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On rings with trivial torsion parts

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In this paper, we exhibit the necessary and sufficient conditions for a ring R to have only the trivial torsion parts with respect to any (hereditary) radical on the category of left R-modules.

0. Introduction

Let R be a ring with identity and r be a (hereditary) radical on the category ${}_R\text{mod}$ of the left R-modules, that is, r is an idempotent subfunctor of identity such that $r(M/r(M)) = 0$ for every $M \in {}_R\text{mod}$ (in addition, r is left exact). In investigations of radical structure on modules, we often need the condition $r(R) = 0$. So it is natural and of interest to study rings having this property for all non-trivial radicals. We shall say that R is a left R-ring (T-ring) if $r(R) = 0$ for every non-trivial (hereditary) radical r on ${}_R\text{mod}$.

In this paper, we exhibit the necessary and sufficient conditions for a ring to be either an R-ring or a T-ring, supplied with interesting counterexamples. The main result of Section 2 is: R is an R-ring (T-ring) iff R_n is an R-ring (T-ring) for every $n \geq 1$. Section 3 applies the ideas of Gardner's work [7] to an extent of a structural investigation of T and R-rings with non-zero socles. Throughout this paper, unless otherwise specified, R stands for a ring with identity and either T or R-rings are considered as the left T or R-rings. Let us recall ([4]) that the existence of a radical r on ${}_R\text{mod}$ is equivalent to the existence of a torsion theory (M, L) where

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$$M = \{M \in {}_R\text{mod} \mid r(M) = M\} = L^+ = \{M \in {}_R\text{mod} \mid \hom_R(M, L) = 0,\ \forall L \in L\}$$

and

$$L = \{M \in {}_R\text{mod} \mid r(M) = 0\} = M^* = \{L \in {}_R\text{mod} \mid \hom_R(M, L) = 0,\ \forall M \in M\} .$$

In particular ([9]), in the case of a hereditary radical it is equivalent to the existence of a radical filter $E \subseteq l(R)$, where $l(R)$ is the set of left ideals of R; that is,

(i) if $I \in E$ then $(I : a) = \{x \in R \mid xa \in I\} \in E$ for every $a \in R$;

(ii) $I \in l(R)$, $J \in E$, and $(I : a) \in E$ for every $a \in J$ imply that $I \in E$.

It is essential to know that if E is a radical filter then the corresponding radical r is defined by $r(M) = \{m \in M \mid (0 : m) \in E\}$ for every $M \in {}_R\text{mod}$ and $I \in E$ iff $r(R/I) = R/I$ ([9]). It is easy to see that our definition of a radical filter is equivalent to that of [9]. Note that if r is a radical on ${}_R\text{mod}$ then $r(R)$ is a two-sided ideal since r is a subfunctor of identity and the right multiplication on R is a left R-homomorphism. It should be remarked that, obviously, simple rings are R-rings and integral domains are T-rings.

We shall frequently use the following notation:

$M \subseteq R$ is right T-nilpotent if $\forall(a_1, a_2, \ldots \in M)$ $\exists(n \geq 1)$ such that $a_n a_{n-1} \ldots a_1 = 0$;

R is a commutative primary ring if the prime radical is a prime ideal;

$I \in l(R)$ is an essential ideal if $I \neq 0$ and $I \cap J \neq 0$ for every $J \in l(R)$, $J \neq 0$;

$R_R = \{M \in {}_R\text{mod} \mid r(M) = 0$ or $r(M) = M$ for every radical r on ${}_R\text{mod}\}$,

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$T_R = \{M \in {}_R\text{mod} \mid r(M) = 0$ or $r(M) = M$ for every hereditary radical r on ${}_R\text{mod}\}$;

$C(R)$ - the center of R ;

$\hat{M}$ - the injective hull of $M \in {}_R\text{mod}$;

$R(+)$ - the underlying abelian group of R ;

$J(R)$ - Jacobson radical of R ;

R^n - direct product of n copies of R ;

R_n - the full ring of matrices of degree n over R .

The scalar matrix corresponding to an element $x \in R$ is the diagonal matrix with all the elements on the diagonal equal to x .

For simplicity, by $M \in \mathsf{T}_R$ or T_R^h or F_R or F_R^h we mean that M is a torsion class, hereditary torsion class, torsion-free class and hereditary torsion-free class respectively.

1. On T and R-rings

THEOREM 1.1. *Let* R *be a ring and* $M \subseteq R$ *be a subset. Then*

$$\mathsf{E}_M = \{I \in l(R) \mid \forall(a_1, a_2, \ldots \in M)\forall(s \in R)\exists(n \geq 1)(a_n a_{n-1} \ldots a_1 s \in I)\}$$

is a radical filter and

(i) if M *is a left ideal then* E_M *is contained in the least radical filter containing* M *,*

(ii) if M *is a two-sided ideal then* E_M *is the least radical filter containing* M *,*

(iii) $\mathsf{E}_M = l(R)$ *iff* M *is right T-nilpotent.*

Proof. Let $I \in \mathsf{E}_M$, $t \in R$ and suppose that $a_1, a_2, \ldots \in M$ and $s \in R$. Then there is $n \geq 1$ such that $a_n a_{n-1} \ldots a_1 st \in I$, that is, $a_n a_{n-1} \ldots a_1 s \in (I : t)$ and consequently $(I : t) \in \mathsf{E}_M$. If K is a left ideal such that for every $k \in I$, $(K : k) \in \mathsf{E}_M$, then there is

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$n \geq 1$ such that $a_n a_{n-1} \ldots a_1 s = u \in I$ and $(K : u) \in E_M$. Hence, there is $m \geq 1$ such that $a_{n+m} \ldots a_{n+1} a_n \ldots a_1 s \in K$.

(i) Let $K \in E_M \backslash C$ where C is the least radical filter containing M. By the definition of radical filter there is $a_1 \in M$ such that $(K : a_1) \notin C$ and consequently there is a sequence $a_1, a_2, \ldots \in M$ such that $((\ldots((K : a_1) : a_2) : \ldots) : a_n) = (K : a_n a_{n-1} \ldots a_1) \notin C$ for every $n \geq 1$, which yields a contradiction with the definition of E_M.

(ii) If M is a two-sided ideal then obviously $M \in E_M$.

(iii) It is easy to show that $E_M = l(R)$ iff $0 \in E_M$.

COROLLARY 1.2. *If R is a commutative ring, I is an ideal in R and $E'_I = \{K \in l(R) \mid K \subseteq I$ and I/K is T-nilpotent in $R/K\}$, then $E_I = \{J \in l(R) \mid \exists (K \in E'_I)(K \subseteq J)\}$ is the least radical filter containing I.*

THEOREM 1.3. *Let R be a ring. If $(0 : a)$ is right T-nilpotent for every $a \in R$, $a \neq 0$ then R is a T-ring. Conversely, if R is a T-ring then $(0 : Ra)$ is right T-nilpotent for every $a \in R$, $a \neq 0$.*

Proof. The sufficient condition follows right from Theorem 1.1. For the necessary condition, since $(0 : Ra)$ is a two-sided ideal, $E_{(0:Ra)}$ is the least radical filter containing $(0 : Ra)$ by Theorem 1.1 *(ii)*. If $a \neq 0$ then $(0 : Ra) \subseteq (0 : a) \in E_{(0:Ra)} = l(R)$, since R is a T-ring; and Theorem 1.1 *(iii)* finishes the proof.

COROLLARY 1.4. *Let R be a commutative ring. Then R is a T-ring iff $(0 : a)$ is T-nilpotent for every $a \in R$, $a \neq 0$.*

COROLLARY 1.5. *Every commutative T-ring is primary.*

PROPOSITION 1.6. *Let R be a T-ring and $e \in R$ be a central idempotent. Then $e = 0$ or $e = 1$.*

Proof. Put $K = eR$. Then $K^2 = K$ and K is a two-sided ideal. If $(0 : e) = 0$ then, obviously, $e = 1$. Suppose that $a \in (0 : e)$,

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$a \neq 0$. Then $K \subseteq (0 : a) \in E$ where $E = \{I \in \mathcal{I}(R) \mid K \subseteq I\}$ is a radical filter containing K (it needs just a tedious checking of the radical filter's properties). Since R is a T-ring, $0 \in E$ and consequently $K = 0$.

REMARK 1.7. (i) By Proposition 1.6, no direct product of 2 rings is a T-ring and consequently T-rings are not closed under quotient rings (for example, consider the ring of integers).

(ii) By Corollary 1.4, the commutative T-rings are closed under the subrings containing the identity. On the other hand, generally it is not so in the non-commutative case. For, consider the full matrix ring of degree $n > 1$ over a field. It is an R-ring which contains an idempotent e different from zero and identity and the subring generated by e and 1 is not a T-ring.

PROPOSITION 1.8. *Let R be a T-ring, $0 \neq a \in C(R)$ and $(0 : a) \neq 0$. Then*

(i) if $0 \neq M \in {}_R\text{mod}$ then there is $m \in M$, $m \neq 0$, such that $a \in (0 : m)$,

(ii) $(0 : a)$ is an essential left ideal of R ,

(iii) $(0 : a)$ is right T-nilpotent,

(iv) a is nilpotent.

Proof. *(i)* Consider $M_a = \{M \in {}_R\text{mod} \mid m \in M, m \neq 0 \Rightarrow am \neq 0\}$. Then $M_a \in F_R^h$. For, it is sufficient to show that M_a is closed under the injective hulls. Let $M \in M_a$. Since $a \in C(R)$, $D = \{m \in \hat{M} \mid am = 0\}$ is a submodule of $\hat{M}$ and $D \cap M = 0$. Hence $D = 0$. Now, by the hypothesis $R \notin M_a$ and since R is a T-ring, $M_a = 0$.

The rest is an easy consequence of *(i)* and Theorem 1.3.

COROLLARY 1.9. *Let R be a T-ring. Then $R(+)$ is either torsion-free or a p-group, for some prime p .*

PROPOSITION 1.10. *Let R be a ring. Then the following are equivalent:*

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(i) R *is an R-ring;*

(ii) *if* $A, B \in {}_R\mathrm{mod}$ *and* $\mathrm{hom}_R(A, B) = 0$ *, then either* $B = 0$ *or* $\mathrm{hom}_R(A, R) = 0$ *;*

(iii) *for every non-zero left ideal* I *and every non-zero* $M \in {}_R\mathrm{mod}$ *,* $\mathrm{hom}_R(I, M) \neq 0$ *;*

(iv) *for every non-trivial left ideal* I *,* $\mathrm{hom}_R(I, R/I) \neq 0$ *,*

(v) *for every non-trivial two-sided ideal* I *,* $\mathrm{hom}_R(I, R/I) \neq 0$.

Proof. *(i)* ⇒ *(ii)* ⇒ *(iii)* ⇒ *(iv)* ⇒ *(v)* is obvious.

(v) ⇒ *(i)*. If r is a radical on ${}_R\mathrm{mod}$ then $r(R)$ is a two-sided ideal and $\mathrm{hom}_R\big(r(R), R/r(R)\big) = 0$. Hence $r(R) = 0$ or $r(R) = R$.

PROPOSITION 1.11. *Let* R *be a ring. Then*

(i) *if* R *is an R-ring then for every non-zero left ideal* I *and every simple module* M *there is a left ideal* K *such that* $K \subseteq I$ *and* $I/K \cong M$ *,*

(ii) *if for every non-trivial two-sided ideal* I *,* I *is projective and there is a left ideal* S *such that* $I \subseteq S$ *and* $\mathrm{hom}_R(I, R/S) \neq 0$ *, then* R *is an R-ring.*

Proof. *(i)* It follows straight from Proposition 1.10 *(iii)*.

(ii) We shall prove condition 1.10 *(v)*. Let I be a non-trivial two-sided ideal. Then we have the exact sequence

$$\mathrm{hom}_R(I, R/I) \to \mathrm{hom}_R(I, R/S) \to \mathrm{ext}_R(I, S/I) = 0 .$$

Since $\mathrm{hom}_R(I, R/S) \neq 0$, $\mathrm{hom}_R(I, R/I) \neq 0$.

PROPOSITION 1.12. *Let* R *be an R-ring and* I *a left ideal such that* $IR \neq R$ *. Then for every left ideal* K *,* $IK = K \Rightarrow K = 0$.

Proof. Put $A_I = \{M \in {}_R\mathrm{mod} \mid IM = M\}$. It is easy work to show that $A_I \in T_R$. Let K be a non-zero left ideal and suppose that $K \in A_I$.

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Then $R \in A_I$ as well, since R is an R-ring, and it yields a contradiction.

PROPOSITION 1.13. *Let R be a ring such that for every non-trivial two-sided ideal I, $I^2 \neq I$. If M is a projective module and $r(M) = M$, for some non-trivial radical r, then $M = 0$.*

Proof. Let $M \neq 0$ be projective and $r(M) = M$ for some non-trivial radical r. Consider the least torsion class $\mathcal{M}$ containing M. Since M is projective, the corresponding torsion-free class $\mathcal{M}^*$ is a hereditary torsion class which is closed under the direct products, which implies that the corresponding radical filter E is closed under intersections, and consequently $\bigcap_{I \in E} I = K$ is an idempotent two-sided ideal. Hence $K = 0$ or $K = R$, a contradiction.

COROLLARY 1.14. *Let R be a ring. If every non-trivial two-sided ideal is projective and not idempotent then R is an R-ring.*

EXAMPLE 1.15. Let G be a subgroup of the additive group of real numbers.such that there exists a sequence $\{a_i\}_{i=1} \subset G \cap (0, 1)$ satisfying $\sum_{i=1}^{\infty} a_i < 1$. Consider the vector space V over a field F having the basis $A = G \cap \langle 0, 1)$. We shall define a binary operation $*$ on $A \cup \{\bar{0}\}$, where $\bar{0}$ is the zero element of V, by the following manner: if $a, b, a+b \in A$ then $a * b = a + b$, $a * b = \bar{0}$ otherwise. We can easily extend the operation $*$ onto the whole V and we get an F-algebra. The following statements are valid:

(i) $(\bar{0} : a)$ is nilpotent for every $a \in V$, $a \neq 0$;

(ii) V is a commutative primary ring;

(iii) V is a T-ring;

(iv) the prime radical P of V is not T-nilpotent and $P^2 = P$;

(v) V is not an R-ring (see Proposition 1.12).

Moreover, it is possible to choose A being countable. This example is based on the ideas of [8].

EXAMPLE 1.16. Consider $S = Z \times Q$, where Z is the additive group of integers and Q the additive group of rational numbers. Define the following binary operation on S:

$$(z_1, q_1) * (z_2, q_2) = (z_1z_2, z_1q_2+z_2q_1) .$$

Then S becomes a commutative primary ring with prime radical nilpotent of degree 2. Hence S is a T-ring which is not an R-ring (see Proposition 1.11). This example is based on the ideas of [5].

2. Full matrix rings over T and R-rings

DEFINITION 2.1. Let R be a ring, $M \in {}_R\text{mod}$ and N be a submodule in M. We shall say that N satisfies the condition (T) in M if $0 \neq N \neq M$ and there exist $x \in N$, $y \in M \backslash N$ such that $(0 : x) \subseteq (N : y)$.

PROPOSITION 2.2. *Let R be a ring, $M \in {}_R\text{mod}$ and N be a submodule in M. Then the following are equivalent:*

(i) there is a hereditary radical r on ${}_R\text{mod}$ such that $r(M) = N$;

(ii) N does not satisfy (T) in M.

Proof. *(i) ⇒ (ii).* Suppose that $0 \neq N \neq M$ and N satisfies (T) in M, that is, there is $x \in N$ and $y \in M \backslash N$ such that $(0 : x) \subseteq (N : y)$. The map $f : Rx \to M/N$, $ax \mapsto ay+N$ is a well-defined homomorphism and it yields a contradiction, since $r(Rx) = Rx$ and $r(M/N) = 0$.

(ii) ⇒ (i). Without loss of generality we can assume that $0 \neq N \neq M$. Consider the least hereditary torsion class M containing N and r be the corresponding hereditary radical. Obviously $N \subseteq r(M)$. If $N \neq r(M)$ then there is a submodule $K \subseteq N$ and a non-zero homomorphism $f : K \to r(M)/N$. Hence there are $k \in K$ and $y \in r(M) \backslash N$ such that $f(k) = y + N$ and consequently $(0 : k) \subseteq (N : y)$, a contradiction.

COROLLARY 2.3. *Let R be a ring and $M \in {}_R\text{mod}$. Then the following are equivalent:*

(i) $M \in T_R$;

(ii) every non-trivial submodule of M *satisfies* (T) .

THEOREM 2.4. *Let* R *be a ring. Then the following are equivalent:*

(i) R *is a* T*-ring;*

(ii) every non-trivial left ideal satisfies (T) *in* R ;

(iii) every non-trivial two-sided ideal satisfies (T) *in* R .

Proof. *(i)* $\Rightarrow$ *(ii)* by Corollary 2.3.

(ii) $\Rightarrow$ *(iii)* obvious.

(iii) $\Rightarrow$ *(i)* by Proposition 2.2, considering the fact that any torsion part of R is a two-sided ideal.

THEOREM 2.5. *Let* R *be a ring. Then*

(i) if R *is a* T*-ring, then for every* $n \geq 1$, *the full matrix ring* R_n *is a* T*-ring,*

(ii) if there is $n \geq 1$ *such that* R_n *is a* T*-ring then* R *is a* T*-ring.*

Proof. *(i)* Let K be a non-trivial two-sided ideal in R_n . It is easy to see that there is a non-trivial two-sided ideal I in R such that $K = I_n$, that is, K is a full matrix ring (possibly without identity) over I . According to Theorem 2.4 *(iii)*, there are $x \in I$ and $y \in R \setminus I$ such that $(0 : x) \subseteq (I : y)$. If $X, Y \in R_n$ are the corresponding scalar matrices then obviously $X \in K$, $Y \in R_n \setminus K$ and $(0 : X) \subseteq (K : Y)$. Now it suffices to use Theorem 2.4 *(iii)*.

(ii) Let R_n be a T-ring, for some $n \geq 1$. There is a bijection f between left ideals of R_n and R-submodules of R^n given by $I \mapsto f(I)$, $f(I)$ is a submodule in R^n consisting of all the rows of matrices from I . If M is a non-trivial submodule of R^n then there are matrices A, B such that $A \in f^{-1}(M)$, $B \in R_n \setminus f^{-1}(M)$ and $(0 : A) \subseteq \left(f^{-1}(M) : B\right)$ (see Theorem 2.4 *(ii)*). Since $B \in R_n \setminus f^{-1}(M)$,

there is $1 \le i \le n$ such that the i-th row of B does not lie in M. Put $C \in R_n$ as follows: $C = (c_{kl})$, $c_{ii} = 1$ and $c_{kl} = 0$ otherwise. Since $(0 : A) \subseteq (f^{-1}(M) : B)$, we get

$$(0 : CA) = ((0 : A) : C) \subseteq ((f^{-1}(M) : B) : C) = (f^{-1}(M) : CB) .$$

Let x be the i-th row of CA and y be the i-th row of CB. Obviously $x \in M$ and $y \in R^n \backslash M$. Consider $a \in (0 : x)$ and denote by D the corresponding scalar matrix. Then $DCA = 0$, hence $DCB \in f^{-1}(M)$ and consequently $ay \in M$. Now, by Corollary 2.3, $R^n \in T_R$ and since T_R is closed under submodules, $R \in T_R$.

PROPOSITION 2.6. *Let R be a ring and N be a submodule of an R-module M. Then the following are equivalent:*

(i) there is a radical r on ${}_R\mathrm{mod}$ such that $r(M) = N$;

(ii) $\mathrm{hom}_R(N, M/N) = 0$.

Proof. *(i)* ⇒ *(ii)* is obvious.

(ii) ⇒ *(i)*. Let A be the least torsion-free class containing M/N and r be the corresponding radical. Obviously $N \subseteq r(M)$. On the other hand, $\mathrm{hom}_R(r(M)/N, M/N) = 0$ implies that $r(M) \subseteq N$.

COROLLARY 2.7. *Let M be an R-module. Then the following are equivalent:*

(i) $M \in \mathsf{R}_R$,

(ii) if N is a non-trivial submodule of M then $\mathrm{hom}_R(N, M/N) \neq 0$.

THEOREM 2.8. *Let R be a ring. Then*

(i) if R is an R-ring then for every $n \ge 1$, the full matrix ring R_n is an R-ring,

(ii) if there is $n \ge 1$ such that R_n is an R-ring then R is an R-ring.

Proof. *(i)* Let K be a two-sided ideal in R_n. Then there is a two-sided ideal $I \subseteq R$ such that $K = I_n$, that is, K is the full matrix ring (possibly without identity) over I and if $S = R/I$ then $R_n/K \cong S_n$ as R_n-modules. Suppose that $0 \neq K \neq R_n$, then $0 \neq I \neq R$ and there is a non-zero $f \in \hom_R(I, R/I)$. Hence we can make f into $\overline{f} \in \hom_{R_n}(K, R_n/K)$ by $\overline{f}((a_{ij})) = (f(a_{ij}))$ and $\overline{f} \neq 0$, so that, with respect to Proposition 1.10 *(v)*, R_n is an R-ring.

(ii) Let M be a non-trivial R-submodule of R^n and I be the corresponding left ideal in R_n. By Proposition 1.10 *(iv)*, there is a non-zero $f \in \hom_{R_n}(I, R_n/I)$ and consequently there is $A = (a_{ij}) \in I$ such that $f((a_{ij})) = (b_{ij}) + I \neq I$. Without loss of generality we can assume that the first row of (b_{ij}) does not lie in M. Hence we can make f into non-zero $\overline{f} \in \hom_R(M, R^n/M)$ by $\overline{f}(m) = (c_{1j}) + M$, where

$$f\left(\begin{pmatrix} m_1, & \ldots, & m_n \\ 0, & \ldots, & 0 \\ & \ldots & \\ 0, & \ldots, & 0 \end{pmatrix}\right) = (c_{ij}) + I ,$$

and an application of Corollary 2.7 shows that $R^n \in \mathrm{R}_R$.

PROPOSITION 2.9. *Let R be such a T-ring that every two-sided ideal I is in the form $I = aR = Ra$, for some $a \in R$. Then R is an R-ring.*

Proof. Suppose that I is a non-trivial two-sided ideal. Then by Theorem 2.4 *(iii)* there is $x \in I$ and $y \in R \backslash I$ such that $(0 : x) \subseteq (I : y)$ and since $I = aR$, $x = ab$ for some $b \in R$. Hence $(0 : a) \subseteq (0 : x)$ and there is a non-zero $f \in \hom_R(I, R/I)$ such that $f(da) = dy + I$; that is, by Proposition 1.10 *(v)*, the proof is finished.

REMARK 2.10. The authors do not know whether, in general, the polynomial rings over T-rings are T-rings. However, the following is

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true.

PROPOSITION 2.11. *Let* R *be a commutative* T*-ring with nilpotent prime radical* $P(R)$. *Then* $R[x]$ *is a* T*-ring.*

Proof. Denote by n the degree of nilpotency of $P(R)$. Let $g \in R[x]$ with $(0 : g) \neq 0$ and $h \in (0 : g)$. It is well known that the coefficients of h are zero divisors in R (see, for example, [1], Chapter 1, exercise 2), and therefore they lie in $P(R)$. Now it is easy to see that $(0 : g)$ is nilpotent of degree n and Theorem 1.3 finishes the proof.

3. On T and R-rings with non-zero socles

THEOREM 3.1. *The following conditions for a ring* R *are equivalent:*

(i) R *is a left* T*-ring with non-zero left socle;*

(ii) all simple left R*-modules are isomorphic and all non-zero left* R*-modules have non-zero socles;*

(iii) ${}_R\mathrm{mod}$ *has only two hereditary torsion theories;*

(iv) R *is isomorphic to a full matrix ring over a local ring having left socle sequence;*

(v) $J(R)$ *is right* T*-nilpotent and* $R/J(R)$ *is a simple semi-simple artinian ring.*

Proof. *(i)* ⇒ *(ii)* Let I be a minimal left ideal in R . By *(i)*, R lies in the least torsion class containing I . Therefore $\hom_R(I, M) \neq 0$ for every non-zero left R-module M and *(ii)* easily follows.

(ii) ⇒ *(iii)*. See [7], Proposition 2.

(iii) ⇒ *(i)*. Obvious.

(iii) ⇔ *(iv)*. See [6], Theorem 1.

(iii) ⇔ *(v)*. See [7], Theorems 4 and 6.

THEOREM 3.2. *Let* R *be a ring. Then the following are equivalent:*

(i) R *is a left* R*-ring with non-zero left socle and* $J(R)$ *is left* T*-nilpotent;*

(i') R *is a right* R*-ring with non-zero right socle and* $J(R)$ *is right* T*-nilpotent;*

(ii) ${}_R\text{mod}$ *has only two torsion theories;*

(ii') mod_R *has only two torsion theories;*

(iii) $J(R)$ *is left and right* T*-nilpotent and* $R/J(R)$ *is a simple semisimple artinian ring;*

(iv) R *is left and right perfect and has only one simple module up to isomorphism;*

(v) R *is isomorphic to a full matrix ring over a left and right perfect local ring.*

Proof. It clearly suffices to prove the equivalence of the left-hand forms, since condition *(iii)* is self-dual.

(i) $\Leftrightarrow$ *(ii)*. It follows from Theorem 3.1 *(v)* and [7], Theorem 3.

(ii) $\Leftrightarrow$ *(iii)*. See [7], Theorems 3 and 6.

(iii) $\Leftrightarrow$ *(iv)*. See [2], Theorem P, (1) $\Leftrightarrow$ (2).

(iii) $\Leftrightarrow$ *(v)*. See [3], the main theorem, (1A) $\Leftrightarrow$ (1F).

REMARK 3.3. These conditions are equivalent to many others; see, for example, [3], [6].

COROLLARY 3.4. *Let* R *be a commutative ring with non-zero socle. Then* R *is a* T*-ring iff it is an* R*-ring.*

PROPOSITION 3.5. *Let* R *be a* T*-ring with non-zero socle. Then the following are equivalent:*

(i) R *is an* R*-ring;*

(ii) all submodules of projective modules contain maximal submodules;

(iii) all left ideals contain maximal submodules.

Proof. *(i)* $\Rightarrow$ *(ii)*. Let $\mathbb{A}$ be the least torsion-free class containing all simple R-modules. Obviously $R \in \mathbb{A}$ and hence every submodule of a projective module has a simple epimorphic image. Thus it contains a maximal submodule.

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(ii) ⇒ *(iii)* is trivial.

(iii) ⇒ *(i)*. By Theorem 3.1 *(ii)*, every non-zero left R-module has a simple submodule unique up to isomorphism, so that *(iii)* gives $\hom_R(I, M) \neq 0$ for every non-zero left ideal I and every non-zero left R-module M. Now it suffices to use Proposition 1.10 *(iii)*.

4. Weakly dense submodules

Let R be a ring and $M \in {}_R\text{mod}$. Then $E(M)$ will be the set consisting of the zero submodule and of all essential submodules of M. Further we shall denote by $\mathbb{M}_M$ the least hereditary torsion class containing M and by r_M the corresponding radical.

DEFINITION 4.1. Let R be a ring and $M \in {}_R\text{mod}$. A submodule $N \subseteq M$ is called weakly dense in M if there are $K \in E(M)$ and $m \in M\backslash K$ such that for every $n \in M$ and $a \in R\backslash(K : m)$, $(N : n) \nsubseteq (K : am)$.

PROPOSITION 4.2. *Let $M \in {}_R\text{mod}$ and $N \subseteq M$ be a submodule. Then N is weakly dense in M iff there are $K \in E(M)$ and $m \in M\backslash K$ such that $\hom_R\big(B/N, R(m+K)\big) = 0$ for every submodule B, $N \subseteq B \subseteq M$.*

Proof. (i) Let N be weakly dense in M and K, m be as in Definition 4.1. Let $f : B/N \to R(m+K)$ be a non-zero homomorphism. There is $b \in B$ such that $f(b+N) = am + K \neq K$. Hence $a \in R\backslash(K : m)$ and $(N : b) \subseteq (K : am)$, a contradiction.

(ii) If N is not weakly dense in M then for every $K \in E(M)$ and $m \in M\backslash K$ there are $n \in M$, $a \in R\backslash(K : m)$ such that $(N : n) \subseteq (K : am)$. Hence $f : (N+Rn)/N \to R(m+K)$ given by $xn + N \mapsto xam + K$, is a non-zero homomorphism.

PROPOSITION 4.3. *Let $M \in {}_R\text{mod}$ and $N \subseteq M$ be a submodule. If N is not weakly dense in M then $M \in \mathbb{M}_{M/N}$.*

Proof. Let $m \in M$ be a non-zero element. As N is not weakly dense in M, there is B, $N \subseteq B \subseteq M$, such that $\hom_R(B/N, Rm) \neq 0$. Hence $r_{M/N}(Rm) \neq 0$, so that $Rm \cap r_{M/N}(M) \neq 0$. Therefore $K = r_{M/N}(M) \in E(M)$.

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Now, from Proposition 4.2, we have $K = M$.

DEFINITION 4.4. Let $M \in {}_R\text{mod}$ and $N \subseteq M$ be a submodule. Then N is called dense in M if $r_{M/N}(M) = 0$, that is, if $\hom_R(B/N, M) = 0$ for all B , $N \subseteq B \subseteq M$.

PROPOSITION 4.5. *Let* $M \in {}_R\text{mod}$ *and* $N \subseteq M$ *be a submodule. Then* N *is dense in* M *iff* $(N : n) \not\subseteq (0 : m)$ *for all* $m, n \in M$, $m \neq 0$.

Proof. This is an immediate consequence of Definition 4.4.

PROPOSITION 4.6. *Let* $M \in {}_R\text{mod}$. *If* $M \in T_R$ *then every weakly dense submodule in* M *is dense in* M .

Proof. It follows from Proposition 4.3 and Definition 4.4.

THEOREM 4.7. *Let* R *be a ring. Then the following are equivalent:*

(i) R *is a* *T-ring;*

(ii) every weakly dense left ideal of R *is dense in* R .

Proof. *(i)* ⇒ *(ii)*. See Proposition 4.6.

(ii) ⇒ *(i)*. Let r be a hereditary radical and E the corresponding radical filter. If E contains only dense left ideals then $r(R) = 0$. Let $I \in E$, I be not dense in R . Then I is not weakly dense in R and hence $r_{R/I}(R) = R$ by Proposition 4.3. However $r(R/I) = R/I$ and therefore $r_{R/I}(M) \subseteq r(M)$ for every $M \in {}_R\text{mod}$. Thus $r(R) = R$.

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