(\xi) ^\frown q\left(\mu_{\alpha_{i+1}}(\xi)\right)\end{aligned}$$

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and the condition $q\left(\mu_{\alpha_{i+1}}(\xi)\right)$ is forced, by

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi)\right)$$

to be in

$$G \setminus \mu_{\alpha_{i+1}}(\xi) \cap e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right).$$

Therefore,

$$\begin{aligned}\{\xi < \kappa \colon & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \restriction \mu_{\alpha_{i+1}}(\xi) \Vdash \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \setminus \mu_{\alpha_{i+1}}(\xi) \in \\ & e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \text{ and} \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \in G\} \in W,\end{aligned}$$

as desired. ⊣

□ of Lemma 4.16. □ of Theorem 4.12.

4.3. Properties of k_α. In this subsection we complete the proof of properties (A) – (D) of k_α. After that, we will prove in Lemma 4.20 that $k_{\kappa^*} \colon M_{\kappa^*} \to M$ is the identity, and conclude the proof of Theorem 4.7.

LEMMA 4.17. *$\mu_\alpha = crit(k_\alpha)$ is measurable in M_α. Moreover, μ_α is the least measurable of M_α above $sup\{\mu_\beta \colon \beta < \alpha\}$ which has cofinality above κ in V.*

PROOF. Write $\mu = [f]_W$ and $\mu = j_\alpha(h)\left(\kappa, j_{0,\alpha}(\beta_1), \dots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)$, for some $f \in V[G]$, $h \in V$, $\beta_1, \dots, \beta_l$ generators of i and $\alpha_1 < \dots < \alpha_k < \alpha$.

Since $\mu < k_\alpha(\mu)$, we can assume that for every $\xi < \kappa$,

$$f(\xi) < h\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right),$$

and let $p \in G$ be a condition which forces this. Given $\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k$, consider the set

$$\begin{aligned} e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k) = \{ & r \in P \setminus \nu_k \colon \text{ for some bounded subset} \\ & A \subseteq h(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k), r \Vdash \underset{\sim}{f}(\xi) \in A\}.\end{aligned}$$

Then $e(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k)$ is $\leq^*$-dense open above conditions which extend p and force that

$$\langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\rangle = \langle \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k\rangle.$$

By Theorem 4.12, the sequence $\langle \beta_1, \dots, \beta_l\rangle$ can be extended to a sequence $\langle \beta'_1, \dots, \beta'_s\rangle$, and p can be extended to a system of conditions,

$$\langle p(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k) \colon \xi, \eta_1, \dots, \eta_s < \kappa, \nu_1 < \dots < \nu_k < \kappa\rangle$$

such that, for a set of ξ-s in W,

$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \restriction \mu_{\alpha_k}(\xi) \Vdash$$
$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \setminus \mu_{\alpha_k}(\xi) \in$$
$$e\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)$$

and

$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \in G.$$

Assume now that $\langle \xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k \rangle$ are given, such that

$$p(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k) \restriction \nu_k \Vdash p(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k) \setminus \nu_k \in e(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k).$$

Let $\underset{\sim}{A}$ be a P_{ν_k}-name, forced by $p(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k) \restriction \nu_k$ to be a witness to the fact that $p(\xi, \vec{\eta}, \vec{\nu}) \setminus \nu_k \in e(\xi, \vec{\eta}, \vec{\nu})$. Namely it is a bounded subset of $h(\xi, \vec{\eta}, \vec{\nu})$, and $p(\xi, \vec{\eta}, \vec{\nu}) \setminus \nu_k \Vdash \underset{\sim}{f}(\xi) \in \underset{\sim}{A}$.

Let $A(\xi, \vec{\eta}, \vec{\nu})$ be the set of ordinals $\gamma < h(\xi, \vec{\eta}, \vec{\nu})$ such that, some $r \geq p(\xi, \vec{\eta}, \vec{\nu}) \restriction \nu_k$ forces that $\gamma \in \underset{\sim}{A}$. Since $\nu_k < h(\xi, \vec{\eta}, \vec{\nu})$, $A(\xi, \vec{\eta}, \vec{\nu})$ is a bounded subset of $h(\xi, \vec{\eta}, \vec{\nu})$. The function $\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})$ lies in V.

By the results of Theorem 4.12, there exists a set of ξ-s in W for which

$$G \ni p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \Vdash$$
$$\underset{\sim}{f}(\xi) \in A\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right).$$

Thus, in $M\left[j_W(G)\right]$,

$$[f]_W \in \left[\xi \mapsto A\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\right]_W =$$
$$k_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})\right)\left(\kappa, j_{0,\alpha}(\beta_1'), \dots, j_{0,\alpha}(\beta_s'), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)\right) \subseteq \operatorname{Im}(k_\alpha),$$

where the last inclusion follows since

$$j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})\right)\left(\kappa, j_{0,\alpha}(\beta_1'), \dots, j_{0,\alpha}(\beta_s'), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)$$

is a bounded subset of

$$\mu_\alpha = j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto h(\xi, \vec{\eta}, \vec{\nu})\right)\left(\kappa, j_{0,\alpha}(\beta_1'), \dots, j_{0,\alpha}(\beta_s'), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)$$

which is $\operatorname{crit}(k_\alpha)$.

Thus we proved that $\mu_\alpha \in \operatorname{Im}(k_\alpha)$, which is a contradiction. ⊣

Lemma 4.18. *μ_α appears in the Prikry sequence added to $k_\alpha(\mu_\alpha)$ in $M\left[j_W(G)\right]$.*

Proof. In $M[H]$, denote by t^* the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ which consists of all the ordinals below μ_α. Denote by n^* the length of t^*.

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Let $\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto t^*(\xi, \vec{\eta}, \vec{\nu})$ be a function in V such that

$$t^* = j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto t^*(\xi, \vec{\eta}, \vec{\nu})\right)\left(\kappa, j_{0,\alpha}(\beta_1), \dots, j_{0,\alpha}(\beta_l), \mu_{\alpha_0}, \dots, \mu_{\alpha_k}\right)$$

(we assumed here that t^* can be represented using the same generators as μ_α. If this is not the case, modify the set of generators).

We can assume that for every $\langle \xi, \vec{\eta}, \vec{\nu} \rangle$, $t^*(\xi, \vec{\eta}, \vec{\nu})$ is a sequence of length n^*. Since $k_\alpha(t^*) = t^*$,

$$\left[\xi \mapsto t^*\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\right]_W = t^*.$$

In $V[G]$, denote, for every $\xi < \kappa$,

$$\mu_\alpha(\xi) = \text{the } (n^*+1)\text{th element in the Prikry sequence of}$$
$$h\left(\vec{\xi}, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right).$$

Clearly $\left[\xi \mapsto \mu_\alpha(\xi)\right]_W \geq \mu_\alpha$.

We argue that equality holds. We will prove that for every $\eta < \left[\xi \mapsto \mu_\alpha(\xi)\right]_W$, $\eta < \mu_\alpha$. Assume that such η is given, and let $f \in V[G]$ be a function such that $[f]_W = \eta$. Then we can assume that for every $\xi < \kappa$,

$$f(\xi) < \mu_\alpha(\xi),$$

and let $p \in G$ be a condition which forces this.

For every $\xi, \vec{\eta}, \vec{\nu}$, consider the set

$$e(\xi, \vec{\eta}, \vec{\nu}) = \{r \in P \setminus \nu_k : \exists \gamma < h(\xi, \vec{\eta}, \vec{\nu}),\ r \Vdash \text{ if } t^*(\xi, \vec{\eta}, \vec{\nu}) \text{ is an initial segment of the Prikry sequence of } h(\xi, \vec{\eta}, \vec{\nu}),\ \text{then } \underset{\sim}{f}(\xi) < \gamma\}$$

then $e\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_k\right)$ is $\leq^*$-dense open above conditions which force that

$$\langle \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi) \rangle = \langle \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_k \rangle.$$

This, since, given a name for an element $\underset{\sim}{f}(\xi)$ which is forced to be strictly below $\mu_\alpha(\xi)$ (which is the element which appears right after $t^*(\xi, \vec{\eta}, \vec{\nu})$ in the Prikry sequence of $h(\xi, \vec{\eta}, \vec{\nu})$), the element can be decided by taking a direct extension.

By Theorem 4.12, the sequence $\langle \beta_1, \dots, \beta_l \rangle$ can be extended to a sequence $\langle \beta_1', \dots, \beta_s' \rangle$, and p can be extended to a system of conditions,

$$\langle p(\xi, \eta_1, \dots, \eta_s, \nu_1, \dots, \nu_k) : \xi, \eta_1, \dots, \eta_s < \kappa, \nu_1 < \dots < \nu_k < \kappa \rangle$$

such that, for a set of ξ-s in W,

$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \restriction \mu_{\alpha_k}(\xi) \Vdash$$
$$p\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \setminus \mu_{\alpha_k}(\xi) \in$$
$$e\left(\xi, \theta_{f_{\beta_1'}(\xi)}, \dots, \theta_{f_{\beta_s'}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)$$

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and

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \in G.$$

Assume now that $\langle \xi, \vec{\eta}, \vec{v} \rangle = \langle \xi, \eta_1, \dots, \eta_s, v_1, \dots, v_k \rangle$ are given, such that

$$p(\xi, \vec{\eta}, \vec{v}) \restriction v_k \Vdash p(\xi, \vec{\eta}, \vec{v}) \setminus v_k \in e(\xi, \vec{\eta}, \vec{v}).$$

Let $\underset{\sim}{\gamma}$ be a P_{v_k}-name, forced by $p(\xi, \eta_1, \dots, \eta_s, v_1, \dots, v_k) \restriction v_k$ to an ordinal below $h(\xi, \widetilde{\vec{\eta}}, \vec{v})$, such that $p(\xi, \vec{\eta}, \vec{v}) \setminus v_k \Vdash \underset{\sim}{f}(\xi) < \underset{\sim}{\gamma}$. Let $\gamma(\xi, \vec{\eta}, \vec{v})$ be the supremum of the set of ordinals $\tau < h(\xi, \vec{\eta}, \vec{v})$ such that, some $r \geq p(\xi, \vec{\eta}, \vec{v}) \restriction v_k$ forces that $\underset{\sim}{\gamma} = \tau$. Since $v_k < h(\xi, \vec{\eta}, \vec{v})$, $\gamma(\xi, \vec{\eta}, \vec{v}) < h(\xi, \vec{\eta}, \vec{v})$. The function $\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto \gamma(\xi, \widetilde{\vec{\eta}}, \vec{v})$ lies in V.

By the results of Theorem 4.12, there exists a set of ξ-s in W for which

$$\begin{aligned} G \ni p &\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \Vdash \\ &\text{if } t^* \left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \\ &\text{is an initial segment of the Prikry sequence of} \\ &h\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right), \text{ then} \\ &\underset{\sim}{f}(\xi) < \gamma\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right). \end{aligned}$$

Thus, in $M\left[j_W(G)\right]$, where indeed t^* is an initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$,

$$\begin{aligned} [f]_W \in &\left[\xi \mapsto \gamma\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\right]_W = \\ &k_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto \gamma(\xi, \vec{\eta}, \vec{v})\right)\left(\kappa, j_{0,\alpha}(\beta'_1), \dots, j_{0,\alpha}(\beta'_s), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)\right) < \mu_\alpha, \end{aligned}$$

as desired. ⊣

LEMMA 4.19. *Let $U_{\mu_\alpha} = \{X \subseteq \mu_\alpha : \mu_\alpha \in k_\alpha(X)\} \cap M_\alpha$. Then $U_{\mu_\alpha} \in M_\alpha$. Furthermore, $k_\alpha\left(U_{\mu_\alpha}\right) = j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha))$, where, for every $\delta \in \Delta$, $U_\delta = W_\delta \cap V$, for W_δ which is the measure used in the Prikry forcing at stage δ in the iteration P.*

PROOF. We first prove that $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) \in \operatorname{Im}(k_\alpha)$. Then, we will prove that the measure $F \in M_\alpha$ for which $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) = k_\alpha(F)$ equals to U_{μ_α}.

In order to prove that $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) \in \operatorname{Im}(k_\alpha)$, we prove that there exists a family $\mathcal{F} \in M_\alpha$ of measures on μ_α, with $|\mathcal{F}| < \mu_\alpha$, such that $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) \in k_\alpha(F) = k''_\alpha \mathcal{F}$.

Fix, in V, an enumeration W of all the normal measures on measurable cardinals below κ. For every $\langle \xi, \vec{\eta}, \vec{v} \rangle$, let $\gamma(\xi, \vec{\eta}, \vec{v})$ be the index of $U_{h(\xi, \vec{\eta}, \vec{v})}$ in this enumeration. Note that each measure $U_{h(\xi, \vec{\eta}, \vec{v})}$ belongs to V, but the sequence $\langle U_{h(\xi, \vec{\eta}, \vec{v})} : \xi, \vec{\eta}, \vec{v} < \kappa \rangle$ might be external to V. So the function $\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto \gamma(\xi, \vec{\eta}, \vec{v})$ doesn't necessarily belong to V.

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Fix $\langle \xi, \vec{\eta}, \vec{v} \rangle$ and consider the set

$$e(\xi, \vec{\eta}, \vec{v}) = \{r \in P \setminus v_k : \text{there exists a set of ordinals } A \text{ of cardinality strictly smaller than } h(\xi, \vec{\eta}, \vec{v}), \text{ such that } r \restriction h(\xi, \vec{\eta}, \vec{v}) \Vdash \gamma(\xi, \vec{\eta}, \vec{v}) \in A\}.$$

Then $e(\xi, \vec{\eta}, \vec{v}) \subseteq P \setminus v_k$ is $\leq^*$-dense open, since $P \restriction h(\xi, \vec{\eta}, \vec{v})$ is $h(\xi, \vec{\eta}, v)$-c.c.

Now apply Theorem 4.12 and argue as in the previous lemma: There exists (in V) a mapping $\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto A(\xi, \vec{\eta}, \vec{v})$ such that, in $M\left[j_W(G)\right]$,

$$\left[\xi \mapsto \gamma\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\right]_W \in$$
$$\left[\xi \mapsto A\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\right]_W =$$
$$k''_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto A(\xi, \vec{\eta}, \vec{v})\right)\left(\kappa, j_{0,\alpha}(\beta'_1), \dots, j_{0,\alpha}(\beta'_s), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)\right).$$

In M_α, let $\mathcal{F}$ be the set of measures on μ_α which are indexed in the enumeration $j_\alpha(W)$ by an index in the set $\mathcal{A} = j_\alpha\left(\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto A(\xi, \vec{\eta}, \vec{v})\right)\left(\kappa, j_{0,\alpha}\left(\beta'_1\right), \dots, j_{0,\alpha}(\beta'_s), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)$. Note that $|\mathcal{A}| < \mu_\alpha$ and thus $|\mathcal{F}| < \mu_\alpha$. Then $j_W(\delta \mapsto U_\delta)$ $\left(k_\alpha(\mu_\alpha)\right)$ is enumerated by the ordinal

$$\left[\xi \mapsto \gamma\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\right]_W \in k''_\alpha \mathcal{A},$$

and thus $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) \in k''_\alpha \mathcal{F}$, as desired.

Let $F \in M_\alpha$ be a measure on μ_α such that

$$j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) = k_\alpha(F).$$

Let us argue that $F = U_{\mu_\alpha}$. It suffices to prove that $F \subseteq U_{\mu_\alpha}$. Fix a set $X \in F$. Assume that

$$X = j_\alpha\left(\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto X(\xi, \vec{\eta}, \vec{v})\right)\left(\kappa, j_{0,\alpha}(\beta_1), \dots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)$$

(we assumed again that X can be represented using the same generators as μ_α. If this is not the case, modify the set of generators of μ_α). Then $k_\alpha(X) \in j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha))$.

As in the previous lemma, let n^* be the length of t^*, the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ below μ_α. For every $\langle \xi, \vec{\eta}, \vec{v} \rangle$, let

$$\begin{aligned} e(\xi, \vec{\eta}, \vec{v}) = \{ & r \in P \setminus v_k : r \restriction h(\xi, \vec{\eta}, \vec{v}) \parallel X(\xi, \vec{\eta}, \vec{v}) \in U_{h(\xi, \vec{\eta}, \vec{v})}, \\ & \text{if it decides positively, then } r \restriction h(\xi, \vec{\eta}, \vec{v}) \Vdash \underset{\sim}{A}^r_{h(\xi, \vec{\eta}, \vec{v})} \subseteq \\ & X(\xi, \vec{\eta}, \vec{v}); \text{ else, } r \restriction h(\xi, \vec{\eta}, \vec{v}) \Vdash \underset{\sim}{A}^r_{h(\xi, \vec{\eta}, \vec{v})} \text{ is disjoint} \\ & \text{from } X(\xi, \vec{\eta}, \vec{v}). \text{ Moreover, } r \restriction h(\xi, \vec{\eta}, \vec{v}) \parallel \text{lh}\left(t^r_{h(\xi, \vec{\eta}, \vec{v})}\right) > n^*, \\ & \text{and if it decides positively, then there exists a bounded subset} \\ & A(\xi, \vec{\eta}, \vec{v}) \subseteq h(\xi, \vec{\eta}, \vec{v}) \text{ for which } r \restriction \xi, \vec{\eta}, \vec{v} \Vdash \text{ the } (n^*+1)\text{th} \\ & \text{element of } t^r_{h(\xi, \vec{\eta}, \vec{v})} \text{ belongs to } A(\xi, \vec{\eta}, \vec{v})\}. \end{aligned}$$

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By Theorem 4.12, there exists a larger set of generators $\beta'_1, \dots, \beta'_s$ and, for every $\langle \xi, \vec{\eta}, \vec{v} \rangle$, a condition $p\left(\langle \xi, \vec{\eta}, \vec{v} \rangle\right)$, such that, for a set of ξ-s in W,

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \restriction \mu_{\alpha_k}(\xi) \Vdash$$
$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \setminus \mu_{\alpha_k}(\xi) \in$$
$$e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)$$

and

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \in G.$$

Let us argue first that for a set of ξ-s in W,

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \restriction \mu_{\alpha_k}(\xi)$$

decides that

$$\mathrm{lh}\left(t^{p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)}_{h\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)}\right) \leq n^*.$$

Indeed, assume otherwise. Let $A^*\left(\xi, \vec{\eta}, \vec{v}\right)$ be the bounded subset of $h\left(\xi, \vec{\eta}, \vec{v}\right)$ which consists of all the ordinals, which are forced by some extension of $p\left(\xi, \vec{\eta}, \vec{v}\right) \restriction v_k$ to be in $A\left(\xi, \vec{\eta}, \vec{v}\right)$ (whenever $p\left(\xi, \vec{\eta}, \vec{v}\right)$ forces that the length of $t^{p(\xi,\vec{\eta},\vec{v})}_{h(\xi,\vec{\eta},\vec{v})}$ is greater than n^*). Then, in $M\left[j_W(G)\right]$,

$$\mu_\alpha \in k_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto A^*\left(\xi, \vec{\eta}, \vec{v}\right)\right)\left(\kappa, j_{0,\alpha}\left(\beta'_1\right), \dots, j_{0,\alpha}\left(\beta'_s\right), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)\right)$$

but this is a contradiction, since $j_\alpha\left(\langle \xi, \vec{\eta}, \vec{v} \rangle \mapsto A^*\left(\xi, \vec{\eta}, \vec{v}\right)\right)\left(\kappa, j_{0,\alpha}\left(\beta'_1\right), \dots, j_{0,\alpha}\left(\beta'_s\right), \mu_{\alpha_1}, \dots, \mu_{\alpha_k}\right)$ is a bounded subset of μ_α.

Therefore, we can assume that

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right) \restriction \mu_{\alpha_k}(\xi)$$

forces that

$$\mathrm{lh}\left(t^{p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)}_{h\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)}\right) \leq n^*.$$

Denote now $p^* = \left[\xi \mapsto p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \dots, \mu_{\alpha_k}(\xi)\right)\right]_W$. Then $p^* \restriction k_\alpha(\mu_\alpha)$ forces that $\mu_\alpha \in \underset{\sim}{A}^{p^*}_{k_\alpha(\mu_\alpha)}$. By the definition of the sets $e\left(\xi, \vec{\eta}, \vec{v}\right)$, the set $\underset{\sim}{A}^{p^*}_{k_\alpha(\mu_\alpha)}$ is forced to be either disjoint or contained in $k_\alpha(X)$. Since $k_\alpha(X) \in$

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$j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha))$, it cannot be disjoint (again, by the definition of $e(\xi, \vec{\eta}, \vec{v})$). Therefore $\mu_\alpha \in k_\alpha(X)$ and thus $X \in U_{\mu_\alpha}$, as desired. ⊣

Finally, let us argue that $j_{\kappa^*} = j_W \restriction V$. Recall that $\kappa^* = i(\kappa)$, and note that $\kappa^* = \sup\{\mu_\alpha : \alpha < \kappa^*\}$.

LEMMA 4.20. *$M = M_{\kappa^*}$, $j_W(\kappa) = i(\kappa)$ and $j_{\kappa^*} = j_W \restriction V$.*

REMARK 4.21. In particular, if $i = j_U$ (namely W is simply generated) then $j_W(\kappa) = j_U(\kappa)$. On the other hand, possibly $j_U(\kappa) < i(\kappa)$, and then $j_W(\kappa) > j_U(\kappa)$.

PROOF. Define, similarly to $k_\alpha : M_\alpha \to M$, the embedding $k_{\kappa^*} : M_{\kappa^*} \to M$ as follows:

$$k_{\kappa^*}\left(j_{\kappa^*}(f)\left(\kappa, j_{0,\kappa^*}(\beta_1), \dots, j_{0,\kappa^*}(\beta_l), \mu_{\alpha_1}, \dots, \mu_{\alpha_m}\right)\right) =$$
$$j_W(f)\left(\kappa, \theta_{\left[f_{\beta_1}(\xi)\right]_W}, \dots, \theta_{\left[f_{\beta_l}(\xi)\right]_W}, \mu_{\alpha_1}, \dots, \mu_{\alpha_m}\right)$$

for every $f \in V$, $\beta_1, \dots, \beta_l$ generators of i and $\alpha_1 < \dots < \alpha_m < \kappa^*$. Clearly crit$(k_{\kappa^*}) \geq \kappa^*$. It suffices to prove that k_{κ^*} is the identity function.

Let τ be an ordinal, and let $f \in V[G]$ be a function such that $[f]_W = \tau$. By the κ-c.c. of P_κ, there exists $F \in V$ such that for every $\xi < \kappa$, $f(\xi) \in F(\xi)$ and $|F(\xi)| < \kappa$. Therefore, in $M\left[j_W(G)\right]$,

$$\tau = [f]_W \in [F]_W = k_{\kappa^*}\left(j_{\kappa^*}(F)(\kappa)\right).$$

But

$$|j_{\kappa^*}(F)(\kappa)| < j_{\kappa^*}(\kappa) = \kappa^* \leq \text{crit}(k_{\kappa^*})$$

so $\tau \in \text{Im}(k_{\kappa^*})$, as desired. ⊣

§5. Further directions and open problems. It is likely that the results of Section 4 can be extended to wider context of Prikry-type forcing notions. The first candidates are one-element Prikry forcings and Prikry forcings with non-normal ultrafilters. For the former it seems that the present arguments can be applied without much changes. The latter looks to require more work since $[id]$ is not κ anymore and additional generators may appear. Another example is Extender-based Prikry forcings. Here some new ideas seem to be needed due to the Cohen parts of the forcings.

Let us state some open questions.

QUESTION 5.1. *Are there other ways to generate normal ultrafilters W in $V[G]$ beyond those given in* 1.1?

Let $W \in V[G]$ be a normal measure on κ. Assuming $\neg o^{\P}$ (or even no inner model with a Woodin cardinal) and exploring the closure of the ultrapower, it seems possible to argue that N of the type of 1.1 should exist (see Section 3.2). So we may extend the above question and ask, whether W must be generated from the embedding $i : V \to N$ as in Section 2.

QUESTION 5.2. *What are the possibilities for non-normal κ-complete ultrafilters in $V[G]$?*

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Recall that, given $i: V \to N$ and a measure W generated from it as in Theorem 1.1, the assumption that $\leq^*_{Q_\alpha} = \leq_{Q_\alpha}$ holds for a final segments of $\alpha \in \Delta$ suffices for $N = M$ (where M is the ground model of $\mathrm{Ult}(V[G], W)$).

QUESTION 5.3. *Suppose that for unboundedly many* $\alpha < \kappa$, $\leq_\alpha \neq \leq^*_\alpha$. *Is then* $M \neq N$?

QUESTION 5.4. *What are the exact conditions on* Q_α*'s that insure* $M = N$?

In Section 3 we studied sufficient and necessary conditions for having $j_W(\kappa) > j_U(\kappa)$. In Proposition 3.1, we proved, under the assumption that κ is a limit of cardinals $\alpha < \kappa$ which are all κ-strong, that there are measures $U \in \mathcal{K}$ and $W \in \mathcal{K}[G]$ on κ extending U, such that $j_W(\kappa) > j_U(\kappa)$.

QUESTION 5.5. *Is the assumption that* κ *is a limit of* κ*-strong cardinals really necessary*?

QUESTION 5.6. *In Theorem 4.7, can we omit the assumption that the normal measures used in the iteration* $P = P_\kappa$ *below* κ *are simply generated*? *How is the structure of* $j_W \restriction V$ *influenced from such a change*?

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