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Rotating periodic solutions for p-Laplacian differential systems

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In this paper, we study existence of rotating periodic solutions for p-Laplacian differential systems. We first build a new continuation theorem by topological degree, and then obtain the existence of rotating periodic solutions for two kinds of p-Laplacian differential systems via this continuation theorem, extend some existing relevant results.

Keywords: p-Laplacian differential systems; rotating periodic solution; continuation theorem; topological degree

2020 *Mathematics Subject Classification:* 34A34; 34C25

1. Introduction

In this paper, we are concerned with the existence of rotating periodic solutions for the following differential system with p-Laplacian operators:

$$-(\phi_p(u'))' = f(t, u(t), u'(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $\phi_p : \mathbb{R}^N \to \mathbb{R}^N$ defined by $\phi_p(x) = |x|^{p-2}x$ if $x \neq 0$, $\phi_p(0) = 0$, $p > 1$, $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory with $f(t+T, x, y) = Qf(t, Q^{-1}x, Q^{-1}y)$, $Q \in O(N)$. Here $O(N)$ denotes the orthogonal group on $\mathbb{R}^N$. Specially, Q may be an $N \times N$ orthogonal matrix.

We say $u(t)$ is a Q-rotating periodic solution of (1.1), if $u(t)$ satisfies (1.1) and $u(t+T) = Qu(t)$ for $t \in \mathbb{R}$. To this end, we first study the existence of solutions for the following p-Laplacian rotating periodic boundary value problem (RPBVP for short):

$$\begin{cases} -(\phi_p(u'))' = f(t, u(t), u'(t)), & 0 \leqslant t \leqslant T, \\ u(T) = Qu(0), & u'(T) = Qu'(0). \end{cases} \tag{H_Q}$$

If $u(t)$ is a solution of RPBVP (H_Q), then we can extend $u(t)$ from $[0, T]$ to $\mathbb{R}$ such that $u(t+T) = Qu(t)$ for $t \in \mathbb{R}$.

Indeed, suppose that $u(t)$ is a solution for RPBVP (H_Q). Let $u(t+T) = Qu(t)$, $t \in [0, T]$. So we have that

$$-\left(|u'(t)|^{p-2}u'(t)\right)' = \left(\left|Q^{-1}u'(t+T)\right|^{p-2}Q^{-1}u'(t+T)\right)'$$
$$= f(t, Q^{-1}u(t+T), Q^{-1}u'(t+T)), \quad t \in [0, T].$$

By $f(t+T, x, y) = Qf(t, Q^{-1}x, Q^{-1}y)$, we obtain

$$-\left(\left|Q^{-1}u'(t+T)\right|^{p-2}Q^{-1}u'(t+T)\right)' = Q^{-1}f(t+T, u(t+T), u'(t+T)), \quad t \in [0, T].$$

As $Q \in O(N)$, then $|Qu'(t)| = |u'(t)|$, furthermore, the above equation deduces to

$$-\left(|u'(t)|^{p-2}u'(t)\right)' = f(t, u(t), u'(t)), \quad t \in [T, 2T].$$

In this way, it is easy to claim that $u(t)$ satisfies (1.1) and $u(t+T) = Qu(t)$ for $t \in \mathbb{R}$.

Hence we may say that the solution $u(t)$ of RPBVP (H_Q) is Q-rotating periodic solution which satisfies $u(t+T) = Qu(t)$ for $t \in \mathbb{R}$. This kind of solutions may be periodic, anti-periodic, subharmonic, or quasi-periodic, if Q is identity matrix $I_{N\times N}$, negative identity matrix $-I_{N\times N}$, a power identity matrix, i.e., $Q^k = I$ for some $k \in \mathbb{N}$, $k \geqslant 2$, or an orthogonal matrix except for the previous cases, i.e., $Q \in O(N)$. So RPBVPs are more general than periodic boundary problems, subharmonic problems and so on.

In recent years, many scholars began to study the rotating periodic solutions for differential systems. In [**1**], Chang and Li proved the existence of rotating periodic solutions for a class of second-order dissipative dynamical system by using the coincidence degree theory. After that, they studied the existence of rotational periodic solutions for singular second-order dissipative dynamical system (see [**2**]). In [**3**], using the fountain theorem, Shen and Liu obtained infinitely many rotating periodic solutions for sup-linear second-order impulsive Hamiltonian system. In [**4**], Xing, Yang and Li built an averaging method for first-order perturbed affine-periodic system and studied the existence of affine-periodic solutions. For more results on rotating periodic solutions, please refer to [**1**–**7**] and references therein. However, it should be pointed out that there is no work on discussing the existence of rotating periodic solutions for p-Laplacian differential systems (1.1).

To our knowledge, p-Laplacian differential equations(systems) with Dirichelt or periodic boundary value conditions have been researched by many scholars. It is well known that Man*á*sevich and Mawhin [**8**] studied the existence of periodic solutions for p-Laplacian-Like systems via building continuation theorem. The nature question is whether new continuation theorem can be established for studying RPBVPs with p-Laplacian operator.

Inspired by [**8**] and above works, our paper aims to give a new continuation theorem for p-Laplacian differential systems with rotating periodic boundary conditions, which should give a criterion for proving the existence of rotating periodic solutions to such problems. The new continuation theorem generalizes and enriches the classical continuation theorem [**1**, **9**]. And then we apply this theorem to obtain

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some existence results for two kinds of p-Laplacian rotating periodic differential systems. Furthermore, if p and Q are special cases, problem (H_Q) is existing classical problem, for example, when $p > 1$ and $Q = I$, (H_Q) is same as [**10**]; when $p = 2$ and $Q \in O(N)$, (H_Q) is same as [**1**]; when $p = 2$ and $Q = I$, (H_Q) is general periodic problem [**9**, **11**]. So, our results extend some existing relevant results.

The paper is organized as follows: we present some preliminary concepts, a new Sobolev inequality and an important proposition in § 2. In § 3, we give a completely continuous operator. By the Leray-Schauder degree, a new continuation theorem will be proved in § 4. In § 5, using the new continuation theorem, we show the existence of rotating periodic solutions for two kinds of p-Laplacian differential systems.

2. Preliminaries

In this section, we present some preliminary concepts, a new Sobolev inequality and an important proposition.

For convenience, we first introduce some necessary basic knowledge and signs. Throughout the paper, $\langle a, b\rangle$ denotes the inner product for any $a, b \in \mathbb{R}^N$, while $|a|$ denotes the Euclidean norm for $a \in \mathbb{R}^N$. $Q \in O(N)$ and $O(N)$ denotes the orthogonal group on $\mathbb{R}^N$.

Set $C = C^0([0, T], \mathbb{R}^N)$ with the norm $\|u\|_0 = \max\limits_{0\leqslant t\leqslant 1} |u(t)|$, $C^m = C^m([0, T], \mathbb{R}^N)$ with the norm $\|u\|_m = \max\{\|u\|_0, \|u'\|_0, \cdots, \left\|u^{(m)}\right\|_0\}$, $L^p = L^p(0, T; \mathbb{R}^N)$ with the norm $\|u\|_{L^p} = (\int_0^T |u(t)|^p \,\mathrm{d}t)^{1/p}$.

Let $C_Q = \{u \in C : u(T) = Qu(0)\}$, $C^1_Q = \{u \in C^1 : u(T) = Qu(0), u'(T) = Qu'(0)\}$,

$X = \left\{u \in C^1_Q : \phi_p(u') \text{ is absolutely continuous}\right\}$ and $Y = L^1(0, T; \mathbb{R}^N)$.

The function $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Carathéodory, which satisfies

(1) the function $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N$ for a.e. $t \in [0, T]$;

(2) the function $f(\cdot, x, y)$ is measurable on $[0, T]$ for each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$;

(3) for each $r > 0$ there exists $a_r \in L^1((0, T); \mathbb{R})$ such that, for a.e. $t \in [0, T]$ and each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leqslant r$, $|y| \leqslant r$, one has

$$|f(t, x, y)| \leqslant a_r(t).$$

Let $Q \in O(N)$ and I be the identity operator. By the orthogonal decomposition theorem in linear algebra, we have

$$\mathbb{R}^N = \ker(\mathrm{I} - \mathrm{Q}) \oplus \mathrm{Im}(\mathrm{I} - \mathrm{Q}). \tag{2.1}$$

Define the orthogonal projector

$$\mathcal{P} : \mathbb{R}^N \to \ker(\mathrm{I} - \mathrm{Q}). \tag{2.2}$$

If ker (I-Q) $\neq \{0\}$ and $Q \neq$ I, define $L_P =$ (I - Q)$|_{\ker \mathcal{P}}$. Then L_P is a bijection from $\ker \mathcal{P}$ to Im(I - Q).

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If $Q = \mathrm{I}$, then let $\mathcal{P} = \mathrm{I}$ and $L_P = \mathrm{I}$.

Let $H^1_{T,Q} = \{u \in H^1 : u(T) = Qu(0)\} \subset H^1$ with the inner product

$$\langle u, v\rangle = \int_0^T \langle u(t), v(t)\rangle + \langle \dot{u}(t), \dot{v}(t)\rangle \, \mathrm{d}t,$$

and corresponding norm $\|u\|^2 = \langle u, u\rangle$, where $Q \in O(N)$. It is easy to show that $H^1_{T,Q}$ is a Hilbert space and the embedding $H^1_{T,Q} \hookrightarrow C$ is compact. Next we will check Wirtinger inequality and Sobolev inequality still hold on $H^1_{T,Q}$.

THEOREM 2.1. *If $u \in H^1_{T,Q}$ and $\int_0^T u(t)\,\mathrm{d}t \in \mathrm{Im}(I - Q)$, then there exist constants $\lambda_1 > 0$, $c_1 > 0$ such that*

$$\int_0^T |u(t)|^2 \,\mathrm{d}t \leqslant \lambda_1 \int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t,$$

(Wirtinger inequality) and

$$\|u\|_0 \leqslant c_1 \|\dot{u}\|_{L^2},$$

(Sobolev inequality).

REMARK 2.2. If $I = Q$, then the theorem 2.1 is the same as the classical result of the periodic case (see [**12**]).

In order to prove theorem 2.1, we first prove the following lemma.

LEMMA 2.3. *Define the functional $J : H^1_{T,Q} \to \mathbb{R}$ by*

$$J(u) = \int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t.$$

Then $c_2 = \min_{u\in E} J(u) > 0$, where

$$E = \left\{ u \in H^1_{T,Q} : \int_0^T |u|^2 \,\mathrm{d}t = 1, \int_0^T u(t)\,\mathrm{d}t \in \mathrm{Im}(I - Q) \right\}.$$

Proof. Let $c_2 = \inf_{u\in E} J(u)$. Obviously, J is coercive on E. Then there exists bounded sequence $\{u_n\} \in E$ such that $J(u_n) \to c_2$. Because $H^1_{T,\,Q}$ is a Hilbert space, there is a subsequence of $\{u_n\}$, which we rename the same, which satisfies $u_n \rightharpoonup u (n \to \infty)$. The set E is weakly sequentially closed, as follows easily from the compact embedding of $H^1_{T,Q}$ in C. Then $u \in E$. Because J is continuous and convex on E, then J is weakly lower semi-continuous on E. It follows that

$$c_2 = \underline{\lim} J(u_n) \geqslant J(u) \geqslant 0.$$

Then we have that J gets the minimum value c_2 at u, i.e., $J(u) = c_2$. If $c_2 = 0$, then $u = a \in \mathbb{R}^N$ with $(I - Q)a = 0$. However $\int_0^T u(t)\,\mathrm{d}t = Ta \in \mathrm{Im(I - Q)}$ from

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the definition of E. Then by (2.1), $a = 0$, which contradicts $\int_0^T |u|^2\,\mathrm{d}t = 1$. Thus, $c_2 > 0$. $\square$

Proof of theorem 2.1. Suppose that $u \in H^1_{T,Q}$ with $\int_0^T u(t)\,\mathrm{d}t \in \mathrm{Im}(\mathrm{I}-\mathrm{Q})$. If $\int_0^T |u(t)|^2\,\mathrm{d}t = 0$, then the result is obviously true. Assume $\int_0^T |u(t)|^2\,\mathrm{d}t \neq 0$. Let

$$v = \frac{u}{\left(\int_0^T |u(t)|^2\,\mathrm{d}t\right)^{1/2}}.$$

Then $v \in E$. By lemma 2.3, we have

$$\int_0^T |\dot{v}(t)|^2\,\mathrm{d}t \geqslant c_2,$$

and hence

$$\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t \geqslant c_2 \int_0^T |u(t)|^2\,\mathrm{d}t.$$

Taking $\lambda_1 = \frac{1}{c_2}$, Wirtinger inequality holds. Because $H^1_{T,Q} \hookrightarrow C$, there exists $c > 0$ such that $\|u\|_0 \leqslant c\|u\|$. Then we obtain Sobolev inequality $\|u\|_0 \leqslant c_1\|\dot{u}\|_{L^2}$. $\square$

Lemma 2.4. *Suppose $Q \in O(N)$. Then*

(a) $\phi_p(Q\alpha) = Q\phi_p(\alpha)$, *for any* $\alpha \in \mathbb{R}^N$;

(b) $\langle \phi_p(\alpha) - \phi_p(\beta), \alpha - \beta\rangle > 0$, *for any* $\alpha, \beta \in \mathbb{R}^N$, $\alpha \neq \beta$.

Proof. (i) As $Q \in O(N)$, for $\alpha \in \mathbb{R}^N$, one has $|Q\alpha| = |\alpha|$, and

$$\phi_p(Q\alpha) = |Q\alpha|^{p-2}(Q\alpha) = |\alpha|^{p-2}(Q\alpha) = Q|\alpha|^{p-2}\alpha = Q\phi_p(\alpha).$$

(ii) It can be checked by simple calculation. $\square$

Lemma 2.5. *Assume $u(t) \in X$. If $u'(t) = \alpha(\alpha \in \mathbb{R}^N)$, then $\alpha = 0$ and $u(t) = \beta \in$ ker (I-Q).*

Proof. If $u'(t) = \alpha$, then $u(t) = \alpha t + \beta(\beta \in \mathbb{R}^N)$. By $u(T) = Qu(0)$, we have $T\alpha + \beta = Q\beta$ which implies $\alpha \in$ Im(I-Q). From $u'(T) = Qu'(0)$ it follows that $\alpha \in$ ker (I-Q). Hence $\alpha = 0$ and $\beta \in$ ker (I-Q) via (2.1). $\square$

Assume that ker (I-Q) $\neq \{0\}$. Consider the simple rotating periodic boundary value problem

$$-(\phi_p(u'))' = f(t), \tag{2.3}$$
$$u(T) = Qu(0), u'(T) = Qu'(0), \tag{2.4}$$

where $f(t) \in Y$ satisfying $f(t + T) = Qf(t)$.

Suppose that $u(t) \in X$ is a solution to (2.3) (2.4). By integrating (2.3) over $[0, T]$, we have that

$$\phi_p(u'(0)) - \phi_p(u'(T)) = \int_0^T (-(\phi_p(u'))')\,\mathrm{d}t = \int_0^T f(t)\,\mathrm{d}t.$$

Using (2.4) and lemma 2.4(i), we get

$$(\mathrm{I} - \mathrm{Q})(\phi_p(u'(0))) = \int_0^T f(t)\,\mathrm{d}t. \tag{2.5}$$

So

$$\mathcal{P}((\mathrm{I} - \mathrm{Q})(\phi_p(u'(0)))) = \mathcal{P}\left(\int_0^T f(t)\,\mathrm{d}t\right),$$

which yields

$$\mathcal{P}\left(\int_0^T f(t)\,\mathrm{d}t\right) = 0. \tag{2.6}$$

On the other hand, since

$$\phi_p(u'(0)) = \mathcal{P}(\phi_p(u'(0))) + (\mathrm{I} - \mathcal{P})(\phi_p(u'(0))),$$

and

$$\int_0^T f(t)\,\mathrm{d}t = \mathcal{P}\left(\int_0^T f(t)\,\mathrm{d}t\right) + (\mathrm{I} - \mathcal{P})\left(\int_0^T f(t)\,\mathrm{d}t\right),$$

then

$$\begin{aligned}(\mathrm{I} - \mathrm{Q})&(\mathcal{P}(\phi_p(u'(0))) + (\mathrm{I} - \mathcal{P})(\phi_p(u'(0))))\\ &= \mathcal{P}\left(\int_0^T f(t)\,\mathrm{d}t\right) + (\mathrm{I} - \mathcal{P})\left(\int_0^T f(t)\,\mathrm{d}t\right).\end{aligned} \tag{2.7}$$

From (2.6),(2.7) and the definition of $\mathcal{P}$, it follows that

$$(\mathrm{I} - \mathrm{Q})\mathcal{P}(\phi_p(u'(0))) = 0 = \mathcal{P}\left(\int_0^T f(t)\,\mathrm{d}t\right),$$

and

$$(\mathrm{I} - \mathrm{Q})(\mathrm{I} - \mathcal{P})(\phi_p(u'(0))) = (\mathrm{I} - \mathcal{P})\left(\int_0^T f(t)\,\mathrm{d}t\right).$$

Taking L_P to act on above equation, one has

$$(\mathrm{I} - \mathcal{P})(\phi_p(u'(0))) = L_P^{-1}(\mathrm{I} - \mathcal{P})\left(\int_0^T f(t)\,\mathrm{d}t\right) = L_P^{-1}\left(\int_0^T f(t)\,\mathrm{d}t\right). \tag{2.8}$$

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Integrating (2.3) and combining (2.8), we have

$$\phi_p(u'(t)) = -\int_0^t f(s)\,\mathrm{d}s + \mathcal{P}(\phi_p(u'(0))) + L_P^{-1}\left(\int_0^T f(s)\,\mathrm{d}s\right),$$

i.e.,

$$u'(t) = \phi_q\left(-\int_0^t f(s)\,\mathrm{d}s + \mathcal{P}(\phi_p(u'(0))) + L_P^{-1}\left(\int_0^T f(s)\,\mathrm{d}s\right)\right) \triangleq a(t), \qquad (2.9)$$

where $\frac{1}{p} + \frac{1}{q} = 1 (p,\, q > 1)$. Integrating (2.9) over $[0,\, T]$, we obtain

$$u(0) - u(T) = \int_0^T (-a(t))\;\mathrm{d}t.$$

Similar to the previous discussion, we have

$$\mathcal{P}\left(\int_0^T (-a(t))\;\mathrm{d}t\right) = 0,$$

i.e.,

$$\mathcal{P}\left(\int_0^T -\phi_q\left(-\int_0^t f(s)\,\mathrm{d}s + \mathcal{P}\phi_p(u'(0)) + L_P^{-1}\int_0^T f(s)\,\mathrm{d}s\right)\;\mathrm{d}t\right) = 0. \qquad (2.10)$$

Following (2.5)–(2.8), and

$$(\mathrm{I}-\mathrm{Q})(\mathrm{I}-\mathcal{P})(u(0)) = (\mathrm{I}-\mathcal{P})\left(\int_0^T (-a(t))\;\mathrm{d}t\right) = \int_0^T (-a(t))\;\mathrm{d}t,$$

we get

$$(\mathrm{I}-\mathcal{P})(u(0)) = L_P^{-1}\left(\int_0^T (-a(t))\;\mathrm{d}t\right). \qquad (2.11)$$

Integrating (2.9), we obtain

$$u(t) = \mathcal{P}(u(0)) + L_P^{-1}\left(\int_0^T (-a(s))\;\mathrm{d}s\right) + \int_0^t a(s)\,\mathrm{d}s. \qquad (2.12)$$

On the basis of (2.10), we define mapping $G_h : \ker(\mathrm{I}-\mathrm{Q}) \to \ker(\mathrm{I}-\mathrm{Q})$ by

$$G_h(\gamma) = \mathcal{P}\int_0^T \phi_q\left(-\int_0^t h(s)\,\mathrm{d}s + \gamma + L_P^{-1}\int_0^T h(s)\,\mathrm{d}s\right)\;\mathrm{d}t, \quad \gamma \in \ker(\mathrm{I}-\mathrm{Q}), \qquad (2.13)$$

where $h \in Y_1 = \left\{h \in Y \,\middle|\, \int_0^T h(s)\,\mathrm{d}s \in \mathrm{Im}(\mathrm{I}\text{ - }\mathrm{Q})\right\}$. Next we discuss the properties of G_h.

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PROPOSITION 2.6. *The mapping G_h has the following properties*:

(a) *For any given $h \in Y_1$, the equation*

$$G_h(\gamma) = 0, \tag{2.14}$$

has a unique solution $\tilde{\gamma}(h) \in \ker(\text{I-Q})$.

(b) *The functional*

$$\tilde{\gamma} : Y_1 \to \ker(\mathrm{I} - \mathrm{Q}),$$

is continuous and sends bounded set into bounded set.

Proof. (i) Because $\mathcal{P} : \mathbb{R}^N \to \ker(\text{I - Q})$ is the orthogonal projector, then for any $\gamma_1,\ \gamma_2 \in \ker(\text{I-Q})$, we have

$$\left\langle (\mathrm{I} - \mathcal{P}) \left(\int_0^T \phi_q \left(-\int_0^t h(s)\,\mathrm{d}s + \gamma_1 + L_P^{-1} \int_0^T h(s)\,\mathrm{d}s \right) \mathrm{d}t \right), \gamma_2 \right\rangle = 0.$$

Let

$$\begin{aligned} K_h(\gamma) &= \int_0^T \phi_q \left(-\int_0^t h(s)\,\mathrm{d}s + \gamma + L_P^{-1} \int_0^T h(s)\,\mathrm{d}s \right) \mathrm{d}t \\ &= (\mathrm{I} - \mathcal{P}) \left(\int_0^T \phi_q \left(-\int_0^t h(s)\,\mathrm{d}s + \gamma + L_P^{-1} \int_0^T h(s)\,\mathrm{d}s \right) \mathrm{d}t \right) \\ &\quad + \mathcal{P} \left(\int_0^T \phi_q \left(-\int_0^t h(s)\,\mathrm{d}s + \gamma + L_P^{-1} \int_0^T h(s)\,\mathrm{d}s \right) \mathrm{d}t \right). \end{aligned}$$

From the definition of G_h and $\mathcal{P}$, it follows that

$$\langle G_h(\gamma_1), \gamma_2 \rangle = \langle K_h(\gamma_1), \gamma_2 \rangle. \tag{2.15}$$

According to lemma 2.4 (ii), for $\gamma_1 \neq \gamma_2$, we have

$$\begin{aligned} &\langle K_h(\gamma_1) - K_h(\gamma_2), \gamma_1 - \gamma_2 \rangle \\ &\quad = \int_0^T \langle \phi_q(\gamma_1 + l_h(t)) - \phi_q(\gamma_2 + l_h(t)), \gamma_1 - \gamma_2 \rangle\,\mathrm{d}t > 0, \end{aligned}$$

where $l_h(t) = -\int_0^t h(s)\,\mathrm{d}s + L_P^{-1} \int_0^T h(s)\,\mathrm{d}s \in C$. Combining (2.15), we obtain

$$\langle G_h(\gamma_1) - G_h(\gamma_2), \gamma_1 - \gamma_2 \rangle = \langle K_h(\gamma_1) - K_h(\gamma_2), \gamma_1 - \gamma_2 \rangle > 0. \tag{2.16}$$

And hence, if (2.14) has a solution then it is unique. □

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To prove existence of solutions, we will show that $\langle G_h(\gamma), \gamma\rangle > 0$ for $|\gamma|$ sufficiently large. Indeed, we have

$$\begin{aligned}\langle G_h(\gamma), \gamma\rangle &= \langle K_h(\gamma), \gamma\rangle \\ &= \int_0^T \langle \phi_q(\gamma + l_h(t)), \gamma\rangle \,\mathrm{d}t \\ &= \int_0^T \langle \phi_q(\gamma + l_h(t)), \gamma + l_h(t)\rangle \,\mathrm{d}t - \int_0^T \langle \phi_q(\gamma + l_h(t)), l_h(t)\rangle \,\mathrm{d}t.\end{aligned}$$

So

$$\langle G_h(\gamma), \gamma\rangle \geqslant \int_0^T \langle \phi_q(\gamma + l_h(t)), \gamma + l_h(t)\rangle \,\mathrm{d}t - \|l_h\|_0 \int_0^T |\phi_q(\gamma + l_h(t))| \,\mathrm{d}t. \tag{2.17}$$

Due to the definition of ϕ_q, we have

$$\langle G_h(\gamma), \gamma\rangle \geqslant \int_0^T (|(\gamma + l_h(t))| - \|l_h\|_0) \, |(\gamma + l_h(t))|^{q-1} \,\mathrm{d}t. \tag{2.18}$$

Since $q > 1$ and $|\gamma + l_h(t)| \to \infty$ as $|\gamma| \to \infty$, there exists $r > 0$ such that

$$\langle G_h(\gamma), \gamma\rangle > 0 \text{ for all} \gamma \in \ker(\mathrm{I} - \mathrm{Q}) \text{ with } |\gamma| \geqslant r. \tag{2.19}$$

It follows from the properties of topological degree that the equation $G_h(\gamma) = 0$ has a solution for each $h \in Y_1$, which is unique by our previous argument.

(ii) From (i), we can define a functional $\tilde{\gamma} : Y_1 \to \ker(\text{I-Q})$ which satisfies

$$G_h(\tilde{\gamma}) = \mathcal{P}\left(\int_0^T \phi_q\left(-\int_0^t h(s)\,\mathrm{d}s + \tilde{\gamma}(h) + L_P^{-1}\int_0^T h(s)\,\mathrm{d}s\right)\mathrm{d}t\right) = 0, \tag{2.20}$$

for any $h \in Y_1$. Hence,

$$0 = \langle G_h(\tilde{\gamma}), \tilde{\gamma}\rangle = \langle K_h(\tilde{\gamma}), \tilde{\gamma}\rangle.$$

Then

$$\int_0^T \langle \phi_q(\tilde{\gamma} + l_h(t)), \tilde{\gamma} + l_h(t)\rangle \,\mathrm{d}t = \int_0^T \langle \phi_q(\tilde{\gamma} + l_h(t)), l_h(t)\rangle \,\mathrm{d}t. \tag{2.21}$$

Let $\Omega \subset Y_1$ be a bounded subset. Then there is $M_1 > 0$ such that $\|h\|_{L^1} \leqslant M_1$ and $\left|\int_0^t h(s)\,\mathrm{d}s\right| \leqslant M_1$, for any $h \in \Omega$. Due to the definition of L_P^{-1} and $l_h(t)$, there exists a constant $M_2 > 0$ such that $\left|L_P^{-1}\int_0^T h(s)\,\mathrm{d}s\right| \leqslant M_2$, and

$$|l_h(t)| \leqslant \left|\int_0^t h(s)\,\mathrm{d}s\right| + \left|L_P^{-1}\int_0^T h(s)\,\mathrm{d}s\right| \leqslant M_1 + M_2,$$

that is, $\|l_h\|_0 \leqslant \sqrt{N}(M_1 + M_2)$, for any $h \in \Omega$.

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Now we show $|\tilde{\gamma}(h)|$ is bounded on Ω. Assume on contrary that $\{\tilde{\gamma}(h) : h \in \Omega\}$ is not bounded. Then for any given $M_3 > \sqrt{N}(M_1 + M_2)$, there is $h \in \Omega$ such that

$$M_3 \leqslant |\tilde{\gamma}(l_h) + l_h(t)|, \quad t \in [0, T].$$

Hence by (2.21), we find that

$$\begin{aligned} M_3 \int_0^T |\tilde{\gamma}(l_h) + l_h(t)|^{q-1}\,\mathrm{d}t &\leqslant \int_0^T |\tilde{\gamma}(l_h) + l_h(t)|^q\,\mathrm{d}t \\ &= \int_0^T \langle \phi_q(\tilde{\gamma} + l_h(t)), \tilde{\gamma} + l_h(t) \rangle \,\mathrm{d}t \\ &= \int_0^T \langle \phi_q(\tilde{\gamma} + l_h(t)), l_h(t) \rangle \,\mathrm{d}t \\ &\leqslant \|l_h\|_0 \int_0^T |\tilde{\gamma}(l_h) + l_h(t)|^{q-1}\,\mathrm{d}t. \end{aligned}$$

Thus $M_3 \leqslant \|l_h\|_0$, a contradiction. Therefore $\tilde{\gamma}$ sends bounded set in Y_1 into bounded set in ker (I-Q).

Finally we show the continuity of $\tilde{\gamma}$. Let $\{h_n\}$ be a convergent sequence in Y_1, i.e., $h_n \to h$, as $n \to \infty$. It is easy to show that $l_{h_n} \to l_h$ in $C[0, T]$ as $n \to \infty$. Since $\{\tilde{\gamma}(l_{h_n})\}$ is bounded sequence, there exists a subsequence $\{\tilde{\gamma}(l_{h_j})\}$ such that $\tilde{\gamma}(l_{h_j}) \to \hat{\gamma}(j \to \infty)$. Letting $j \to \infty$ in

$$\mathcal{P}\left(\int_0^T \phi_q\left(\tilde{\gamma}(l_{h_j}) + l_{h_j}(t)\right)\,\mathrm{d}t\right) = 0,$$

we find that

$$\mathcal{P}\left(\int_0^T \phi_q\left(\hat{\gamma} + l_h(t)\right)\,\mathrm{d}t\right) = 0,$$

and $\tilde{\gamma}(l_{h_j}) = \hat{\gamma}$ from the definition of $\hat{\gamma}$, which show the continuity of $\tilde{\gamma}$.

Define the projectors $\hat{\mathcal{P}} : X \to X$ and $\hat{\mathcal{Q}} : Y \to Y$ respectively by

$$\begin{aligned} \hat{\mathcal{P}}(u) &= \mathcal{P}(u(0)), \\ \hat{\mathcal{Q}}(f) &= \mathcal{P}\left(\frac{1}{T}\int_0^T f(t)\,\mathrm{d}t\right). \end{aligned}$$

For $h \in Y$, let $\gamma : Y \to \ker(\text{I-Q})$ be defined by

$$\gamma(h) = \tilde{\gamma}((\mathrm{I} - \hat{\mathcal{Q}})h). \tag{2.22}$$

Then, it is clear that γ is a continuous function and sends bounded set into bounded set. Noting that $\dim\ker(\text{I-Q}) < \infty$, so γ is a completely continuous mapping.

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3. An equivalent operator equation

In this section, we give an equivalent operator equation with RPBVP (H_Q).

Firstly, set Nemytski operator $N_f : X \to Y$ by

$$N_f u = f(t, u(t), u'(t)), \tag{3.1}$$

where $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory with $f(t+T, x, y) = Qf(t, Q^{-1}x, Q^{-1}y)$.

Next, we define the operator H on X by

$$(Hu)(t) = \hat{\mathcal{P}}(u) + \hat{\mathcal{Q}}(N_f u) - L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right) + \int_0^t c(s)\,\mathrm{d}s, \tag{3.2}$$

where

$$c(s) = \phi_q\left(-\int_0^s (\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t \right.$$
$$\left. + L_P^{-1}\int_0^T (\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t + \tilde{\gamma}\left((\mathrm{I} - \hat{\mathcal{Q}})(N_f u)\right)\right),$$

$\tilde{\gamma}((\mathrm{I} - \hat{\mathcal{Q}})(N_f u))$ is defined in (2.22) and $\frac{1}{p} + \frac{1}{q} = 1 (p,\, q > 1)$.

By (2.20) and the definition of $\tilde{\gamma}$, we have $\mathcal{P}(\int_0^T c(s)\,\mathrm{d}s) = G_{(\mathrm{I} - \hat{\mathcal{Q}})(N_f u)}(\tilde{\gamma}) = 0$. So the definition of H is fine.

Lemma 3.1. *The mapping H is a continuous operator from X to X.*

Proof. Obviously, H is continuous in C from the continuity of $\hat{\mathcal{P}}$, $\hat{\mathcal{Q}}$, N_f and L_P^{-1}. Writing $H(t) \triangleq (H(u))(t)$ and $\tilde{\gamma} = \tilde{\gamma}((\mathrm{I} - \hat{\mathcal{Q}})(N_f u))$, we have that

$$H'(t) = \phi_q\left(-\int_0^t (\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(s)\,\mathrm{d}s + L_p^{-1}\int_0^T (\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t + \tilde{\gamma}\right),$$

and

$$\phi_p\left(H'(t)\right) = -\int_0^t (\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(s)\,\mathrm{d}s + L_p^{-1}\int_0^T (\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t + \tilde{\gamma}, \tag{3.3}$$

which show $H \in C^1$ and $\phi_p(H'(t))$ is absolutely continuous, where $\frac{1}{p} + \frac{1}{q} = 1 (p,\, q > 1)$. □

Next, we will prove $H(T) = QH(0)$ and $H'(T) = QH'(0)$.

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By (3.2), we have

$$H(T)=\hat{\mathcal{P}}(u)+\hat{\mathcal{Q}}(N_fu)-L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right)+\int_0^T c(s)\,\mathrm{d}s, \tag{3.4}$$

and

$$H(0)=\hat{\mathcal{P}}(u)+\hat{\mathcal{Q}}(N_fu)-L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right).$$

Then

$$QH(0)=Q\left(\hat{\mathcal{P}}(u)+\hat{\mathcal{Q}}(N_fu)-L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right)\right). \tag{3.5}$$

From the definition of $\tilde{\gamma}$, we find that

$$\begin{aligned}&\mathcal{P}\left(\int_0^T c(s)\,\mathrm{d}s\right)\\ &\quad=\mathcal{P}\left(\int_0^T\phi_q\left(-\int_0^s(\mathrm{I}-\hat{\mathcal{Q}})(N_fu)(t)\,\mathrm{d}t+L_P^{-1}\int_0^T(\mathrm{I}-\hat{\mathcal{Q}})(N_fu)(t)\,\mathrm{d}t+\tilde{\gamma}\right)\mathrm{d}s\right)\\ &\quad=0.\end{aligned}$$

The definitions of $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ yield that

$$(\mathrm{I}-\mathrm{Q})\hat{\mathcal{P}}(u)=0,\quad (\mathrm{I}-\mathrm{Q})\hat{\mathcal{Q}}(N_fu)=0,$$

i.e.,

$$\hat{\mathcal{P}}(u)=Q(\hat{\mathcal{P}}(u)),\quad \hat{\mathcal{Q}}(N_fu)=Q(\hat{\mathcal{Q}}(N_fu)). \tag{3.6}$$

According to the definition of L_P^{-1}, we get that

$$\begin{aligned}QL_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right)&=-(\mathrm{I}-\mathrm{Q})L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right)+L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right)\\ &=-\int_0^T c(s)\,\mathrm{d}s+L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right).\end{aligned} \tag{3.7}$$

Substituting (3.6)–(3.7) into (3.5), we obtain

$$QH(0)=H(T).$$

On the other hand, we see from (3.3) that

$$\phi_p\left(H'(T)\right)=-\int_0^T(\mathrm{I}-\hat{\mathcal{Q}})(N_fu)(t)\,\mathrm{d}t+L_P^{-1}\int_0^T(\mathrm{I}-\hat{\mathcal{Q}})(N_fu)(t)\,\mathrm{d}t+\tilde{\gamma},$$

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and

$$\phi_p\left(H'(0)\right) = L_P^{-1}\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t + \tilde{\gamma}.$$

Because $\tilde{\gamma} \in \ker(\text{I-Q})$, i.e., $Q\tilde{\gamma} = \tilde{\gamma}$, then we have that

$$\begin{aligned} Q\phi_p\left(H'(0)\right) &= QL_P^{-1}\left(\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t\right) + Q\tilde{\gamma} \\ &= -(\mathrm{I}-\mathrm{Q})L_P^{-1}\left(\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t\right) \\ &\quad + L_P^{-1}\left(\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t\right) + \tilde{\gamma} \\ &= -\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t + L_P^{-1}\left(\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t\right) + \tilde{\gamma}. \end{aligned}$$

It follows from lemma 2.4 that

$$\phi_p\left(H'(T)\right) = Q\phi_p\left(H'(0)\right) = \phi_p\left(QH'(0)\right),$$

which yields that $H'(T) = QH'(0)$. Hence the H is a continuous operator from X to X.

Lemma 3.2. *The mapping H is a completely continuous operator on X.*

Proof. We only need to prove that H is a compact operator. Let $S \subset X$ be an open bounded subset such that $\|u\|_1 \leqslant M_1$ for any $u \in S$. It is easy to see that N_f is continuous and sends bounded set into equi-integrable set. Then there exists $l(t) \in L^1((0,T);\mathbb{R})$ such that $|N_f(u)| \leqslant l(t)$ for any $u \in S$. Taking $\{u_n\} \subset S$, we have

$$(Hu_n)(t) = \hat{\mathcal{P}}(u_n) + \hat{\mathcal{Q}}(N_f u_n) - L_P^{-1}\left(\int_0^T c_n(s)\,\mathrm{d}s\right) + \int_0^t c_n(s)\,\mathrm{d}s, \tag{3.8}$$

where

$$c_n(s) = \phi_q\left(-\int_0^s (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t + L_p^{-1}\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t + \tilde{\gamma}\right),$$

and $\tilde{\gamma} = \tilde{\gamma}((\text{I -}\hat{\mathcal{Q}})(N_f u_n))$ is defined in (2.22). Obviously, $\left|\hat{\mathcal{P}}(u_n)\right| \leqslant M_1$ and

$$\left|\hat{\mathcal{Q}}(N_f u_n)\right| \leqslant \left|\frac{1}{T}\int_0^T (N_f u_n)(\tau)\,\mathrm{d}\tau\right| \leqslant \frac{1}{T}\int_0^T l(\tau)\,\mathrm{d}\tau = \frac{1}{T}M_2.$$

Then

$$\left|\int_0^s (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t\right| \\ \leqslant \int_0^T |(N_f u_n)(t)|\ \mathrm{d}t + \int_0^T \left|\hat{\mathcal{Q}}(N_f u_n)(t)\right|\ \mathrm{d}t \leqslant 2\int_0^T l(\tau)\,\mathrm{d}\tau = 2M_2,$$

uniformly in $s \in [0, T]$. By the continuity of the L_P^{-1} and $\tilde{\gamma}$, there exists $M_3 > 0$ such that

$$\left|L_P^{-1}\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t\right| \leqslant M_3,$$

and

$$|\tilde{\gamma}| \leqslant M_3.$$

Hence we have that

$$\begin{aligned}|c_n(s)| &= \left|-\int_0^s (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t + L_P^{-1}\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t + \tilde{\gamma}\right|^{q-1} \\ &\leqslant \left|\left|\int_0^s (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t\right| + \left|L_P^{-1}\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t\right| + |\tilde{\gamma}|\right|^{q-1} \\ &\leqslant (2(M_2+M_3))^{q-1},\end{aligned}$$

uniformly in $s \in [0, T]$. Therefore, we obtain that there is a constant $M_4 > 0$ such that

$$|(Hu_n)(t)| \leqslant M_4, \quad \forall\ u_n \in S,$$

uniformly in $t \in [0, T]$, which shows that $\{H(u_n)\}$ is uniformly bounded in C. Since $(Hu_n)'(t) = c_n(t)$, then $\{H(u_n)'(t)\}$ is uniformly bounded in C. Hence, $\{H(u_n)\}$ is equi-continuous. According to the Arzelà-Ascoli theorem, $\{H(u_n)\}$ is sequentially compact.

For any $u \in S$ and $s_1, s_2 \in [0, T]$, we have

$$\begin{aligned}|w(s_2)-w(s_1)| &\leqslant \left|\int_{s_1}^{s_2} (N_f u)(t)\,\mathrm{d}t\right| + \left|\hat{\mathcal{Q}}(N_f u)\right| |s_2-s_1| \\ &\leqslant \int_{s_1}^{s_2} l(\tau)\,\mathrm{d}\tau + |s_2-s_1| \int_0^T l(\tau)\,\mathrm{d}\tau,\end{aligned}$$

where $w(s) = \int_0^s (\mathrm{I} - \widehat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t$.

Taking sequence $\{u_n\} \subset S$, then $\left\{-\int_0^s (\mathrm{I} - \hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t\right\}$ is uniformly bounded and equi-continuous. By Arzelà-Ascoli theorem there is a subsequence of

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$\left\{-\int_0^s (\mathrm{I}\text{ -}\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t\right\}$, which we rename the same, which is convergent in C. Then, passing to a subsequence if necessary, we obtain that the sequence

$$\left\{-\int_0^s (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t + L_p^{-1}\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u_n)(t)\,\mathrm{d}t + \tilde{\gamma}\right\},$$

is convergent in C. Using that $\phi_q : C \to C$ is continuous it follows that $\{c_n\}$ is convergent in C. Hence the mapping H is a completely continuous operator. □

LEMMA 3.3. *The fixed point of operator H is equivalent to the solution of RPBVP* (H_Q)*.*

Proof. Assume that $u \in X$ is a fixed point of H: $H(u) = u$, i.e.,

$$u(t) = \hat{\mathcal{P}}(u) + \hat{\mathcal{Q}}(N_f u) - L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right) + \int_0^t c(s)\,\mathrm{d}s), \tag{3.9}$$

where

$$c(s) = \phi_q\Bigg(- \int_0^s (\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t \\ + L_P^{-1}\int_0^T (\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t + \tilde{\gamma}\left((\mathrm{I}-\hat{\mathcal{Q}})(N_f u)\right)\Bigg),$$

and $\tilde{\gamma}((\mathrm{I}\text{ -}\hat{\mathcal{Q}})(N_f u))$ as (2.22). Hence

$$u(0) = \hat{\mathcal{P}}(u) + \hat{\mathcal{Q}}(N_f u) - L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right).$$

From the definitions of $\hat{\mathcal{P}}$, $\hat{\mathcal{Q}}$ and L_P^{-1}, it follows that

$$\begin{aligned}\mathcal{P}(u(0)) &= \mathcal{P}\left(\mathcal{P}u(0) + \frac{1}{T}\mathcal{P}\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau - L_P^{-1}\left(\int_0^T c(s)\,\mathrm{d}s\right)\right) \\ &= \mathcal{P}(u(0)) + \frac{1}{T}\mathcal{P}\left(\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau\right),\end{aligned}$$

which yields

$$\frac{1}{T}\mathcal{P}\left(\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau\right) = 0. \tag{3.10}$$

By (3.9), (3.10) and the definition of $\hat{\mathcal{Q}}$, we have

$$(\phi_p(u'(t)))' = -(\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t) = -f(t, u(t), u'(t)) + \hat{\mathcal{Q}}(N_f u) = -f(t, u(t), u'(t)).$$

Noting the definition of X, we know that u is a solution of (H_Q). □

On the other hand, assume that $u \in X$ is a solution of RPBVP (H_Q), i.e.,

$$\begin{cases} -(\phi_p(u'))' = f(t, u, u'), \\ u(T) = Qu(0), u'(T) = Qu'(0). \end{cases}$$

Similar to the previous discussion, we obtain that

$$\mathcal{P} \int_0^T (N_f(u))(t)\, \mathrm{d}t = 0, \tag{3.11}$$

$$\mathcal{P} \left(\int_0^T \phi_q \left(- \int_0^t (N_f(u))(s)\, \mathrm{d}s + \mathcal{P}\phi_p(u'(0)) + L_P^{-1} \int_0^T (N_f(u))(s)\, \mathrm{d}s \right) \mathrm{d}t \right) = 0, \tag{3.12}$$

and

$$u(t) = \hat{\mathcal{P}}(u) - L_P^{-1} \left(\int_0^T a(s)\, \mathrm{d}s \right) + \int_0^t a(s)\, \mathrm{d}s, \tag{3.13}$$

where

$$a(t) = \phi_q \left(- \int_0^t (N_f(u))(s)\, \mathrm{d}s + \mathcal{P}\phi_p(u'(0)) + L_P^{-1} \int_0^T (N_f(u))(s)\, \mathrm{d}s \right).$$

Due to (3.11), we have

$$(\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t) = (N_f u)(t).$$

According to the definition of $\tilde{\gamma}$ and (3.12), we get that

$$\mathcal{P}\phi_p(u'(0)) = \tilde{\gamma} \left((\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t) \right).$$

From (3.13), it follows that

$$u(t) = \hat{\mathcal{P}}(u) + \hat{\mathcal{Q}}(N_f u) - L_P^{-1} \left(\int_0^T c(s)\, \mathrm{d}s \right) + \int_0^t c(s)\, \mathrm{d}s,$$

where

$$c(s) = \phi_q \left(- \int_0^s (\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t)\, \mathrm{d}t \right.$$
$$\left. + L_P^{-1} \int_0^T (\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t)\, \mathrm{d}t + \tilde{\gamma} \left((\mathrm{I} - \hat{\mathcal{Q}})(N_f u)(t) \right) \right).$$

Hence, we obtain

$$u = H(u),$$

i.e., u is a fixed point of operator H.

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4. A new continuation theorem

In this section, we build a new continuation theorem for studying the existence of solutions of RPBVP (H_Q).

THEOREM 4.1. *Suppose that* Ω *is an open bounded set in* X *such that the following conditions hold.*

(a) *For* $\forall\ \lambda \in (0,\ 1)$, *the problem*

$$\begin{cases} -(\phi_p(u'))' = \lambda f(t, u, u'), \\ u(T) = Qu(0), u'(T) = Qu'(0), \end{cases} \tag{4.1}$$

has no solution on $\partial\Omega$.

(b) *Assume that* $\ker(\mathrm{I} - \mathrm{Q}) \neq \{0\}$, *the equation*

$$F(a) := \frac{1}{T}\mathcal{P}\left(\int_0^T f(t, a, 0)\, \mathrm{d}t\right) = 0, \tag{4.2}$$

has no solution on $\partial\Omega \cap \ker(\mathrm{I} - \mathrm{Q})$, *and the Brouwer degree*

$$\deg_B(F, \Omega \cap \ker(\mathrm{I} - \mathrm{Q}), 0) \neq 0, \tag{4.3}$$

where the orthogonal projector $\mathcal{P} : \mathbb{R}^N \to \ker(\mathrm{I} - \mathrm{Q})$.

Then RPBVP (H_Q) has at least one solution in $\bar{\Omega}$.

Proof. Consider the following homotopy boundary value problem with (H_Q)

$$\begin{cases} -(\phi_p(u'))' = \lambda(N_f u)(t) + (1-\lambda)\frac{1}{T}\mathcal{P}\int_0^T (N_f u)(\tau)\, \mathrm{d}\tau, \\ u(T) = Qu(0), u'(T) = Qu'(0), \end{cases} \tag{4.4}$$

where $(N_f u)(t) = f(t,\ u,\ u')$, $\lambda \in [0,\ 1]$. For $\lambda \in (0,\ 1]$, if u is a solution to problem (4.4), then by integrating both sides of (4.4) over $[0,\ T]$, we have

$$\begin{aligned} \phi_p(u'(0)) - \phi_p(u'(T)) &= (\mathrm{I} - \mathrm{Q})(\phi_p(u'(0))) \\ &= \lambda \int_0^T (N_f u)(\tau)\, \mathrm{d}\tau + (1-\lambda)\mathcal{P}\left(\int_0^T (N_f u)(\tau)\, \mathrm{d}\tau\right). \end{aligned}$$

Taking $\mathcal{P}$ to act on above equation, one has

$$\mathcal{P}\left(\int_0^T (N_f u)(\tau)\, \mathrm{d}\tau\right) = 0. \tag{4.5}$$

Similarly, if u is a solution to problem (4.1), then

$$\mathcal{P}\left(\int_0^T (N_f u)(\tau)\, \mathrm{d}\tau\right) = 0.$$

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Hence, for $\lambda \in (0,\, 1]$, problems (4.1) and (4.4) have the same solutions. Define homotopy operator $N : X \times [0,\, 1] \to Y$ by

$$N(u, \lambda) = \lambda(N_f u)(t) + (1-\lambda)\frac{1}{T}\mathcal{P}\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau = \lambda N_f u + (1-\lambda)\hat{\mathcal{Q}}(N_f u)(t).$$

From lemma 3.3, problem (4.4) can be written by the equivalent operator equation

$$u = H_\lambda u, \tag{4.6}$$

where

$$H_\lambda u = \hat{\mathcal{P}}(u) + \hat{\mathcal{Q}}(N_f u) - L_P^{-1}\left(\int_0^T c_\lambda(s)\,\mathrm{d}s\right) + \int_0^t c_\lambda(s)\,\mathrm{d}s),$$

and

$$c_\lambda(s) = \phi_q\left(-\int_0^s \lambda(\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t + L_P^{-1}\int_0^T \lambda(\mathrm{I}-\hat{\mathcal{Q}})(N_f u)(t)\,\mathrm{d}t + \tilde{\gamma}\right),$$

$\tilde{\gamma} = \tilde{\gamma}(\lambda(\mathrm{I}\text{ -}\hat{\mathcal{Q}})(N_f u))$ is defined in (2.22). □

Assume that for $\lambda = 1$, the problem (4.6) has no solution on $\partial\Omega$ otherwise the proof is complete. Due to hypothesis (i) we know that (4.6) has no solutions for $(u,\, \lambda) \in \partial\Omega \times (0,\, 1]$. For $\lambda = 0$, (4.4) is the form

$$\begin{cases} -(\phi_p(u'))' = \dfrac{1}{T}\mathcal{P}\left(\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau\right), \\ u(T) = Qu(0), u'(T) = Qu'(0). \end{cases} \tag{4.7}$$

Now we claim the problem (4.7) has no solution on $\partial\Omega \times 0$. If u is a solution of problem (4.7), then u satisfies (4.5), which shows $u'(t) = \phi_q(\alpha)$, where $\alpha \in \mathbb{R}^N$. By lemma 2.5, we have $\alpha = 0$ and $u(t) = \beta\,(\beta \in \ker(\mathrm{I}\text{ - Q}))$. It follows from (4.5) that

$$\mathcal{P}\left(\int_0^T f(\tau, \beta, 0)\,\mathrm{d}\tau\right) = 0,$$

which, together with hypothesis (ii), implies that $u = \beta \notin \partial\Omega$. Thus we have proved that (4.6) has no solution $(u,\, \lambda) \in \partial\Omega \times [0,\, 1]$.

By lemma 3.2, H_λ is a completely continuous operator. Then we have that for each $\lambda \in [0,\, 1]$, the Leray-Schauder degree $\deg_{LS}(I - H_\lambda,\, \Omega,\, 0)$ is well defined, and

$$\deg_{LS}(I - H_1, \Omega, 0) = \deg_{LS}(I - H_0, \Omega, 0). \tag{4.8}$$

It is clear that the operator equation

$$u = H_1(u) \tag{4.9}$$

is equivalent to the problem (H_Q). Now, we only prove that $\deg_{LS}(I - H_0,\, \Omega,\, 0) \neq 0$.

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Because

$$(I - H_0)u = u - \hat{\mathcal{P}}(u) - \frac{1}{T}\mathcal{P}\left(\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau\right),$$

$(I - H_0)u = 0$ deduces to $u = \hat{\mathcal{P}}(u) - \frac{1}{T}\mathcal{P}(\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau)$, which from the definitions of $\mathcal{P}$ and $\hat{\mathcal{P}}$ yields $u = c$ in Ω. On the basis of lemma 2.5, we have $c \in \ker(\mathrm{I} - \mathrm{Q})$. Hence by the properties of the Leray-Schauder degree and (4.3), we get that

$$\begin{aligned}\deg_{LS}(I - H_0, \Omega, 0) &= \deg_{LS}(I - H_0, \Omega \cap \ker(\mathrm{I} - \mathrm{Q})), 0) \\ &= \deg_B(-F, \Omega \cap \ker(\mathrm{I} - \mathrm{Q})), 0) \neq 0,\end{aligned}$$

where the function F is defined in (4.2). Then $\deg_{LS}(I - H_1,\, \Omega,\, 0) \neq 0$, that is, RPBVP (H_Q) has at least one solution in $\bar{\Omega}$.

REMARK 4.2. If $Q = I$, then $\ker(\mathrm{I}$ - $\mathrm{Q}) = \mathbb{R}^N$ and $\mathcal{P} = \mathrm{I}$. Theorem 4.1 is the same as the continuation theorem [8] for periodic boundary value problems.

THEOREM 4.3. *Suppose that* $\ker(\mathrm{I}$ - $\mathrm{Q}) = \{0\}$, *and* Ω *is an open bounded set in* X *such that* $0 \in \Omega$ *and the problem*

$$\begin{cases} -(\phi_p(u'))' = \lambda f(t, u, u'), \\ u(T) = Qu(0), u'(T) = Qu'(0), \end{cases} \tag{4.10}$$

has no solution on $\partial\Omega$, *for* $\forall\ \lambda \in (0,\, 1)$.

Then RPBVP (H_Q) *has at least one solution in* $\bar{\Omega}$.

Proof. Since $\ker(\mathrm{I}$ - $\mathrm{Q}) = \{0\}$, there exists $(\ \mathrm{I}$ - $\mathrm{Q}\ )^{-1}$. Define $\tilde{H} : X \to X$ by

$$\begin{aligned}\tilde{H}u = &\int_0^t \phi_q\left(-\int_0^s (N_f u)(\tau)\,\mathrm{d}\tau + (I - Q)^{-1}\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &- (I - Q)^{-1}\int_0^T \phi_q\left(-\int_0^s (N_f u)(\tau)\,\mathrm{d}\tau + (I - Q)^{-1}\int_0^T (N_f u)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.\end{aligned} \tag{4.11}$$

Then the RPBVP (H_Q) is equivalent to the operator equation

$$u = \tilde{H}u.$$

Similar to the lemmas 3.1–3.2, we can prove that $\tilde{H}$ is a completely continuous operator from X to X. Furthermore, define $\tilde{H}_\lambda$ by

$$\begin{aligned}\tilde{H}_\lambda u = &\int_0^t \phi_q\left(-\int_0^s \lambda(N_f u)(\tau)\,\mathrm{d}\tau + (I - Q)^{-1}\int_0^T \lambda(N_f u)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &- (I - Q)^{-1}\int_0^T \phi_q\left(-\int_0^s \lambda(N_f u)(\tau)\,\mathrm{d}\tau + (I - Q)^{-1}\int_0^T \lambda(N_f u)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,\end{aligned} \tag{4.12}$$

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for $\lambda \in [0, 1]$. We assume that for $\lambda = 1$, (4.10) has no solution on $\partial\Omega$ otherwise the proof is complete. For $\lambda = 0$, the Eqn (4.10) only has zero solution via lemma 2.5 and ker (I - Q) $= \{0\}$. By hypothesis, for each $\lambda \in [0, 1]$, the Leray-Schauder degree $\deg_{LS}(I - \tilde{H}_\lambda, \Omega, 0)$ is well defined, and

$$\deg_{LS}(I - \tilde{H}_1, \Omega, 0) = \deg_{LS}(I - \tilde{H}_0, \Omega, 0) = \deg_{LS}(I, \Omega, 0) = 1.$$

Hence, the RPBVP (H_Q) has at least one solution in $\bar{\Omega}$. □

5. Applications

In this section, we take useful of theorem 4.1 to further discuss the sufficient conditions of existence of solutions for the two kinds of RPBVP (H_Q).

5.1. Existence of solutions for a kind of the RPBVP (H_Q)

THEOREM 5.1. *Assume that* $\ker(\mathrm{I} - \mathrm{Q}) \neq \{0\}$ *and the following conditions hold.*

(f_1) *There exist* $h \in L^1([0, T], \mathbb{R}_+)$ *and* $n \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *satisfying* $n'(x)$ *is negative semi-definite and* $n(Qx) = Qn(x)$ *for each* $x \in \mathbb{R}^N$, *such that*

$$|f(t, x, y)| \leqslant \langle f(t, x, y), n(x)\rangle + h(t), \tag{5.1}$$

for any $x, y \in \mathbb{R}^N$, *and a.e.* $t \in [0, T]$.

(f_2) f satisfies a generalized Villari-type condition, i.e. there exists a constant $M > 0$ such that for all $u \in X$ with $\min_{t\in[0,T]} |u(t)| > M$,

$$\mathcal{P}\left(\int_0^T f(t, u, u')\, \mathrm{d}t\right) \neq 0, \tag{5.2}$$

where $\mathcal{P} : \mathbb{R}^N \to$ ker (I - Q).

Then the problem (H_Q) has at least one solution.

Proof. First we take a priori estimate for solutions of (4.1). Let $(u, \lambda) \in X \times (0, 1)$ be a solution to problem (4.1). Then we have

$$\phi_p(u'(t)) = -\int_0^t \lambda f(s, u, u')\, \mathrm{d}s + \mathcal{P}(\phi_p(u'(0))) + L_P^{-1}\int_0^T \lambda f(s, u, u')\, \mathrm{d}s, \tag{5.3}$$

and

$$(\mathrm{I} - \mathrm{Q})u(0) = \int_0^T \phi_q\left(-\int_0^t \lambda f(s, u, u')\, \mathrm{d}s + \mathcal{P}(\phi_p(u'(0))) + L_P^{-1}\int_0^T \lambda f(s, u, u')\, \mathrm{d}s\right) \mathrm{d}t \in \mathrm{Im}(I - Q),$$

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where $\frac{1}{p}+\frac{1}{q}=1(p,\, q>1)$. Hence,

$$\mathcal{P}\left(\int_0^T \phi_q\left(-\int_0^t \lambda f(s,u,u')\,\mathrm{d}s+\mathcal{P}(\phi_p(u'(0)))+L_P^{-1}\int_0^T \lambda f(s,u,u')\,\mathrm{d}s\right)\mathrm{d}t\right)=0. \tag{5.4}$$

□

Because $n'(u)$ is negative semi-definite, we obtain that

$$\begin{aligned}0 &\geqslant \int_0^T \langle \phi_p(u'(t)), n'(u(t))u'(t)\rangle\ \mathrm{d}t\\ &= \langle \phi_p(u'(t)), n(u(t))\rangle\,|_0^T - \int_0^T \left\langle (\phi_p(u'(t)))', n(u(t))\right\rangle\ \mathrm{d}t\\ &= \langle \phi_p(u'(T)), n(u(T))\rangle - \langle \phi_p(u'(0)), n(u(0))\rangle - \int_0^T \left\langle (\phi_p(u'(t)))', n(u(t))\right\rangle\ \mathrm{d}t\\ &= \langle Q\phi_p(u'(0)), Qn(u(0))\rangle - \langle \phi_p(u'(0)), n(u(0))\rangle + \int_0^T \langle \lambda f(t,u,u'), n(u(t))\rangle\ \mathrm{d}t\\ &= \int_0^T \langle \lambda f(t,u,u'), n(u(t))\rangle\ \mathrm{d}t.\end{aligned} \tag{5.5}$$

Furthermore, we have that

$$\mathcal{P}\left(\int_0^T \lambda f(t,u,u')\,\mathrm{d}t\right)=0. \tag{5.6}$$

By (5.1) and (5.5), we have that

$$\lambda\int_0^T |f(t,u,u')|\ \mathrm{d}t \leqslant \int_0^T \langle f(t,u,u'), n(u)\rangle\ \mathrm{d}t + \int_0^T h(t)\,\mathrm{d}t \leqslant \int_0^T h(t)\,\mathrm{d}t \triangleq M_1. \tag{5.7}$$

According to the definition of L_P , there exists $M_2>0$ such that

$$\left|L_P^{-1}\int_0^T \lambda f(t,u,u')\,\mathrm{d}t\right| \leqslant M_2. \tag{5.8}$$

From (5.6) and (5.7), it follows that $\lambda f(t,\, u,\, u')\in Y_1$ is L^1-bounded for any solution of (4.1). According to the definition of $G_h(\gamma)$, proposition 2.6 , (5.4),(5.7) and (5.8), we have that $\tilde{\gamma}=\mathcal{P}(\phi_p(u'(0)))$ is bounded, i.e., there exists $M_3>0$ such that

$$|\mathcal{P}(\phi_p(u'(0)))| \leqslant M_3.$$

Hence for any $t\in[0,\, T]$, we have

$$\begin{aligned}|\phi_p(u'(t))| &\leqslant \left|\int_0^t \lambda f(s,u,u')\,\mathrm{d}s\right| + |\mathcal{P}(\phi_p(u'(0)))| + \left|L_P^{-1}\int_0^T \lambda f(s,u,u')\,\mathrm{d}s\right|\\ &\leqslant M_1+M_2+M_3.\end{aligned}$$

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In the light of the definition of ϕ_q , there exists $M_4 > 0$ such that

$$\|u'\|_0 \leqslant M_4.$$

Thanks to (5.6) and hypothesis (f_2), there exists $t_j \in [0, T]$ such that $|u(t_j)| < M$, and

$$|u(t)| = |u(t_j)| + \left|\int_{t_j}^{t} u'(s)\,\mathrm{d}s\right| \leqslant M + TM_4 = M_5.$$

It follows that

$$\|u\|_1 \leqslant M_4 + M_5 \triangleq r.$$

Let $\Omega_0 = \{u \in X \,|\, \|u\|_1 < r+1\}$. Then condition (i) of theorem 4.1 is satisfied.

Take constant $\alpha \in X$, then $\alpha \in \ker$ (I-Q). By hypothesis (f_2), one of the following conditions holds:

$$\left\langle \mathcal{P}\left(\int_0^T f(t,\alpha,0)\,\mathrm{d}t\right), \alpha \right\rangle > 0, \quad |\alpha| > M, \tag{5.9}$$

or

$$\left\langle \mathcal{P}\left(\int_0^T f(t,\alpha,0)\,\mathrm{d}t\right), \alpha \right\rangle < 0, \quad |\alpha| > M. \tag{5.10}$$

In the case (5.9), define the following homotopy mapping:

$$H_\mu(\alpha) = \mu\alpha + (1-\mu)\mathcal{P}\left(\int_0^T f(t,\alpha,0)\,\mathrm{d}t\right);$$

in the case (5.10), define the following homotopy mapping:

$$H_\mu(\alpha) = -\mu\alpha + (1-\mu)\mathcal{P}\left(\int_0^T f(t,\alpha,0)\,\mathrm{d}t\right),$$

where $\mu \in [0, 1]$. It is easy to check that the solution of $H_\mu(\alpha) = 0$ must be in $\Omega_1 \cap \ker(\mathrm{I}-\mathrm{Q})$, where $\Omega_1 = \{u \in X \,|\, \|u\|_1 < M+1\}$. Then we have that

$$\begin{aligned}\deg_B(H_\mu(\alpha), \Omega_1 \cap \ker(\mathrm{I}-\mathrm{Q}), 0) &= \deg_B(\mathcal{P}\left(\int_0^T f(t,\alpha,0)\,\mathrm{d}t\right), \Omega_1 \cap \ker(\mathrm{I}-\mathrm{Q}), 0)\\ &= \deg_B(\pm\mathrm{I}, \Omega_1 \cap \ker(\mathrm{I}-\mathrm{Q}), 0) \neq 0.\end{aligned}$$

Thus the condition (ii) of theorem 4.1 is satisfied with Ω_1.

Finally, take

$$\Omega = \{u \in X \,|\, \|u\|_1 < \max\{r+1, M+1\}\}.$$

Then conditions (i) and (ii) of theorem 4.1 are satisfied on Ω, which leads to the problem (H_Q) has at least one solution.

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REMARK 5.2. The Villari condition was first introduced for the scalar case by Villari in [13], i.e., there exists a $k > 0$ such that for all $u \in C^1([0, T], \mathbb{R})$ with $\min_{t\in[0,T]} |u(t)| \geqslant k$,

$$\operatorname{sgn}(u) \int_0^T f(t, u, u')\, \mathrm{d}t \geqslant 0.$$

Obviously, the above condition requires u and $\int_0^T f(t, u, u')\, \mathrm{d}t$ to be the same sign. But our conditions do not require that.

In [8], Manásevich and Mawhin gave the generalized Villari condition for periodic problem, i.e., there exists a $k > 0$ such that for all $u \in C_T^1$, $u = (u_1, \cdots, u_N)$, with $\min_{t\in[0,T]} |u_j(t)| \geqslant k$, for some $j \in \{1, \cdots, N\}$,

$$\int_0^T f_i(t, u, u')\, \mathrm{d}t \neq 0,$$

for some $i \in \{1, \cdots, N\}$. However, this does not lead to the condition (ii) of theorem 4.1.

COROLLARY 5.3. *Assume that* $\ker(\mathrm{I} - \mathrm{Q}) \neq \{0\}$ *and the following conditions hold.*

(1) *The condition* (f_1) *of theorem* 5.1 *holds.*

(2) *There exist* $h_1 \in L^1([0, T], \mathbb{R}_+)$ *and* $\alpha : [0, +\infty) \to [0, +\infty)$ *such that* $\alpha(s) \to +\infty$ *as* $s \to +\infty$ *and*

$$\alpha(|x|) - h_1(t) \leqslant |\mathcal{P} f(t, x, y)| \tag{5.11}$$

for almost all $t \in [0, T]$ *and all* $x, y \in \mathbb{R}^N$.

(3) *Condition* (4.3) *holds.*

Then the problem (H_Q) has at least one solution.

Proof. Let (u, λ), $\lambda \in (0, 1)$ be a solution for problem (4.1). As in the proof of theorem 5.1, it follows from 1) that there is $M_1 > 0$ such that $\|u'\|_0 \leqslant M_1$. We claim that 1) and (5.11) imply that there exists $M_2 > 0$ such that $\|u\|_0 \leqslant M_2$. In fact, by (5.1) and (5.5), we have $\int_0^T |f(t, u, u')|\, \mathrm{d}t \leqslant \|h\|_{L^1}$. From (5.11) and $|\mathcal{P}x| \leqslant |x|$, it follows that

$$\begin{aligned}\int_0^T \alpha(|u|)\, \mathrm{d}t &\leqslant \int_0^T |\mathcal{P} f(t, u, u')|\, \mathrm{d}t + \|h_1\|_{L^1} \leqslant \int_0^T |f(t, u, u')|\, \mathrm{d}t + \|h_1\|_{L^1} \\ &\leqslant \|h\|_{L^1} + \|h_1\|_{L^1}.\end{aligned}$$

Since $\alpha(s) \to +\infty$ as $s \to +\infty$, we find the required bound for $\|u\|_0$.

Now let $a \in \ker(\mathrm{I} - \mathrm{Q})$ such that $\mathcal{P} \int_0^T f(t, a, 0)\, \mathrm{d}t = 0$. By (5.11), we get that $\alpha(|a|) \leqslant M_3$, and hence $|a| \leqslant M_4$. Here M_3 and M_4 are positive constants. Thus there is $r > 0$ such that all solution to (4.2) belongs to

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$\Omega = \{a \in \ker(I - Q) : |a| < r\}$. The rest of the proof follows from the theorem 5.1. $\square$

EXAMPLE 5.4. Now, we give a simple example for corollary 5.3. Consider the following p-Laplacian differential systems

$$\begin{cases} -\left(|u'|^{p-2}u_1'\right)' + u_1\left(1+u_1^2\right) + u_1\left(1+u_2^2\right) \\ \quad = e_1(t),\ -\left(|u'|^{p-2}u_2'\right)' + u_2\left(1+u_2^2\right) = e_2(t), \end{cases} \tag{5.12}$$

with rotating periodic boundary conditions

$$u(T) = Qu(0), \quad u'(T) = Qu'(0).$$

where $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$, $e(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \in L^1(0, T; \mathbb{R}^2)$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Then we have $(\mathrm{I} - \mathrm{Q}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $\ker(\mathrm{I} - \mathrm{Q}) = a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathrm{Im}(\mathrm{I} - \mathrm{Q}) = a\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $a \in \mathbb{R}$.

Let $\mathcal{P}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, $n(x) = \begin{pmatrix} -2x_1 - 2x_2 \end{pmatrix}$ and $\alpha(|x|) = |x|$, for $x \in \mathbb{R}^2$.

Set

$$f(t,x) = \left(-x_1(1+x_1^2) - x_1(1+x_2^2) + e_1(t) - x_2(1+x_2^2) + e_2(t)\right).$$

Obviously, for a.e. $t \in [0, T]$ and $x \in \mathbb{R}^2$, we have

$$-|e(t)| + |x| \leqslant |\mathcal{P}f(t,x)| \leqslant |f(t,x)| \leqslant 2|x|^3 + |x| + |e(t)|,$$

as $|x_1| \geqslant 1$ and $|x_2| \geqslant 1$. Hence for some $l(t) \in L^1([0, T], \mathbb{R}_+)$, we have

$$|f(t,x)| \leqslant 2|x|^3 + |x| + |l(t)|,$$

for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^2$. On the other hand,

$$\langle f(t,x), n(x)\rangle \geqslant |x|^4 + 2|x|^2 - 2|x|\,|e|,$$

for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^2$. Thus for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^2$, we can choose $h(t) \in L^1([0, T], \mathbb{R}_+)$ such that

$$|f(t,x)| \leqslant \langle f(t,x), n(x)\rangle + h(t).$$

Next, for $b \in \ker(\mathrm{I} - \mathrm{Q})$, we have

$$F(b) = \begin{pmatrix} -b_1(1+b_1^2) - b_1 + \frac{1}{T}\int_0^T e_1(t)\,\mathrm{d}t \\ 0 \end{pmatrix}.$$

By the properties of the Brouwer degree, we have for sufficiently large $r > 0$

$$\deg_B(F(b), \Omega(r), 0) = 1,$$

where $\Omega(r) = \{b \in \ker(I - Q) : |b| < r\}$. Hence, the RPBVP (5.12) has at least one solution.

5.2. Existence of solutions of RPBVP (H_Q) for the p-Laplacian Liénard-type system

Consider the following p-Laplacian Liénard-type system with the rotating periodic boundary conditions:

$$\begin{cases}(\phi_p(u'(t)))' + (\nabla F(u))' + Au(t) = e(t),\\ u(T) = Qu(0), u'(T) = Qu'(0),\end{cases} \tag{5.13}$$

where $p \geqslant 2$, Q is an $N \times N$ orthogonal matrix with ker (I-Q) $\neq \{0\}$, A is an $N \times N$ matrix with $AQ = QA$, $F \in C^2(\mathbb{R}^N, \mathbb{R})$ with $F(u) = F(|u|)$, $e \in L^2$ with $e(t + T) = Qe(t)$.

In [10], Mawhin studied the T-periodic solutions of the following p-Laplacian Liénard system:

$$\begin{cases}(\phi_p(u'))' + (\nabla F(u))' + Au = e(t),\\ u(0) = u(T), u'(0) = u'(T).\end{cases}$$

And the author obtained some existence theorems for the above problem.

Next, we extend periodic boundary value conditions to rotating periodic boundary conditions and give some existence results for (5.13).

THEOREM 5.5. *Assume that A is a negative definite matrix and satisfies $\mathcal{P}A(\alpha) = A\mathcal{P}(\alpha)$ for any $\alpha \in \mathbb{R}^N$, where $\mathcal{P} : \mathbb{R}^N \to$ ker (I - Q). Then for each $e \in L^2$, problem (5.13) has at least one solution.*

Proof. To apply theorem 4.1, we consider the auxiliary RPBVP:

$$\begin{cases}(\phi_p(u'(t)))' + \lambda(\nabla F(u))' + \lambda Au(t) = \lambda e(t),\\ u(T) = Qu(0), u'(T) = Qu'(0),\end{cases} \tag{5.14}$$

where $\lambda \in (0, 1]$. □

First, we make a prior estimate. Let $(u, \lambda) \in X \times (0, 1]$ be solution of (5.14). Integrating (5.14) over $[0, T]$, we get that

$$(\mathrm{I} - \mathrm{Q})\phi_p(u'(0)) + (\mathrm{I} - \mathrm{Q})\lambda\frac{dF}{du}(|u(0)|)\frac{u(0)}{|u(0)|} + \lambda\int_0^T e(t)\,\mathrm{d}t = \lambda A\int_0^T u(t)\,\mathrm{d}t.$$

Taking $\mathcal{P}$ to act on the above equation, we have

$$\mathcal{P}\left(\int_0^T e(t)\,\mathrm{d}t\right) = A\mathcal{P}\left(\int_0^T u(t)\,\mathrm{d}t\right).$$

Let $\bar{e} = \mathcal{P}(\frac{1}{T}\int_0^T e(t)\,\mathrm{d}t)$ and $\bar{u} = \mathcal{P}(\frac{1}{T}\int_0^T u(t)\,\mathrm{d}t)$, then

$$|\bar{u}| = \left|A^{-1}\bar{e}\right| \leqslant \left|A^{-1}\right| |\bar{e}|. \tag{5.15}$$

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Now taking the inner product for the both side of (5.14) by u and integrating over $[0,\,T]$, we obtain

$$\begin{aligned}\int_0^T \left\langle (\phi_p(u'(t)))', u(t)\right\rangle \,\mathrm{d}t &= \langle (\phi_p(u'(t)))\,, u(t)\rangle\,\big|_0^T - \int_0^T \langle (\phi_p(u'(t)))\,, u'(t)\rangle \,\mathrm{d}t \\ &= \langle Q\phi_p(u'(0)), Qu(0)\rangle - \langle \phi_p(u'(0)), u(0)\rangle \\ &\quad - \int_0^T |u'(t)|^p \,\mathrm{d}t \\ &= -\int_0^T |u'(t)|^p \,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\int_0^T \left\langle (\nabla F(u))', u(t)\right\rangle \,\mathrm{d}t &= \langle \nabla F(u), u(t)\rangle\,\big|_0^T - \int_0^T \nabla F(u) du \\ &= \left\langle Q\frac{dF}{du}(|u(0)|)\frac{u(0)}{|u(0)|}, Qu(0)\right\rangle \\ &\quad - \left\langle \frac{dF}{du}(|u(0)|)\frac{u(0)}{|u(0)|}, u(0)\right\rangle \\ &\quad - F(u(T)) + F(u(0)) \\ &= F(u(0)) - F(|Qu(0)|) = 0,\end{aligned}$$

and

$$\int_0^T \langle e(t), u(t)\rangle \,\mathrm{d}t = T\,\langle \bar{e}, \bar{u}\rangle + \int_0^T \langle \tilde{e}(t), \tilde{u}(t)\rangle \,\mathrm{d}t.$$

Then we have

$$\int_0^T |u'(t)|^p \,\mathrm{d}t - \lambda \int_0^T \langle Au(t), u(t)\rangle \,\mathrm{d}t = -\lambda T\,\langle \bar{e}, \bar{u}\rangle - \lambda \int_0^T \langle \tilde{e}(t), \tilde{u}(t)\rangle \,\mathrm{d}t, \quad (5.16)$$

where $\tilde{e}(t) = e(t) - \bar{e} = e(t) - \mathcal{P}(\frac{1}{T}\int_0^T e(t)\,\mathrm{d}t)$ and $\tilde{u}(t) = u(t) - \bar{u} = u(t) - \mathcal{P}$ $(\frac{1}{T}\int_0^T u(t)\,\mathrm{d}t)$, which yield $\int_0^T \tilde{e}(t)\,\mathrm{d}t$, $\int_0^T \tilde{u}(t)\,\mathrm{d}t \in \mathrm{Im(I - Q)}$. According to the assumption of A and (5.15), we get that

$$\int_0^T |u'(t)|^p \,\mathrm{d}t \leqslant T\left|A^{-1}\right| |\bar{e}|^2 + N\|\tilde{e}\|_{L^1}\|\tilde{u}\|_0. \quad (5.17)$$

For $\tilde{u}$, it follows from Sobolev inequality that

$$\|\tilde{u}\|_0 \leqslant M\|\tilde{u}'\|_{L^2} = M\|u'\|_{L^2}. \quad (5.18)$$

Next we claim there exists $M_2 > 0$ such that

$$\|u'\|_{L^2} \leqslant M_2. \quad (5.19)$$

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If this is false, there are $\lambda_n \in (0, 1]$ $(n = 1, 2, \cdots)$ such that corresponding solutions u_n satisfy $\|u_n'\|_{L^2} \to \infty$ $(n \to \infty)$. By (5.17), (5.18) and $p \geqslant 2$, we have

$$\|u_n'\|_{L^2}^2 \leqslant (T)^{(p-2)/p} \left(\int_0^T |u_n'(t)|^p \, \mathrm{d}t \right)^{2/p}$$
$$\leqslant (T)^{(p-2)/p} \left(T \left| A^{-1} \right| |\bar{e}|^2 + NM \|\tilde{e}\|_{L^1} \|u_n'\|_{L^2} \right)^{2/p},$$

which is a contradiction as $n \to \infty$. From (5.15), (5.18) and (5.19), together with $u(t) = \tilde{u}(t) + \bar{u}$, it follows that there exists $M_3 > 0$ such that

$$\|u\|_0 \leqslant M_3. \tag{5.20}$$

(5.14) implies that

$$\left| (\phi_p(u'(t)))' \right| \leqslant \left| \frac{d^2 F(u(t))}{du_i du_j} u'(t) \right| + |A| \, |u(t)| + \sum_{i=1}^N |e_i(t)|,$$

for a.e. $t \in [0, T]$. And owing to (5.20) and the quality of $F(u)$, we obtain

$$\left| (\phi_p(u'(t)))' \right| \leqslant M_4 \sum_{i=1}^N |u_i{}'(t)| + |A| \, M_3 + \sum_{i=1}^N |e_i(t)|,$$

where $\left| \frac{d^2 F(u(t))}{du_i du_j} \right| \leqslant M_4$. By Hölder inequality, we have

$$\left| (\phi_p(u'(t)))' \right|^2 \leqslant 3N(M_4)^2 \sum_{i=1}^N |u_i{}'(t)|^2 + 3|A|^2 (M_3)^2 + 3N \sum_{i=1}^N |e_i(t)|^2.$$

Furthermore,

$$\int_0^T \left| (\phi_p(u'(t)))' \right|^2 \, \mathrm{d}t \leqslant 3(M_4)^2 N^2 \|u'\|_{L^2}^2 + 3T|A|^2 (M_3)^2 + 3N^2 \|e\|_{L^2}^2$$
$$\leqslant 3(M_4)^2 N^2 (M_2)^2 + 3T|A|^2 (M_3)^2 + 3N^2 \|e\|_{L^2}^2 \triangleq M_5. \tag{5.21}$$

Write $v(t) = \phi_p(u'(t))$ and decompose it as $v(t) = \tilde{v}(t) + \bar{v}$. We have $\int_0^T \tilde{v}(t) \, \mathrm{d}t \in$ Im(I - Q) and

$$u'(t) = \phi_q \left(\tilde{v}(t) + \bar{v} \right),$$

where $\frac{1}{p} + \frac{1}{q} = 1 (p, \, q > 1)$. Hence,

$$\mathcal{P} \left(\int_0^T \phi_q \left(\tilde{v}(t) + \bar{v} \right) \, \mathrm{d}t \right) = 0.$$

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We deduce, from (5.21) and Sobolev inequality, that

$$\|\tilde{v}\|_0^2 \leqslant M^2 \int_0^T |\tilde{v}'(t)|^2 \, \mathrm{d}t = M^2 \int_0^T \left|(\phi_p(u'(t)))'\right|^2 \, \mathrm{d}t \leqslant M^2 M_5 \triangleq M_6.$$

From proposition 2.6, it follows that $|\bar{v}| = |\tilde{\gamma}(\tilde{v}(t))|$ is bounded. Therefore

$$\|\phi_p(u')\|_0 = \|v\|_0 \leqslant \|\tilde{v}\|_0 + |\bar{v}| \leqslant M_7.$$

Then

$$\|u'\|_0 \leqslant M_8.$$

So there exists $M_0 > 0$ independent of λ such that

$$\|u\|_1 = \max\{\|u\|_0, \|u'\|_0\} \leqslant M_0.$$

Secondly, to check the condition (ii) of theorem 4.1, we see that

$$F(\alpha) := \mathcal{P}\left(\frac{1}{T}\int_0^T (e(t) - A\alpha) \, \mathrm{d}t\right) = \bar{e} - A\alpha,$$

where $\alpha \in \ker(\mathrm{I} - \mathrm{Q})$. Then $F(\alpha) = 0$ has the unique solution $\alpha = A^{-1}\bar{e}$ which trivially yields that $\deg_B(F, B(r), 0)$ is well defined and equal to ± 1 for all sufficiently large $r > 0$, so that condition (ii) of theorem 4.1 is satisfied.

Corollary 5.6. *If A is a negative definite matrix and satisfies $\mathcal{P}A(\alpha) = A\mathcal{P}(\alpha)$ for any $\alpha \in \mathbb{R}^N$, then for each $e \in L^2$, the RPBVP*

$$\begin{cases} (\phi_p(u'(t)))' + Au(t) = e(t), \\ u(T) = Qu(0), u'(T) = Qu'(0), \end{cases} \tag{5.22}$$

has an unique solution.

Proof. Only the uniqueness has to proved. Let u and v be solutions of (5.22). Then we have

$$\begin{aligned} &(\phi_p(u'(t)))' - (\phi_p(v'(t)))' + A(u - v) = 0, \\ &u(T) = Qu(0), u'(T) = Qu'(0), \quad v(T) = Qv(0), v'(T) = Qv'(0). \end{aligned}$$

And hence, after multiplication by $u - v$, and integration by parts over $[0, T]$, we get

$$\int_0^T \langle \phi_p(u'(t)) - \phi_p(v'(t)), u'(t) - v'(t) \rangle \, \mathrm{d}t - \int_0^T \langle A(u(t) - v(t)), (u(t) - v(t)) \rangle \, \mathrm{d}t = 0.$$

The above formula and lemma 2.4 yield that $u = v$. □

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Corollary 5.7. *If A is a negative semi-definite matrix with $\mathcal{P}A(\alpha) = A\mathcal{P}(\alpha)$ for any $\alpha \in \mathbb{R}^N$, then for each $e \in L^2$ with $\bar{e} = \mathcal{P}(\frac{1}{T}\int_0^T e(t)\,\mathrm{d}t) = 0$, the RPBVP (5.13) has at least one solution u such that $\bar{u} = \mathcal{P}(\frac{1}{T}\int_0^T u(t)\,\mathrm{d}t) = 0$.*

Proof. Consider the auxiliary RPBVP:

$$\begin{cases}(\phi_p(u'(t)))' + (\nabla F(u))' + Au(t) - \dfrac{1}{n}u(t) = e(t),\\ u(T) = Qu(0), u'(T) = Qu'(0),\end{cases} \tag{5.23}$$

where $n > 0$. By integrating the equation over $[0, T]$ and using $\mathcal{P}$ to act, then each solution u of (5.23) satisfies

$$(A - \frac{1}{n}I)\mathcal{P}\left(\frac{1}{T}\int_0^T u(t)\,\mathrm{d}t\right) = \mathcal{P}\left(\frac{1}{T}\int_0^T e(t)\,\mathrm{d}t\right) = \bar{e} = 0.$$

Notice that $(A - \frac{1}{n}I)$ is negative definite for each n. So $\bar{u} = 0$. It follows from theorem 5.5 and its proof that, RPBVP (5.23) has at least one solution $u_n(t)$ for each n. Further there is $r_0 > 0$ independent of n such that $\|u_n\|_1 \leqslant r_0$. From lemma 3.3, it follows that those u_n are fixed points of the equivalent completely continuous operator. So there exists a subsequence converging to a solution of (5.13) with $\bar{u} = 0$. □

Corollary 5.8. *If $A \triangleq a < 0$ is a constant, then for each $e \in L^2$, the problem (5.13) has at least one solution u.*

Remark 5.9. If $Q = I$, then we immediately deduce Theorem 6.1 in [**10**] from theorem 5.5.

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