Canad. Math. Bull. Vol. **47** (4), 2004 pp. 504–514

High Frequency Resolvent Estimates and Energy Decay of Solutions to the Wave Equation

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Abstract. We prove an uniform Hölder continuity of the resolvent of the Laplace-Beltrami operator on the real axis for a class of asymptotically Euclidean Riemannian manifolds. As an application we extend a result of Burq on the behaviour of the local energy of solutions to the wave equation.

1 Introduction and Statement of Results

Let (M, g) be an n-dimensional unbounded, connected Riemannian manifold with a Riemannian metric g of class $C^\infty(\overline{M})$ and a compact C^∞-smooth boundary ∂M (which may be empty), of the form $M = X_0 \cup X$, where X_0 is a compact, connected Riemannian manifold with a metric $g_{|X_0}$ of class $C^\infty(\overline{X}_0)$ with a compact boundary $\partial X_0 = \partial M \cup \partial X$, $\partial M \cap \partial X = \varnothing$, $X = [r_0, +\infty) \times S$, $r_0 \gg 1$, with metric $g_{|X} := dr^2 + \sigma(r)$. Here $(S, \sigma(r))$ is an $n-1$ dimensional compact Riemannian manifold without boundary equipped with a family of Riemannian metrics $\sigma(r)$ depending smoothly on r which can be written in any local coordinates $\theta \in S$ in the form

$$\sigma(r) = \sum_{i,j} g_{ij}(r,\theta)\,d\theta_i d\theta_j, \quad g_{ij} \in C^\infty(X).$$

Denote $X_r = [r, +\infty) \times S$. Clearly, ∂X_r can be identified with the Riemannian manifold $(S, \sigma(r))$ with the Laplace-Beltrami operator $\Delta_{\partial X_r}$ written as follows

$$\Delta_{\partial X_r} = -p^{-1} \sum_{i,j} \partial_{\theta_i}(p g^{ij} \partial_{\theta_j}),$$

where (g^{ij}) is the inverse matrix to (g_{ij}) and $p = (\det(g_{ij}))^{1/2} = (\det(g^{ij}))^{-1/2}$. Let Δ_g denote the Laplace-Beltrami operator on (M, g). We have

$$\Delta_X := \Delta_g|_X = -p^{-1}\partial_r(p\partial_r) + \Delta_{\partial X_r} = -\partial_r^2 - \frac{p'}{p}\partial_r + \Delta_{\partial X_r},$$

where $p' = \partial p/\partial r$. We have the identity

$$\Delta_X^\sharp := p^{1/2}\Delta_X p^{-1/2} = -\partial_r^2 + \Lambda_r + q(r,\theta), \tag{1.1}$$

Received by the editors November 19, 2002; revised February 12, 2003.
The first author was partially supported by CNPq (Brazil). The authors have also been partially supported by the agreement Brazil-France in Mathematics–Proc. 69.0014/01-5.
AMS subject classification: 35B37, 35J15, 47F05.

where

$$\Lambda_r = -\sum_{i,j} \partial_{\theta_i}(g^{ij}\partial_{\theta_j}),$$

and q is an effective potential given by

$$q(r,\theta) = (2p)^{-2}\Big(\frac{\partial p}{\partial r}\Big)^2 + (2p)^{-2}\sum_{i,j}\frac{\partial p}{\partial\theta_i}\frac{\partial p}{\partial\theta_j}g^{ij} + 2^{-1}p\Delta_X(p^{-1}).$$

We make the following assumptions:

$$\Big|\frac{\partial^k q}{\partial r^k}(r,\theta)\Big| \le Cr^{-k-\delta_0}, \quad k=0,1, \tag{1.2}$$

with constants $C, \delta_0 > 0$. Set $g_\flat^{ij} := r^2 g^{ij}$ and denote

$$h^\flat(r,\theta,\xi) = \sum_{i,j} g_\flat^{ij}(r,\theta)\xi_i\xi_j, \quad (\theta,\xi)\in T^*S.$$

We suppose that

$$\Big|\frac{\partial h^\flat}{\partial r}(r,\theta,\xi)\Big| \le Cr^{-1-\delta_0}h^\flat(r,\theta,\xi), \quad \forall(\theta,\xi)\in T^*S, \tag{1.3}$$

with constants $C, \delta_0 > 0$.

Denote by G the selfadjoint realization of Δ_g on the Hilbert space

$$H = L^2(M, d\mathrm{Vol}_g)$$

with Dirichlet or Neumann boundary conditions on ∂M. Given a real $s > 1/2$, choose a real-valued function $\chi_s \in C^\infty(\overline{M})$, $\chi_s = 1$ on $M \setminus X_{r_0+1}$, $\chi_s = r^{-s}$ on X_{r_0+2}. Also, given $a > r_0$ choose a real-valued positive function $\eta_a \in C^\infty(\overline{M})$, $\eta_a = 0$ on $M \setminus X_a$, $\eta_a = 1$ on X_{a+1}.

It was proved in [3] (in a more general situation) that (for $z \ge C_0$, $0 < \varepsilon \le 1$, and the constant $a > r_0$ big enough) the following estimates hold true

$$\|\chi_s(G - z \pm i\varepsilon)^{-1}\chi_s\|_{\mathcal{L}(H)} \le e^{Cz^{1/2}}, \tag{1.4}$$

$$\|\eta_a\chi_s(G - z \pm i\varepsilon)^{-1}\chi_s\eta_a\|_{\mathcal{L}(H)} \le C'z^{-1/2}, \tag{1.5}$$

with some constants $C_0, C, C' > 0$ independent of z and ε. One of the purposes of the present paper is to prove the following

Theorem 1.1 *Under the assumptions* (1.2) *and* (1.3)*, for every $s > 1/2$, there exist constants $a > r_0$ and $C_0, C, C' > 0$ so that for $z \ge C_0$, the limit*

$$R_s^\pm(z) := \lim_{\varepsilon\to 0^+}\chi_s(G - z \pm i\varepsilon)^{-1}\chi_s \colon H \to H$$

exists and satisfies the estimates, for $C_0 \le z_1 \le z$, $C_0 \le z_2 \le z$,

$$\|R_s^{\pm}(z_2) - R_s^{\pm}(z_1)\|_{\mathcal{L}(H)} \le C'|z_2 - z_1|^{\mu} e^{Cz^{1/2}}, \tag{1.6}$$

$$\|\eta_a R_s^{\pm}(z_2)\eta_a - \eta_a R_s^{\pm}(z_1)\eta_a\|_{\mathcal{L}(H)} \le C'|z_2 - z_1| e^{Cz^{1/2}} + C'|z_2 - z_1|^{\mu}, \tag{1.7}$$

where $0 < \mu < 1$ *is a constant depending only on* s *and* δ_0.

We will use this theorem to extend a result by Burq [1] on the behaviour of the local energy of the solutions of the mixed problem for the wave equation

$$\begin{cases} (\partial_t^2 + \Delta_g)u(t,x) = 0 & \text{in } \mathbf{R} \times M, \\ Bu(t,x) = 0 & \text{on } \mathbf{R} \times \partial M, \\ u(0,x) = f_1(x),\ \partial_t u(0,x) = f_2(x), & \text{for } x \in M, \end{cases} \tag{1.8}$$

where B denotes either Dirichlet or Neumann boundary conditions. Recall that the solutions to (1.8) can be expressed by the formula

$$u = \cos\left(t\sqrt{G}\right) f_1 + \frac{\sin\left(t\sqrt{G}\right)}{\sqrt{G}} f_2. \tag{1.9}$$

Our main result is the following:

Theorem 1.2 *Under the assumptions* (1.2) *and* (1.3), *for every* $s > 1/2$, *and* $m > 0$, *the following estimates hold for* $t \gg 1$:

$$\|\chi_s \cos\left(t\sqrt{G}\right) \psi(G)(G+1)^{-m/2}\chi_s\|_{\mathcal{L}(H)} \le C_{m,s}(\log t)^{-m}, \tag{1.10}$$

$$\|\chi_s \sin\left(t\sqrt{G}\right) \psi(G)(G+1)^{-m/2}\chi_s\|_{\mathcal{L}(H)} \le C_{m,s}(\log t)^{-m}, \tag{1.11}$$

with a constant $C_{m,s} > 0$, *where* ψ *denotes the characteristic function of the interval* $[C_0', +\infty)$ *and* $C_0' > C_0$ *is arbitrary and fixed.*

Remark 1 It follows easily by an interpolation argument that we have analogues of (1.10) and (1.11) for $0 < s \le 1/2$ as well, but with $O_\epsilon\left((\log t)^{-m(2s)^2+\epsilon}\right)$, $0 < \epsilon \ll 1$, in place of $(\log t)^{-m}$.

Remark 2 Clearly, the above results still hold true for the selfadjoint realization of $\Delta_g + V(x)$, where V is a real-valued potential, $V(x) \ge 0$, provided the assumption (1.2) is satisfied with q replaced by $q + V|_X$.

Remark 3 When $\partial M = \varnothing$ and the metric g is nontrapping (that is, every geodesic reaches the region X_r, $\forall r > r_0$, in a finite time), one can easily show by the methods of [3] (see also [4] where a similar bound is proved in a semi-classical setting) that (1.4) holds with $O(z^{-1/2})$ in place of the exponential term. As a consequence, our proof of the above theorems gives that in this case one can improve (1.6) and (1.7) replacing the exponential terms by constants, and have (1.10) and (1.11) with $O\left(t^{-\nu m/(m+2)}\right)$ in the right-hand side, where $0 < \nu < 1$ is independent of m but depending on s.

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Remark 4 We can take $\psi \equiv 1$ in Theorem 1.2 if the resolvent satisfies the following estimates:

$$\|\lambda R_s^{\pm}(\lambda^2)\|_{\mathcal{L}(H)} \leq C,$$
$$\|\lambda_2 R_s^{\pm}(\lambda_2^2) - \lambda_1 R_s^{\pm}(\lambda_1^2)\|_{\mathcal{L}(H)} \leq C|\lambda_2 - \lambda_1|^{\mu},$$

for all $0 < \lambda, \lambda_1, \lambda_2 \leq \sqrt{C_0}$, with some constants $C, \mu > 0$, where C_0 is as in Theorem 1.1.

It is easy to see that a long-range perturbation of the Euclidean metric on $\mathbf{R}^n$, $n \geq 2$, provides an example of a manifold of the kind described above and satisfying the assumptions (1.2) and (1.3), and hence to which our results apply. More precisely, let $\mathcal{O} \subset \mathbf{R}^n$ be a bounded domain with a C^∞-smooth boundary and a connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Let g be a Riemannian metric in Ω of the form

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j, \quad g_{ij}(x) \in C^\infty(\overline{\Omega}),$$

satisfying the estimates

$$|\partial_x^\alpha (g_{ij}(x) - \delta_{ij})| \leq C_\alpha \langle x \rangle^{-\gamma_0 - |\alpha|}, \tag{1.12}$$

for every multi-index α, with constants $C_\alpha, \gamma_0 > 0$, where $\langle x \rangle := (1 + |x|^2)^{1/2}$ and δ_{ij} denotes the Kronecker symbol. It is easy to see that (Ω, g) is isometric to a Riemannian manifold of the form described above satisfying assumptions (1.2) (with $\delta_0 = 2$) and (1.3) (with $\delta_0 = \gamma_0$) because of (1.12) and the fact that they are satisfied for the Euclidean metric on $\mathbf{R}^n$.

In the case when $g_{ij} = \delta_{ij}$ for $|x| \geq \rho_0$ with some $\rho_0 \gg 1$, Burq [1] proved (1.4) with χ_s replaced by a cutoff function $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi(x) = 1$ for $|x| \leq \rho_0 + 1$. As a consequence he obtained (1.10) and (1.11) with χ_s replaced by χ. His proof is based on the fact that in this case the exponential bound of the cutoff resolvent on the real axis implies that it extends analytically to a region of the form

$$\{z \in \mathbf{C} : |\operatorname{Im} z| \leq e^{-C_1|z|^{1/2}}, \operatorname{Re} z \geq C_2\}$$

for some constants $C_1, C_2 > 0$. In [2] he extended these results to long-range metrics analytic outside some compact. His approach, however, does not work anymore when the metric is not analytic outside a compact or when we have a weighted function instead of a cutoff. We show in the present paper that a uniform Hölder continuity of $R_s^{\pm}(z)$ suffices to establish the time decay in Theorem 1.2.

Usually, the Hölder continuity of the weighted resolvent near the real axis is proved by Mourre's method. To prove Theorem 1.1, however, we do not use this method. Instead, we show that this property follows from the estimate (1.4) and the Hölder continuity of the weighted resolvent of the Dirichlet self-adjoint realization of the operator Δ_X on the Hilbert space $L^2(X, d\mathrm{Vol}_g)$. Thus we are reduced to studying the resolvent of a much simpler operator. This is carried out in Section 3. The main point in our analysis is that, roughly speaking, the operator $2\Delta_X^\sharp + [r\partial_r, \Delta_X^\sharp]$ is of order $O(r^{-\delta_0})$ for $r \gg 1$, because of the assumptions (1.2) and (1.3). The only place where this fact is used is in the proof of the boundedness of the operator $\mathcal{B}$ introduced in Section 3. All the other arguments work out under the less restrictive assumptions of [3]. It is worth noticing that Mourre's method does not work in our situation without extra assumptions, because it requires some information about the double commutator $[r\partial_r, [r\partial_r, \Delta_X^\sharp]]$. Therefore, an application of this method would require making assumptions on $\partial_r^k g_\flat^{ij}$ for $k = 0, 1, 2$, and hence restricting the class of the Riemannian manifolds to which our results apply.

2 Proof of Theorems 1.1 and 1.2

Denote by G_0 the Dirichlet selfadjoint realization of Δ_X on $H_0 = L^2(X, d\,\mathrm{Vol}_g)$. Recall that $r \geq r_0 > 0$ on X, so the function r^{-s} belongs to $C^\infty(\overline{X})$ for all real s. We will derive Theorem 1.1 from the bounds (1.4), (1.5) and the following:

Proposition 2.1 *Under the assumptions* (1.2) *and* (1.3)*, for every $s > 1/2$, there exist constants $C_0, C_1 > 0$, $0 < \mu < 1$, so that for $C_0 \leq \operatorname{Re} z_1 \leq z$, $0 < \operatorname{Im} z_1 \leq 1$, $C_0 \leq \operatorname{Re} z_2 \leq z$, $0 < \operatorname{Im} z_2 \leq 1$, we have*

$$\|r^{-s}(G_0 - z_2)^{-1}r^{-s} - r^{-s}(G_0 - z_1)^{-1}r^{-s}\|_{\mathcal{L}(H_0)} \leq C_1|z_2 - z_1|^\mu. \tag{2.1}$$

Let $\rho \in C^\infty(\overline{M})$, $\rho = 1$ on $M \setminus X_{a+1}$, $\rho = 0$ on X_{a+2}. Given any $u \in D(G)$, we have $(1-\rho)u \in D(G_0)$, and $G(1-\rho)u = G_0(1-\rho)u$. Therefore, we have the following identity

$$\begin{aligned}
\chi_s(G - z_2)^{-1}\chi_s - \chi_s(G - z_1)^{-1}\chi_s
&= (z_2 - z_1)\chi_s(G - z_2)^{-1}\rho(2-\rho)(G - z_1)^{-1}\chi_s \\
&\quad + \chi_s(G - z_2)^{-1}(1-\rho)^2(G - z_1)^{-1}\chi_s \\
&= (z_2 - z_1)\chi_s(G - z_2)^{-1}\rho(2-\rho)(G - z_1)^{-1}\chi_s \\
&\quad + \big(\chi_s(G - z_2)^{-1}[G_0, \rho] + (1-\rho)\chi_s\big)\big((G_0 - z_2)^{-1} - (G_0 - z_1)^{-1}\big) \\
&\qquad \times \big(\chi_s(1-\rho) + [\rho, G_0](G - z_1)^{-1}\chi_s\big).
\end{aligned} \tag{2.2}$$

On the other hand, it is easy to see that (1.4) and (1.5) imply, respectively,

$$[\rho, G_0](G - z_j)^{-1}\chi_s = O\big(e^{Cz^{1/2}}\big) : H \to H,$$
$$[\rho, G_0](G - z_j)^{-1}\chi_s\eta_a = O(1) : H \to H,$$

where $j = 1, 2$. Thus, for these values of z_1 and z_2, (1.6) and (1.7) follow from (2.1), (2.2), (1.4) and (1.5). This in turn implies the existence of the limit, and hence (1.6) and (1.7) hold for real z_1 and z_2.

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In what follows in this section we will show that the bounds (1.4)–(1.6) imply Theorem 1.2. We let $\|\cdot\|$ denote the norm in $\mathcal{L}(H)$. Let $A_0 = \sqrt{C_0'}$ and let $A > A_0$ be a big parameter to be fixed later on. We can write

$$
\begin{aligned}
J(t) &:= \chi_s \cos\left(t\sqrt{G}\right)\psi(G)(G+1)^{-m/2}\chi_s \\
&= \sum_{j=1}^{2} \chi_s \cos\left(t\sqrt{G}\right)\psi_j(G)(G+1)^{-m/2}\chi_s := J_1(t) + J_2(t),
\end{aligned} \tag{2.3}
$$

where ψ_1 is the characteristic function of the interval $[A_0^2, A^2)$ and ψ_2 is the characteristic function of the interval $[A^2, +\infty)$. Clearly, by the spectral theorem we have

$$
\|J_2(t)\| \leq \|\psi_2(G)(G+1)^{-m/2}\| \leq \max_{\sigma} |\psi_2(\sigma)(\sigma+1)^{-m/2}| \leq A^{-m}. \tag{2.4}
$$

On the other hand,

$$
J_1(t) = \int_{A_0}^{A} \cos(t\lambda)F(\lambda)\,d\lambda, \tag{2.5}
$$

where

$$
F(\lambda) = (\pi i)^{-1}\lambda(1+\lambda^2)^{-m/2}\left(R_s^+(\lambda^2) - R_s^-(\lambda^2)\right)
$$

satisfies the bound (in view of (1.4) and (1.6))

$$
\|F(\lambda_2) - F(\lambda_1)\| \leq |\lambda_2 - \lambda_1|^{\mu} e^{CA}, \tag{2.6}
$$

for $A_0 \leq \lambda_1 \leq A$, $A_0 \leq \lambda_2 \leq A$, with possibly a new constant $C > 0$. Let $\phi \in C_0^\infty(\mathbf{R})$ be a real-valued function, $\phi \geq 0$, such that $\int \phi(\sigma)d\sigma = 1$. The function

$$
F_\epsilon(\lambda) = \epsilon^{-1}\int F(\lambda - \sigma)\phi(\sigma/\epsilon)\,d\sigma, \quad 0 < \epsilon \ll 1,
$$

is smooth with values in $\mathcal{L}(H)$ and, in view of (2.6), satisfies the bound (for $A_0 \leq \lambda \leq A$)

$$
\begin{aligned}
\|F_\epsilon(\lambda) - F(\lambda)\| &\leq \epsilon^{-1}\int \|F(\lambda) - F(\lambda-\sigma)\|\,\phi(\sigma/\epsilon)\,d\sigma \\
&\leq e^{CA}\epsilon^{-1}\int \sigma^{\mu}\phi(\sigma/\epsilon)\,d\sigma \leq O(\epsilon^{\mu})e^{CA}.
\end{aligned} \tag{2.7}
$$

Hence,

$$
\left\| J_1(t) - \int_{A_0}^{A} \cos(t\lambda)F_\epsilon(\lambda)\,d\lambda \right\| \leq O(\epsilon^{\mu})e^{CA}, \tag{2.8}
$$

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with possibly a new constant $C > 0$. On the other hand, integrating by parts gives

$$t\int_{A_0}^{A}\cos(t\lambda)F_\epsilon(\lambda)\,d\lambda = F_\epsilon(A_0)\sin(A_0t) - F_\epsilon(A)\sin(At) - \int_{A_0}^{A}\sin(t\lambda)\frac{dF_\epsilon(\lambda)}{d\lambda}\,d\lambda. \tag{2.9}$$

By (1.4) we have (with $k = 0, 1$)

$$\left\|\frac{d^kF_\epsilon(\lambda)}{d\lambda^k}\right\| \le O(\epsilon^{-k})e^{CA}, \tag{2.10}$$

for $A_0 \le \lambda \le A$. By (2.9) and (2.10), we conclude

$$\left\|\int_{A_0}^{A}\cos(t\lambda)F_\epsilon(\lambda)\,d\lambda\right\| \le O(\epsilon^{-1})t^{-1}e^{CA}. \tag{2.11}$$

Choosing $\epsilon = t^{-1/(1+\mu)}$, we get from (2.8) and (2.11),

$$\|J_1(t)\| \le O(t^{-\nu})e^{CA}, \tag{2.12}$$

where $\nu = \mu/(1+\mu)$. Choose now $A = O(\log t)$ so that $e^{CA} = t^{\nu/2}$. Then it is clear that (1.10) follows from (2.3), (2.4) and (2.12). The estimate (1.11) is treated in the same way.

3 Proof of Proposition 2.1

Denote by $G_0^\sharp$ the Dirichlet self-adjoint realization of the operator $\Delta_X^\sharp$ on the Hilbert space $H_0^\sharp = L^2(X, drd\theta)$. Clearly, it suffices to prove (2.1) with G_0 replaced by $G_0^\sharp$, and H_0 replaced by $H_0^\sharp$. In what follows $\|\cdot\|$ will denote the norm in $\mathcal{L}(H_0^\sharp)$. It is easy to see that (1.3) implies

$$-[\partial_r, \Lambda_r] \ge \frac{C}{r}\Lambda_r, \quad C > 0,$$

for r big enough. Therefore, it follows from Theorem 2.1 of [5] that we have the estimate (for $s > 1/2$, $0 < \varepsilon \le 1$, $k = 0, 1$)

$$\left\|r^{-s}\mathcal{D}_r^k(G_0^\sharp - z \pm i\varepsilon)^{-1}r^{-s}\right\| \le Cz^{-1/2}, \quad z \ge C_0, \tag{3.1}$$

where $\mathcal{D}_r = -iz^{-1/2}\partial_r$, with constants $C_0, C > 0$ independent of z and ε. Thus, it suffices to prove (2.1) when $0 < |z_2 - z_1| \le 1$. Obviously, if (2.1) holds true for some $s_0 > 1/2$ with $\mu = \mu_0 > 0$, it also holds for every $s > s_0$ with $\mu = \mu_0$. Let us see that it holds for $1/2 < s < s_0$, too. Given any $A \gg 1$, denote by $\chi(r \le A)$ (resp.

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$\chi(r \geq A)$) the characteristic function of the set $r \leq A$ (resp. $r \geq A$). In view of (3.1), we have

$$\begin{aligned}
&\left\| r^{-s}(G_0^\sharp - z_2)^{-1} r^{-s_0} - r^{-s}(G_0^\sharp - z_1)^{-1} r^{-s_0} \right\| \\
&\quad \leq \left\| r^{s_0 - s}\chi(r \leq A)\bigl(r^{-s_0}(G_0^\sharp - z_2)^{-1} r^{-s_0} - r^{-s_0}(G_0^\sharp - z_1)^{-1} r^{-s_0}\bigr) \right\| \\
&\qquad + \left\| r^{-(2s-1)/4}\chi(r \geq A) r^{-(2s+1)/2}(G_0^\sharp - z_2)^{-1} r^{-s_0} \right\| \\
&\qquad + \left\| r^{-(2s-1)/4}\chi(r \geq A) r^{-(2s+1)/2}(G_0^\sharp - z_1)^{-1} r^{-s_0} \right\| \\
&\quad \leq C A^{s_0 - s} |z_2 - z_1|^{\mu_0} + C A^{-(2s-1)/4} = O\bigl(|z_2 - z_1|^{\mu} \bigr),
\end{aligned}$$

if we choose $A = |z_2 - z_1|^{-4\mu_0/(4s_0 - 2s - 1)}$, where $\mu = \mu_0(2s-1)/(2s_0 - s - 1)$. Proceeding in the same way once more we can replace s_0 on the right by s as well. Thus, it suffices to prove (2.1) for $s > 3/2$. Let us see now that it would follow from the following estimate

$$\left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-2} r^{-s} \right\| \leq C\varepsilon^{-1+\alpha}, \tag{3.2}$$

for real $z \geq C_0$, with constants $C_0, C, \alpha > 0$ independent of z and ε. Fix $z \in \mathbf{C}$, $\operatorname{Im} z > 0$, $\operatorname{Re} z \geq C_0$, and let $0 < \varepsilon \leq 1$. Clearly, (3.2) implies

$$\left\| r^{-s}(G_0^\sharp - z - i\varepsilon)^{-1} r^{-s} - r^{-s}(G_0^\sharp - z)^{-1} r^{-s} \right\| \leq C\varepsilon^{\alpha}. \tag{3.3}$$

Therefore, if z_1 and z_2 are as in Proposition 2.1, we have

$$\begin{aligned}
&\left\| r^{-s}(G_0^\sharp - z_2)^{-1} r^{-s} - r^{-s}(G_0^\sharp - z_1)^{-1} r^{-s} \right\| \\
&\quad \leq \left\| r^{-s}(G_0^\sharp - z_2 - i\varepsilon)^{-1} r^{-s} - r^{-s}(G_0^\sharp - z_2)^{-1} r^{-s} \right\| \\
&\qquad + \left\| r^{-s}(G_0^\sharp - z_1 - i\varepsilon)^{-1} r^{-s} - r^{-s}(G_0^\sharp - z_1)^{-1} r^{-s} \right\| \\
&\qquad + \left\| r^{-s}(G_0^\sharp - z_2 - i\varepsilon)^{-1} r^{-s} - r^{-s}(G_0^\sharp - z_1 - i\varepsilon)^{-1} r^{-s} \right\| \\
&\quad \leq 2C\varepsilon^{\alpha} + |z_2 - z_1|\varepsilon^{-2} = O\bigl(|z_2 - z_1|^{\mu} \bigr),
\end{aligned} \tag{3.4}$$

if we take $\varepsilon = |z_2 - z_1|^{1/(2+\alpha)}$, where $\mu = \alpha/(\alpha + 2)$.

Proof of (3.2) Set

$$\mathcal{A} := 2\Delta_g^\sharp + [r\partial_r, \Delta_g^\sharp] = 2q + r\frac{\partial q}{\partial r} - r^{-1}\sum_{i,j} \partial_{\theta_i}\Bigl(\frac{\partial g_\flat^{ij}}{\partial r} \partial_{\theta_j} \Bigr).$$

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We have

$$
\begin{aligned}
2(z \mp i\varepsilon) r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-2} r^{-s} &= -2r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \\
&\quad + r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} 2\Delta_g^\sharp (G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \\
&= -2r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \\
&\quad - r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} [r\partial_r, \Delta_g^\sharp](G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \\
&\quad + r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \mathcal{A} (G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \\
&= -r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} - r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \partial_r r^{-s+1} \\
&\quad + r^{-s+1}\partial_r (G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \\
&\quad + r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \mathcal{A} (G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s},
\end{aligned}
$$

and since $s > 3/2$, in view of (3.1), we obtain

$$
(3.5)\quad 2z\left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-2} r^{-s} \right\| \le C + \left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \mathcal{A} (G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \right\|.
$$

We need now the following

Lemma 3.1 *For $s > 1/2$, $0 < \varepsilon \le 1$, we have*

$$
(3.6)\qquad \left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \right\| \le C\varepsilon^{-1/2} z^{-1/4}, \quad z \ge C_0,
$$

with constants $C, C_0 > 0$ independent of ε and z.

By (3.1) and (3.6), $\forall\, 0 < \delta \le 1/2$, we have

$$
(3.7)\quad
\begin{aligned}
&\left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-\delta} \right\| \\
&\quad \le \left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \chi(r \le A) r^{-\delta} \right\| + \left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \chi(r \ge A) r^{-\delta} \right\| \\
&\quad \le A^{s-\delta} \left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \right\| + A^{-\delta} \left\| r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \right\| \\
&\quad \le CA^{s-\delta} + CA^{-\delta}\varepsilon^{-1/2} = O\big(\varepsilon^{-1/2+\nu}\big),
\end{aligned}
$$

if we choose $A = \varepsilon^{-1/(2s)}$, where $\nu = \delta/(2s)$. Furthermore, we have

$$
(3.8)\quad
\begin{aligned}
&r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} \mathcal{A} (G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} = \\
&\qquad \big(r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} [\partial_r^2, r^{-\delta_0/2}] \\
&\qquad\qquad + (z \mp i\varepsilon + i) r^{-s}(G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-\delta_0/2} + r^{-s-\delta_0/2} \big) \\
&\qquad \times \mathcal{B} \big(r^{-s-\delta_0/2} + (z \mp i\varepsilon - i) r^{-\delta_0/2} (G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \\
&\qquad\qquad + [r^{-\delta_0/2}, \partial_r^2](G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \big),
\end{aligned}
$$

https://doi.org/10.4153/CMB-2004-050-3 Published online by Cambridge University Press

where

$$\mathcal{B} = (G_0^\sharp - i)^{-1} r^{\delta_0} \mathcal{A} (G_0^\sharp + i)^{-1}$$

is easily seen to be a bounded operator on $H_0^\sharp$. Indeed, it follows from the assumptions (1.2) and (1.3) that the real-valued quadratic form

$$\beta(v,v) = \langle r^{\delta_0} \mathcal{A} u, u \rangle_{H_0^\sharp}, \quad u = (G_0^\sharp + i)^{-1} v \in D(G_0^\sharp), \quad v \in H_0^\sharp,$$

satisfies the estimate

$$|\beta(v,v)| \le C \|u\|_{H_0^\sharp}^2 + C\langle \Lambda_r u, u\rangle_{H_0^\sharp} \le C\|u\|_{H_0^\sharp}^2 + C\langle \Delta_g^\sharp u, u\rangle_{H_0^\sharp} \le C\|v\|_{H_0^\sharp}^2, \quad C > 0.$$

Therefore, by the Riesz theorem there exists a self-adjoint operator $\mathcal{B}_1 \in \mathcal{L}(H_0^\sharp)$ such that

$$\beta(v,v) = \langle \mathcal{B}_1 v, v\rangle_{H_0^\sharp}.$$

Thus we get $\mathcal{B} = \mathcal{B}_1$ and the desired property follows. Now, by (3.1), (3.7) and (3.8) we conclude that

$$\left\| r^{-s} (G_0^\sharp - z \pm i\varepsilon)^{-1} \mathcal{A} (G_0^\sharp - z \pm i\varepsilon)^{-1} r^{-s} \right\| \le C z \varepsilon^{-1+\alpha}, \tag{3.9}$$

with $\alpha = \delta_0/(2s)$ if $\delta_0 \le 1$, and $\alpha = 1$ if $\delta_0 > 1$. Clearly, (3.2) follows from (3.5) and (3.9).

Proof of Lemma 3.1 It is actually contained in the proof of (3.1) (see the proof of Theorem 2.1 of [5] or the proof of Proposition 2.4 of [3]). We will only sketch the main points. Denote $\lambda = z^{1/2}$, $\mathcal{D}_r = (i\lambda)^{-1}\partial_r$, $P = \lambda^{-2}\Delta_g^\sharp - 1 + i\varepsilon\lambda^{-2}$, where $0 < \varepsilon \ll 1$. It is proved in the above articles that, $\forall u \in D(G_0^\sharp)$, we have

$$\begin{aligned} \left\| r^{-s} u \right\|_{H_0^\sharp}^2 &\le C \left\| Pu \right\|_{H_0^\sharp}^2 + C\varepsilon\lambda^{-1} \left| \langle u, \mathcal{D}_r u\rangle_{H_0^\sharp} \right| + C\lambda \left| \langle Pu, \mathcal{D}_r u\rangle_{H_0^\sharp} \right| \\ &\le O\big(\lambda^3 \varepsilon^{-1}\big) \|Pu\|_{H_0^\sharp}^2 + O\big(\varepsilon\lambda^{-1}\big) \left(\|u\|_{H_0^\sharp}^2 + \|\mathcal{D}_r u\|_{H_0^\sharp}^2 \right). \end{aligned} \tag{3.10}$$

On the other hand,

$$\varepsilon\lambda^{-2} \|u\|_{H_0^\sharp}^2 = \operatorname{Im} \langle Pu, u\rangle_{H_0^\sharp} \le O\left(\lambda^2 \varepsilon^{-1}\right) \|Pu\|_{H_0^\sharp}^2 + 2^{-1}\varepsilon\lambda^{-2}\|u\|_{H_0^\sharp}^2, \tag{3.11}$$

$$\begin{aligned} \|\mathcal{D}_r u\|_{H_0^\sharp}^2 - 2\|u\|_{H_0^\sharp}^2 &\le \|\mathcal{D}_r u\|_{H_0^\sharp}^2 + \langle \Lambda_r u, u\rangle_{H_0^\sharp} - \langle (1 - \lambda^{-2} q) u, u\rangle_{H_0^\sharp} \\ &= \operatorname{Re} \langle Pu, u\rangle_{H_0^\sharp} \le \|Pu\|_{H_0^\sharp}^2 + \|u\|_{H_0^\sharp}^2. \end{aligned} \tag{3.12}$$

By (3.10), (3.11) and (3.12),

$$\|r^{-s} u\|_{H_0^\sharp}^2 \le O\left(\lambda^3 \varepsilon^{-1}\right) \|Pu\|_{H_0^\sharp}^2, \tag{3.13}$$

which clearly implies (3.6).

Acknowledgements A part of this work was carried out while the second author was visiting Mittag-Leffler Institut in September–October 2002 and the first author was visiting University of Nantes in November 2002. The authors would like to thank these institutions for the hospitality.

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https://doi.org/10.4153/CMB-2004-050-3 Published online by Cambridge University Press