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RADII OF HARMONIC MAPPINGS IN THE PLANE

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Abstract

In this paper, for the convolution and convex combination of harmonic mappings, the radii of univalence, full convexity and starlikeness of order α are explored. All results are sharp. By way of application, the univalent radius and the Bloch constant of the convolution of two bounded harmonic mappings are obtained.

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1. Introduction

Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions f in the unit disk $\mathbb{D}$ normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in \mathcal{H}$ can be decomposed as $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$ such that

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

Denote by $\mathcal{S}_H$ the class of univalent and orientation-preserving functions $f \in \mathcal{H}$. Then the Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to Lewy's theorem [12], f is locally univalent in $\mathbb{D}$ if and only if $J_f(z) \neq 0$ for any $z \in \mathbb{D}$.

Let $\mathcal{K}_H$, $\mathcal{S}^*_H$ and C_H be the subclass of $\mathcal{S}_H$ mapping $\mathbb{D}$ onto convex, starlike and close-to-convex domains, respectively. Denoted by $\mathcal{K}^o_H$, $\mathcal{S}^{*o}_H$, C^o_H and $\mathcal{S}^o_H$ the class consisting of those functions f in $\mathcal{K}_H$, $\mathcal{S}^*_H$, C_H and $\mathcal{S}_H$ respectively, for which $f_{\bar{z}}(0) = b_1 = 0$.

In [5], it was conjectured that if $f \in \mathcal{S}^o_H$ then

$$|a_n| \leq \tfrac{1}{6}(2n+1)(n+1) \quad \text{and} \quad |b_n| \leq \tfrac{1}{6}(2n-1)(n-1)$$

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for all $n \geq 1$. This coefficient conjecture remains an open problem for the full class $\mathcal{S}_H^o$. However, it was verified for some subclasses of $\mathcal{S}_H^o$, such as typically real functions [5], starlike functions [17] and close-to-convex functions [18]. The extremal function is the harmonic Koebe function

$$K(z) = \sum_{n=1}^{\infty} \frac{1}{6}(2n+1)(n+1)z^n + \overline{\sum_{n=1}^{\infty} \frac{1}{6}(2n-1)(n-1)z^n}$$
$$= \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}}.$$

If $f \in \mathcal{K}_H^o$, then Clunie and Sheil-Small [5] proved that

$$|a_n| \leq \frac{n+1}{2} \quad \text{and} \quad |b_n| \leq \frac{n-1}{2}$$

for all $n \geq 1$. Equality occurs for the harmonic half plane mapping

$$L(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n + \overline{\sum_{n=1}^{\infty} \Big(-\frac{n-1}{2}\Big) z^n}$$
$$= \frac{z - \frac{1}{2}z^2}{(1-z)^2} + \overline{\frac{-\frac{1}{2}z^2}{(1-z)^2}}.$$

The convexity and starlikeness are hereditary properties for conformal mappings. That is to say, if an analytic function maps $\mathbb{D}$ onto a convex domain or starlike domain, then it also maps each concentric subdisk onto a convex domain or starlike domain, respectively. However, these hereditary properties do not generalize to harmonic mappings (see [16, 17]). The failure of the hereditary property of starlike and convex harmonic mapping leads to the notion of fully starlike and fully convex functions, which was discussed in [4].

A harmonic mapping f of $\mathbb{D}$ is said to be fully convex of order α, $0 \leq \alpha < 1$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve satisfying

$$\frac{\partial}{\partial\theta}\Big(\arg\Big(\frac{\partial}{\partial\theta} f(re^{i\theta})\Big)\Big) > \alpha, \quad 0 \leq \theta < 2\pi,\ 0 < r < 1.$$

If $\alpha = 0$, then f is said to be fully convex.

Similarly, a harmonic mapping f of $\mathbb{D}$ with $f(0) = 0$ is said to be fully starlike of order α, $0 \leq \alpha < 1$, if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin satisfying

$$\frac{\partial}{\partial\theta}(\arg(f(re^{i\theta}))) > \alpha, \quad 0 \leq \theta < 2\pi,\ 0 < r < 1.$$

If $\alpha = 0$, then f is said to be fully starlike.

Let $\mathcal{FK}_H(\alpha)$ and $\mathcal{FS}_H^*(\alpha)$ denote the subclass of $\mathcal{K}_H$ consisting of fully convex functions of order α and the subclass of $\mathcal{S}_H^*$ consisting of fully starlike functions of order α, respectively. The following two lemmas give a sufficient condition for functions $f \in \mathcal{H}$ to be $\mathcal{FK}_H(\alpha)$ and $\mathcal{FS}_H^*(\alpha)$, respectively.

Lemma 1.1 [9]. *Let $f = h + \bar{g}$, where h and g are given by (1.1). Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 1$$

and $0 \leq \alpha < 1$. Then f is harmonic univalent in $\mathbb{D}$, and $f \in \mathcal{FK}_H(\alpha)$.

Lemma 1.2 [10]. *Let $f = h + \bar{g}$, where h and g are given by (1.1). Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha}|b_n| \leq 1$$

and $0 \leq \alpha < 1$. Then f is harmonic univalent in $\mathbb{D}$, and $f \in \mathcal{FS}_H^*(\alpha)$.

According to the Radó–Kneser–Choquet theorem, a fully convex harmonic mapping of order α is necessarily univalent in $\mathbb{D}$. However, unlike fully convex mappings, a fully starlike mapping need not be univalent.

The radius of harmonic functions is an interesting and important problem. In [16, 17], it was given that the radius of full convexity of the class $\mathcal{K}_H^o$ is $\sqrt{2} - 1$, while the radius of full convexity of class $\mathcal{S}_H^{*o}$ is $3 - \sqrt{8}$. In [11], the radius of close-to-convexity and full starlikeness of harmonic mappings was determined. There results are generalized in the context of fully starlike and fully convex functions of order α in [1, 15].

Denote by $\mathcal{B}_H^M$ the class of harmonic mappings f of the unit disk $\mathbb{D}$ with $f(0) = f_{\bar{z}}(0) = f_z(0) - 1 = 0$, and $|f(z)| < M$ for $z \in \mathbb{D}$. There are two important constants: one is the radius of univalence, while the other is the Bloch constant. Estimates were given in [2, 3, 7, 13]. But these results are not sharp. In [11], a better estimate for the radius of close-to-convexity and full starlikeness of $\mathcal{B}_H^M$ is given. The following lemma is given in [11].

Lemma 1.3. *Let h and g be given by (1.1) with $|b_1| < 1$ and*

$$|a_n| + |b_n| \leq C$$

for all $n \geq 2$. Then $f = h + \bar{g}$ satisfies the inequality

$$|h'(z) - 1| < 1 - |g'(z)| \tag{1.2}$$

in the disk $|z| < r_s$ and is fully starlike in $|z| < r_s$, where

$$r_s = 1 - \sqrt{\frac{C}{C + 1 - |b_1|}}.$$

The result is sharp.

Remark 1.4. If $f = h + \bar{g}$ satisfy (1.2), then f is close-to-convex in $\mathbb{D}$ (see [3, 5]). Therefore, r_s given in Lemma 1.3 is the radius of close-to-convexity (univalence) at the same time.

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Linear combination is an important method to construct a new function. MacGregor [14] showed that the linear combination $tf + (1-t)g$ for $0 \le t \le 1$ of analytic functions need not be univalent even if f and g are convex functions.

The convolution of two harmonic mappings in a simply connected domain is defined as

$$f * F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n},$$

where

$$f = h + \bar{g} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n},$$
$$F = H + \bar{G} = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}.$$

There have been some results about harmonic convolution; see [5, 6, 8]. The harmonic convolution $f * F$ of two harmonic functions f and F may not preserve the properties of f or F, such as convexity or even univalence.

The main results of this paper are shown in Sections 2–4. In Section 2 the radius of linear convex combinations of harmonic mappings is studied. In Section 3 we explore the radius of convolution of harmonic mappings. In Section 4 we consider the radius of univalency and Bloch constant of the convolution of two harmonic mappings. All these results are sharp.

2. Radius of linear convex combination of harmonic mappings

The following identities are quite useful in the following proofs of the theorems:

$$\begin{gathered}
\sum_{n=1}^{\infty} n r^{n-1} = \frac{1}{(1-r)^2}, \quad \sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{1+r}{(1-r)^3}, \\
\sum_{n=1}^{\infty} n^3 r^{n-1} = \frac{1+4r+r^2}{(1-r)^4}, \quad \sum_{n=1}^{\infty} n^4 r^{n-1} = \frac{(1+r)(1+10r+r^2)}{(1-r)^5}, \\
\sum_{n=1}^{\infty} n^5 r^{n-1} = \frac{1+26r+66r^2+26r^3+r^4}{(1-r)^6}, \\
\sum_{n=1}^{\infty} n^6 r^{n-1} = \frac{1+57r+302r^2+302r^3+57r^4+r^5}{(1-r)^7}.
\end{gathered} \tag{2.1}$$

In the following, we denote the radius of full starlikeness and full convexity (of order α) by r_s and r_c , respectively.

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Theorem 2.1. *Let* $f_j = h_j + \overline{g_j} \in \mathcal{H},\ j = 1, 2,$ *where*

$$h_j = z + \sum_{n=2}^{\infty} a_{j_n} z^n, \quad g_j = \sum_{n=2}^{\infty} b_{j_n} z^n$$

with

$$\begin{aligned} |a_{1_n}| \le \frac{n+1}{2}, \quad |b_{1_n}| \le \frac{n-1}{2}, \\ |a_{2_n}| \le \tfrac{1}{6}(2n+1)(n+1) \quad \text{and} \quad |b_{2_n}| \le \tfrac{1}{6}(2n-1)(n-1) \end{aligned} \tag{2.2}$$

for $n \ge 2$*. Then for* $F = (1-t)f_1 + tf_2$, $t \in (0, 1]$,

(1) *the radius of full starlikeness of order* α *is* r_s*, where* $r_s = r_s(\alpha, t)$ *is the unique real root of the equation*

$$2(1-\alpha)(1-r)^4 - (1-t)[(1+r)(1-r) - \alpha(1-r)^3] - t[(1+r)^2 - \alpha(1-r)^2] = 0 \tag{2.3}$$

in the interval $(0, 1)$*;*

(2) *the univalent radius is* r_u*, where* r_u *is the unique real root of the equation*

$$2(1-r)^4 - (1-t)(1+r)(1-r) - t(1+r)^2 = 0 \tag{2.4}$$

in the interval $(0, 1)$.

Furthermore, all the results are sharp.

Proof. By assumption, we have

$$\begin{aligned} F &= (1-t)f_1 + tf_2 \\ &= z + \sum_{n=2}^{\infty} [(1-t)a_{1_n} + ta_{2_n}]z^n + \overline{\sum_{n=2}^{\infty} [(1-t)b_{1_n} + tb_{2_n}]z^n}. \end{aligned}$$

For $r \in (0, 1)$, it is sufficient to show that $F_r(z) \in \mathcal{FS}_H^*(\alpha)$ in $\mathbb{D}$, where

$$\begin{aligned} F_r(z) &= \frac{F(rz)}{r} \\ &= z + \sum_{n=2}^{\infty} [(1-t)a_{1_n} + ta_{2_n}]r^{n-1}z^n + \overline{\sum_{n=2}^{\infty} [(1-t)b_{1_n} + tb_{2_n}]r^{n-1}z^n}. \end{aligned} \tag{2.5}$$

Consider the sum

$$S = \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |(1-t)a_{1_n} + ta_{2_n}| r^{n-1} + \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha} |(1-t)b_{1_n} + tb_{2_n}| r^{n-1}.$$

According to Lemma 1.2, it is enough to show that $S \le 1$. Considering conditions (2.2), we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} \left[(1-t)\frac{n+1}{2} + \frac{1}{6}t(2n+1)(n+1) \right] r^{n-1} \\ &\quad + \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha} \left[(1-t)\frac{n-1}{2} + \frac{1}{6}t(2n-1)(n-1) \right] r^{n-1} \le 1. \end{aligned}$$

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Using the identities (2.1), the last inequality reduces to

$$2(1-\alpha)(1-r)^4-(1-t)[(1+r)(1-r)-\alpha(1-r)^3]$$
$$-t[1-\alpha+2(1+\alpha)r+(1-\alpha)r^2]\geq 0.$$

Thus, $F_r(z)\in\mathcal{FS}_H^*(\alpha)$ for $r\leq r_s$ where r_s is the unique real root of (2.3). The existence and uniqueness of the root will be shown by Lemma 5.1.

Note that for fixed $t\in(0,1]$ the roots of Equation (2.3) in $(0,1)$ are decreasing as a function of $\alpha\in[0,1)$. Consequently, $r_s(\alpha,t)\leq r_s(0,t)$. Therefore, taking $\alpha=0$, Equation (2.3) reduces to (2.4). Then by Lemma 1.2, we know that f is harmonic univalent in $|z|\leq r_u$, where $r_u=r_s(0,t)$.

To prove sharpness, we take

$$\begin{aligned}
f_{1_0}(z)&=z-\sum_{n=2}^{\infty}\frac{n+1}{2}z^n+\overline{\sum_{n=1}^{\infty}\frac{n-1}{2}z^n}\\
&=2z-\frac{z-\frac{1}{2}z^2}{(1-z)^2}+\overline{\frac{\frac{1}{2}z^2}{(1-z)^2}}=2z-L(z),\\
f_{2_0}(z)&=z-\sum_{n=2}^{\infty}\frac{1}{6}(2n+1)(n+1)z^n+\overline{\sum_{n=1}^{\infty}\frac{1}{6}(2n-1)(n-1)z^n}\\
&=2z-\frac{z-\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3}+\overline{\frac{\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3}}.
\end{aligned}$$

Then

$$\begin{aligned}
F_0&=(1-t)f_{1_0}+tf_{2_0}\\
&=2z-(1-t)\frac{z-\frac{1}{2}z^2}{(1-z)^2}-t\frac{z-\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3}+\overline{(1-t)\frac{\frac{1}{2}z^2}{(1-z)^2}+t\frac{\frac{1}{2}z^2+\frac{1}{6}z^3}{(1-z)^3}}\\
&=H_0(z)+\overline{G_0(z)}.
\end{aligned}$$

Direct computation leads to

$$H_0'(r)=2-(1-t)\frac{1}{(1-r)^3}-t\frac{1+r}{(1-r)^4},$$
$$G_0'(r)=(1-t)\frac{r}{(1-r)^3}+t\frac{r+r^2}{(1-r)^4}.$$

Considering Equation (2.4), we have

$$[H_0'(r)-G_0'(r)]|_{r=r_u}=\frac{1}{(1-r)^4}[2(1-r)^4-(1-t)(1+r)(1-r)-t(1+r)^2]\Big|_{r=r_u}=0.$$

Hence,

$$J_{F_0}(r)|_{r=r_u}=[H_0'(r)+G_0'(r)][H_0'(r)-G_0'(r)]|_{r=r_u}=0.$$

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Therefore, in view of Lewy's theorem, the function F_0 is not univalent in $|z| < r$ if $r > r_u$. This shows that r_u is sharp.

Furthermore,

$$\frac{\partial}{\partial\theta}(\arg(F_0(re^{i\theta})))|_{\theta=0} = \frac{rH_0'(r) - rG_0'(r)}{H_0(r) + G_0(r)} = \frac{2tr(1+r) + 1 - r^2 - 2(1-r)^4}{tr(1-r)^2 + (1-r)^3 - 2(1-r)^4}. \quad (2.6)$$

At the same time, from Equation (2.3), we have

$$\alpha = \frac{2tr(1+r) + 1 - r^2 - 2(1-r)^4}{tr(1-r)^2 + (1-r)^3 - 2(1-r)^4}. \quad (2.7)$$

Thus it follows form (2.6) and (2.7) that

$$\frac{\partial}{\partial\theta}(\arg(F_0(re^{i\theta})))|_{\theta=0, r=r_s(\alpha)} = \alpha.$$

This shows that the bound r_s is the best possible. □

Theorem 2.2. *Under the hypothesis of Theorem 2.1, for $t \in (0,1]$, $F = (1-t)f_1 + tf_2$ is fully convex of order α in $|z| < r_c$, where $r_c = r_c(\alpha, t)$ is the unique real root of the equation*

$$\begin{aligned} 2(1-\alpha)(1-r)^5 - (1-t)[(1+4r+r^2)(1-r) - \alpha(1-r)^3] \\ - t(1+r)[1-\alpha+(6+2\alpha)r + (1-\alpha)r^2] = 0 \end{aligned} \quad (2.8)$$

in the interval $(0,1)$. *Moreover, the result is sharp.*

Proof. Let $r \in (0,1)$, it is sufficient to show that $F_r(z) \in \mathcal{FK}_H^*(\alpha)$ in $\mathbb{D}$, where $F_r(z)$ is defined by (2.5).

Consider the sum

$$S = \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|(1-t)a_{1_n} + ta_{2_n}|r^{n-1} + \sum_{n=2}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|(1-t)b_{1_n} + tb_{2_n}|r^{n-1}.$$

According to Lemma 1.1, it is enough to show $S \le 1$. Therefore, considering assumption (2.2), we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}\left[(1-t)\frac{n+1}{2} + \frac{1}{6}t(2n+1)(n+1)\right]r^{n-1} \\ &\quad + \sum_{n=2}^{\infty} \frac{n(n+\alpha)}{1-\alpha}\left[(1-t)\frac{n-1}{2} + \frac{1}{6}t(2n-1)(n-1)\right]r^{n-1} \le 1. \end{aligned}$$

Using identities (2.1), the last inequality reduces to

$$\begin{aligned} 2(1-\alpha)(1-r)^5 - (1-t)[(1+4r+r^2)(1-r) - \alpha(1-r)^3] \\ - t(1+r)[1-\alpha+(6+2\alpha)r + (1-\alpha)r^2] \ge 0. \end{aligned}$$

Therefore, $F_r(z) \in \mathcal{FK}_H^*(\alpha)$ for $r \le r_c$, where r_c is the unique real root of (2.8) in $(0, 1)$. The existence and uniqueness of r_c will be shown by Lemma 5.2.

To prove sharpness, we take

$$\begin{aligned} f_{1_0}(z) &= z - \sum_{n=2}^{\infty} \frac{n+1}{2} z^n - \overline{\sum_{n=1}^{\infty} \frac{n-1}{2} z^n} \\ &= 2z - \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \overline{\frac{\frac{1}{2}z^2}{(1-z)^2}}, \\ f_{2_0}(z) &= z - \sum_{n=2}^{\infty} \frac{1}{6}(2n+1)(n+1)z^n - \overline{\sum_{n=1}^{\infty} \frac{1}{6}(2n-1)(n-1)z^n} \\ &= 2z - \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} - \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}} = 2z - K(z). \end{aligned}$$

Then

$$\begin{aligned} F_0 &= (1-t)f_{1_0} + t f_{2_0} \\ &= 2z - (1-t)\frac{z - \frac{1}{2}z^2}{(1-z)^2} - t\frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} - \overline{(1-t)\frac{\frac{1}{2}z^2}{(1-z)^2} + t\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}} \\ &= H_0(z) + \overline{G_0(z)}. \end{aligned}$$

By direct computation, we have

$$\begin{aligned} &\frac{\partial}{\partial\theta}\Big(\arg\Big(\frac{\partial}{\partial\theta}F_0(re^{i\theta})\Big)\Big)\Big|_{\theta=0} \\ &\quad = \frac{H_0'(r) + G_0'(r) + r(H_0''(r) + G_0''(r))}{H_0' - G_0'(r)} \\ &\quad = \frac{2(1-r)^5 - (1-t)(1-r)(1+4r+r^2) - t(1+r)(1+6r+r^2)}{2(1-r)^5 - (1-t)(1-r)^3 - t(1+r)(1-r)^2}. \end{aligned} \tag{2.9}$$

Meanwhile, from Equation (2.8), we have

$$\alpha = \frac{2(1-r)^5 - (1-t)(1-r)(1+4r+r^2) - t(1+r)(1+6r+r^2)}{2(1-r)^5 - (1-t)(1-r)^3 - t(1+r)(1-r)^2}. \tag{2.10}$$

Thus, from (2.9) and (2.10), we have

$$\frac{\partial}{\partial\theta}\Big(\arg\Big(\frac{\partial}{\partial\theta}F_0(re^{i\theta})\Big)\Big)\Big|_{\theta=0, r=r_c} = \alpha.$$

This shows that the bound r_c given by Equation (2.8) is sharp. □

From Theorems 2.1 and 2.2 we have the following corollary.

Corollary 2.3. *Let $f_1 \in \mathcal{K}_H^o$ and $f_2 \in C_H^o$. Then for $F = (1-t)f_1 + tf_2$, $t \in (0, 1]$,*

(1) *the univalent radius is r_u, where r_u is the real root of the Equation (2.4) in the interval $(0, 1)$;*

(2) *the radius of full starlikeness of order α is r_s, where $r_s = r_s(\alpha, t)$ is the real root of Equation (2.3) in the interval $(0, 1)$;*

(3) *the radius of full convexity of order α is r_c, where $r_c = r_c(\alpha, t)$ is the real root of Equation (2.8) in the interval $(0, 1)$.*

Furthermore, all the results are sharp.

Remark 2.4. If $t = 0$, then Theorems 2.1 and 2.2 reduce to [15, Theorems 3.5 and 3.7], respectively. If $t = 1$, then Theorems 2.1 and 2.2 reduce to [15, Theorems 3.1 and 3.3], respectively.

3. Radius of convolution of harmonic mappings

Theorem 3.1. *Let $f = h + \overline{g} \in \mathcal{H}$ be given by (1.1) with*

$$\begin{aligned} |a_n| &\le \tfrac{1}{36}(2n+1)^2(n+1)^2, \quad n \ge 2, \\ |b_n| &\le \tfrac{1}{36}(2n-1)^2(n-1)^2, \quad n \ge 1. \end{aligned} \tag{3.1}$$

Then, for f,

(1) *the radius of full starlikeness of order α is r_s, where $r_s = r_s(\alpha)$ is the real root of the equation*

$$\begin{aligned} &36(1-\alpha)(1-r)^6 - 4(1 + 26r + 66r^2 + 26r^3 + r^4) \\ &\quad - (13 - 12\alpha)(1 + 4r + r^2)(1-r)^2 - (1 - 6\alpha)(1-r)^4 = 0 \end{aligned} \tag{3.2}$$

in the interval $(0, 1)$;

(2) *the univalent radius is r_u, where r_u is the real root of the equation*

$$36(1-r)^6 - 4(1 + 26r + 66r^2 + 26r^3 + r^4) - 13(1 + 4r + r^2)(1-r)^2 - (1-r)^4 = 0 \tag{3.3}$$

in the interval $(0, 1)$.

Furthermore, all the results are sharp.

Proof. The proof is similar to that of Theorem 2.1.

For $r \in (0, 1)$, it is sufficient to show that $f_r(z) \in \mathcal{FS}_H^*(\alpha)$ in $\mathbb{D}$, where

$$f_r(z) = \frac{f(rz)}{r} = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=1}^{\infty} b_n r^{n-1} z^n}. \tag{3.4}$$

Consider the sum

$$S = \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| r^{n-1} + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n| r^{n-1}.$$

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According to Lemma 1.2, it is enough to show that $S \le 1$. Considering conditions (3.1), we have

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{36(1-\alpha)}(2n+1)^2(n+1)^2 r^{n-1} + \sum_{n=1}^{\infty} \frac{n+\alpha}{36(1-\alpha)}(2n-1)^2(n-1)^2 r^{n-1} \le 1.$$

Using identities (2.1), the last inequality reduces to

$$\begin{aligned}&36(1-\alpha)(1-r)^6 - 4(1+26r+66r^2+26r^3+r^4)\\ &\quad - (13-12\alpha)(1+4r+r^2)(1-r)^2 - (1-6\alpha)(1-r)^4 \ge 0.\end{aligned}$$

Thus, $F_r(z) \in \mathcal{FS}_H^*(\alpha)$ for $r \le r_s$, where r_s is the real root of (3.2).

Taking $\alpha = 0$, Equation (3.2) reduces to (3.3). In other words, $r_u = r_s(0)$.

To prove sharpness, we take the function $f_0 = h_0 + \overline{g_0}$ with

$$h_0(z) = 2z - \sum_{n=1}^{\infty} \frac{1}{36}(2n+1)^2(n+1)^2 z^n,$$

$$g_0(z) = \sum_{n=2}^{\infty} \frac{1}{36}(2n-1)^2(n-1)^2 z^n.$$

Then direct computation and Equation (3.3) lead to

$$[h_0'(r) - g_0'(r)]|_{r=r_u} = \frac{1}{18(1-r)^6} M(r)\bigg|_{r=r_u} = 0,$$

where $M(r) = 36(1-r)^6 - 4(1+26r+66r^2+26r^3+r^4) - 13(1+4r+r^2)(1-r)^2 - (1-r)^4$. Hence,

$$J_{f_0}(r)|_{r=r_u} = [h_0'(r) + g_0'(r)][h_0'(r) - g_0'(r)]|_{r=r_u} = 0.$$

Therefore the function f_0 is not univalent in $|z| < r$ if $r > r_u$. This shows that r_u is sharp.

Furthermore,

$$\begin{aligned}\frac{\partial}{\partial\theta}(\arg(f_0(re^{i\theta})))|_{\theta=0} &= \frac{rh_0'(r) - rg_0'(r)}{h_0(r) + g_0(r)}\\ &= \frac{M(r)}{36(1-r)^6 - 6(1-r)^4 - 12(1+4r+r^2)(1-r)^2}.\end{aligned} \tag{3.5}$$

At the same time, from Equation (3.2), we have

$$\alpha = \frac{M(r)}{36(1-r)^6 - 6(1-r)^4 - 12(1+4r+r^2)(1-r)^2}. \tag{3.6}$$

Thus it follows form (3.5) and (3.6) that

$$\frac{\partial}{\partial\theta}(\arg(f_0(re^{i\theta})))|_{\theta=0, r=r_s(\alpha)} = \alpha.$$

This shows that the bound r_s is the best possible. □

THEOREM 3.2. *Under the hypothesis of Theorem 3.1, for* $\alpha \in [0, 1)$, $f = h + \bar{g}$ *is fully convex of order* α *in* $|z| < r_c$, *where* $r_c = r_c(\alpha)$ *is the real root of the equation*

$$\begin{aligned} &36(1-\alpha)(1-r)^7 - 4(1+57r+302r^2+302r^3+57r^4+r^5) \\ &\quad -(13-12\alpha)(1+r)(1+10r+r^2)(1-r)^2 - (1-6\alpha)(1+r)(1-r)^4 = 0 \end{aligned} \tag{3.7}$$

in the interval $(0, 1)$. *Moreover, the result is sharp for* $\alpha \in [0, 1)$.

PROOF. The proof is similar to that of Theorem 2.2.

For $r \in (0, 1)$, it is sufficient to show that $f_r(z) \in \mathcal{FK}_H^*(\alpha)$ in $\mathbb{D}$, where $f_r(z)$ is given by (3.4). Lemma 1.1 and assumption (3.1) lead to

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{36(1-\alpha)}(2n+1)^2(n+1)^2 r^{n-1} + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{36(1-\alpha)}(2n-1)^2(n-1)^2 r^{n-1} \leq 1.$$

Using identities (2.1), the last inequality reduces to

$$\begin{aligned} &36(1-\alpha)(1-r)^7 - 4(1+57r+302r^2+302r^3+57r^4+r^5) \\ &\quad -(13-12\alpha)(1+r)(1+10r+r^2)(1-r)^2 - (1-6\alpha)(1+r)(1-r)^4 \geq 0. \end{aligned}$$

Thus, $f_r(z) \in \mathcal{FK}_H^*(\alpha)$ for $r \leq r_c$, where r_c is the real root of (3.7).

To prove sharpness, we take $f_0 = h_0 + \overline{g_0}$, where

$$h_0(z) = 2z - \sum_{n=1}^{\infty} \frac{1}{36}(2n+1)^2(n+1)^2 z^n,$$

$$g_0(z) = -\sum_{n=2}^{\infty} \frac{1}{36}(2n-1)^2(n-1)^2 z^n.$$

By direct computation, we have

$$\frac{\partial}{\partial\theta}\Big(\arg\Big(\frac{\partial}{\partial\theta} f_0(re^{i\theta})\Big)\Big)\Big|_{\theta=0} = \frac{h_0'(r) + g_0'(r) + r(h_0''(r) + g_0''(r))}{h_0' - g_0'(r)} = \frac{H(r)}{G(r)}. \tag{3.8}$$

where $H(r) = 36(1-r)^7 - (1+r)(1-r)^4 - 13(1+r)(1+10r+r^2)(1-r)^2 - 4(1+57r+302r^2+302r^3+57r^4+r^5)$ and $G(r) = 36(1-r)^7 - 6(1+r)(1-r)^4 - 12(1+r)(1+10r+r^2)(1-r)^2$. Meanwhile, from Equation (3.7), we have

$$\alpha = \frac{H(r)}{G(r)}. \tag{3.9}$$

Thus, from (3.8) and (3.9), we have

$$\frac{\partial}{\partial\theta}\Big(\arg\Big(\frac{\partial}{\partial\theta} f_0(re^{i\theta})\Big)\Big)\Big|_{\theta=0, r=r_c} = \alpha.$$

This shows that the bound r_c given by Equation (3.7) is sharp. □

From Theorems 3.1 and 3.2 we have the following corollary.

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COROLLARY 3.3. *Let $f_1, f_2 \in C_H^o$ and $\alpha \in [0,1)$. Then for $F = f_1 * f_2$,*

(1) *the univalent radius r_u is the real root of the Equation (3.3) in the interval $(0,1)$;*

(2) *the radius of full starlikeness of order α is r_s, where $r_s = r_s(\alpha)$ is the real root of Equation (3.2) in the interval $(0,1)$;*

(3) *the radius of full convexity of order α is r_c, where $r_c = r_c(\alpha)$ is the real root of Equation (3.7) in the interval $(0,1)$.*

4. Univalent radius and Bloch constant of convolution of bounded harmonic mappings

THEOREM 4.1. *Let $f_j \in \mathcal{B}_H^{M_j}$, $j = 1,2$. Then for $F = f_1 * f_2$, the radius of close-to-convexity (univalence) and full starlikeness is r_0, where*

$$r_0 = 1 - \sqrt{\frac{8(M_1^2 + M_2^2)}{\pi^2 + 8(M_1^2 + M_2^2)}}. \tag{4.1}$$

Furthermore, $F(\mathbb{D}_{r_0})$ contains a univalent disk of radius at least

$$R_0 = r_0 - \frac{8(M_1^2 + M_2^2) r_0^2}{\pi^2 (1 - r_0)}, \tag{4.2}$$

where $\mathbb{D}_{r_0} = \{z : |z| < r_0\}$.

PROOF. Let

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{j_n} z^n + \overline{\sum_{n=2}^{\infty} b_{j_n} z^n}, \quad j = 1,2.$$

Then

$$f_1 * f_2 = z + \sum_{n=2}^{\infty} a_{1_n} a_{2_n} z^n + \overline{\sum_{n=2}^{\infty} b_{1_n} b_{2_n} z^n}.$$

Because of $f_j \in \mathcal{B}_H^{M_j}$, $j = 1,2$, according to [3], we can obtain the sharp estimates

$$|a_{j_n}| + |b_{j_n}| \le \frac{4M_j}{\pi}$$

for any $n \ge 1$ and $j = 1,2$. Then it follows that

$$\begin{aligned}
|a_{1_n} a_{2_n}| + |b_{1_n} b_{2_n}| &\le \frac{|a_{1_n}|^2 + |a_{2_n}|^2}{2} + \frac{|b_{1_n}|^2 + |b_{2_n}|^2}{2} \\
&= \frac{1}{2}[|a_{1_n}|^2 + |b_{1_n}|^2 + |a_{2_n}|^2 + |b_{2_n}|^2] \le \frac{1}{2}[(|a_{1_n}| + |b_{1_n}|)^2 + (|a_{2_n}| + |b_{2_n}|)^2] \\
&\le \frac{1}{2}\Big[\Big(\frac{4M_1}{\pi}\Big)^2 + \Big(\frac{4M_1}{\pi}\Big)^2\Big] = \frac{8(M_1^2 + M_2^2)}{\pi^2}
\end{aligned}$$

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for any $n \geq 1$. Noting that $b_{1_1} b_{2_1} = 0$, by Lemma 1.3 with $C = 8(M_1^2 + M_2^2)/\pi^2$ and Remark 1.4, we conclude that, for $f_1 * f_2$, the radius of close-to-convexity and full starlikeness is

$$r_0 = 1 - \sqrt{\frac{C}{C+1}} = 1 - \sqrt{\frac{8(M_1^2 + M_2^2)}{\pi^2 + 8(M_1^2 + M_2^2)}}.$$

In particular, the radius of univalence of $f_1 * f_2$ is r_0.

Furthermore, for $|z| = r_0$, we have

$$\begin{aligned}
|f_1 * f_2(z)| &= \left| z + \sum_{n=2}^{\infty} a_{1_n} a_{2_n} z^n + \overline{\sum_{n=2}^{\infty} b_{1_n} b_{2_n} z^n} \right| \\
&\geq |z| - \left| \sum_{n=2}^{\infty} (a_{1_n} a_{2_n} z^n + \overline{b_{1_n} b_{2_n} z^n}) \right| \\
&\geq r_0 - \sum_{n=2}^{\infty} (|a_{1_n} a_{2_n}| + |b_{1_n} b_{2_n}|) r_0^n \\
&\geq r_0 - \frac{8(M_1^2 + M_2^2)}{\pi^2} \sum_{n=2}^{\infty} r_0^n \\
&\geq r_0 - \frac{8(M_1^2 + M_2^2) r_0^2}{\pi^2 (1 - r_0)}. \qquad \square
\end{aligned}$$

REMARK 4.2. (1) Equation (4.1) reduces to

$$\frac{8(M_1^2 + M_2^2)}{\pi^2} = \frac{(1 - r_0)^2}{1 - (1 - r_0)^2}. \tag{4.3}$$

Substituting (4.3) into (4.2), we have

$$R_0 = \frac{r_0}{2 - r_0}.$$

This gives the relationship between r_0 and R_0, directly.

(2) Let r_j be the univalent radius of f_j for $j = 1, 2$. Then according to [11, Theorem 1.7], we have

$$r_j = 1 - \sqrt{\frac{4M_j}{\pi + 4M_j}}.$$

This reduces to

$$M_j = \frac{\pi (1 - r_j)^2}{4[1 - (1 - r_j)^2]}. \tag{4.4}$$

Substituting (4.4) into (4.3), we have

$$\frac{(1 - r_0)^2}{1 - (1 - r_0)^2} = \frac{1}{2} \left[\frac{(1 - r_1)^4}{[1 - (1 - r_1)^2]^2} + \frac{(1 - r_2)^4}{[1 - (1 - r_2)^2]^2} \right].$$

The last equation shows the relationship between r_1, r_2 and r_0, directly.

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5. Two lemmas

In this section, we will prove the existence and uniqueness of the roots of Equations (2.3) and (2.8) in detail in the following two lemmas.

LEMMA 5.1. *For any $\alpha \in [0,1)$ and $t \in (0,1]$, the equation*

$$2(1-\alpha)(1-r)^4 - (1-t)[(1+r)(1-r) - \alpha(1-r)^3] - t[(1+r)^2 - \alpha(1-r)^2] = 0$$

has a unique real root $r = r(\alpha, t)$ in the interval $(0,1)$.

PROOF. Let

$$f(r) = 2(1-\alpha)(1-r)^4 - (1-t)[(1+r)(1-r) - \alpha(1-r)^3] - t[(1+r)^2 - \alpha(1-r)^2].$$

Then direct computations lead to

$$f(0) = 1-\alpha > 0, \tag{5.1}$$
$$f(1) = -4t < 0, \tag{5.2}$$
$$f'(r) = g(r) + th(r), \tag{5.3}$$

where $g(r) = -3\alpha(1-r)^2 - 8(1-\alpha)(1-r)^3 + 2r$ and $h(r) = 3\alpha(1-r)^2 - 2(1+2r) - 2\alpha(1-r)$. It follows that

$$f'(0) = -(2-\alpha)t + 5\alpha - 8 < 0, \tag{5.4}$$
$$f'(1) = 2 - 6t, \tag{5.5}$$
$$f''(1) = 2[1 - t(2-\alpha)], \tag{5.6}$$
$$f'''(r) = -48(1-\alpha)(1-r) - 6\alpha(1-t) < 0. \tag{5.7}$$

We divide the proof into two subclasses.

Subclass 1. $t \in (0, \frac{1}{3})$. Equations (5.5) and (5.6) lead to

$$f'(1) > 0 \tag{5.8}$$

and

$$f''(1) > 0, \tag{5.9}$$

for any $\alpha \in [0,1)$ and $t \in (0, \frac{1}{3})$, respectively. Thus, inequalities (5.7) and (5.9) lead to

$$f''(r) > 0 \tag{5.10}$$

for all $r \in (0,1)$. From inequalities (5.4), (5.8) and (5.10), it is easy to see that there exists a unique $\xi_1 \in (0,1)$, such that $f'(r) < 0$ for $r \in (0, \xi_1)$ and $f'(r) > 0$ for $r \in (\xi_1, 1)$. Therefore, it follows from inequalities (5.1) and (5.2), together with the monotonicity of f, that there exists a unique $\xi_2 \in (0, \xi_1)$ such that $f(\xi_2) = 0$.

Subclass 2. $t \in [\frac{1}{3}, 1]$. Since $h(r) = 3\alpha(1-r)^2 - 2(1+2r) - 2\alpha(1-r) = -2 + \alpha - 4r - 4\alpha r + 3\alpha r^2 < 0$, Equation (5.3) leads to

$$f'(r) \le g(r) + \tfrac{1}{3}h(r) \tag{5.11}$$

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for all $t \in [\frac{1}{3}, 1]$. Direct computations yields

$$g(r) + \tfrac{1}{3}h(r) = \tfrac{2}{3}(1 - r)[-12(1 - \alpha)(1 - r)^2 - (4 - 3r)\alpha - 1] < 0. \tag{5.12}$$

Therefore, it follows form inequalities (5.1), (5.2), (5.11) and (5.12) that there exists a unique $\xi_3 \in (0, 1)$ such that $f(\xi_3) = 0$. $\square$

LEMMA 5.2. *For any $\alpha \in [0, 1)$ and $t \in (0, 1]$, the equation*

$$\begin{aligned} 2(1 - \alpha)(1 - r)^5 - (1 - t)[(1 + 4r + r^2)(1 - r) - \alpha(1 - r)^3] \\ - t(1 + r)[1 - \alpha + (6 + 2\alpha)r + (1 - \alpha)r^2] = 0 \end{aligned}$$

has a unique real root $r = r(\alpha, t)$ in the interval $(0, 1)$.

PROOF. Let

$$\begin{aligned} f(r) = 2(1 - \alpha)(1 - r)^5 - (1 - t)[(1 + 4r + r^2)(1 - r) - \alpha(1 - r)^3] \\ - t(1 + r)[1 - \alpha + (6 + 2\alpha)r + (1 - \alpha)r^2]. \end{aligned}$$

Direct computations lead to

$$f(0) = 1 - \alpha > 0, \tag{5.13}$$

$$f(1) = -16t < 0, \tag{5.14}$$

$$f'(r) = g(r) + t\,h(r), \tag{5.15}$$

where $g(r) = -10(1 - \alpha)(1 - r)^4 - [3\alpha(1 - r)^2 - 1 - 4r - r^2 + 2(1 - r)(2 + r)]$ and

$$h(r) = 2[-2 + \alpha - 10r - 4\alpha r - 3r^2 + 3\alpha r^2] < 0. \tag{5.16}$$

It follows that

$$f'(1) = 5 - 30t,$$

$$f''(t) = G(r) + tH(r), \tag{5.17}$$

where $G(r) = 8 - 2(1 - r) + 6\alpha(1 - r) + 40(1 - \alpha)(1 - r)^3 + 4r$ and

$$H(r) = 4[-5 - 3r + \alpha(-2 + 3r)] < 0. \tag{5.18}$$

We divide the proof into two subclasses.

Subclass 1. $t \in (0, \frac{1}{5})$. Equation (5.17) and inequality (5.18) imply that

$$f''(r) > G(r) + \tfrac{1}{5}H(r) \tag{5.19}$$

holds for all $t \in (0, \frac{1}{5})$. Direct computations lead to

$$G(r) + \tfrac{1}{5}H(r) = \tfrac{2}{5}[10 + 6\alpha - (1 - \alpha)\phi(r)], \tag{5.20}$$

where $\phi(r) = 100r^3 - 300r^2 + 291r - 95$. It is easy to show that

$$\phi(r) \le \phi\left(1 - \frac{\sqrt{3}}{10}\right) < -3 \tag{5.21}$$

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for all $r \in (0, 1)$. Thus, it follows from (5.19)–(5.21) that

$$f''(r) > \tfrac{2}{5}[10 + 6\alpha + 3(1 - \alpha)] > 0 \tag{5.22}$$

holds for all $t \in (0, \frac{1}{5})$. Therefore, from (5.13), (5.14) and (5.22), we know there exists a unique $\xi_1 \in (0, 1)$ such that $f(\xi_1) = 0$.

Subclass 2. $t \in [\frac{1}{5}, 1]$. Equation (5.15) and inequality (5.16) imply that

$$f'(r) \le g(r) + \tfrac{1}{5}h(r) \tag{5.23}$$

for all $t \in [\frac{1}{5}, 1]$. By direct computation,

$$g(r) + \tfrac{1}{5}h(r) = -\tfrac{1}{5}(1 - r)[22(1 + \alpha) + (1 - \alpha)\psi(r)], \tag{5.24}$$

where $\psi(r) = -50r^3 + 150r^2 - 141r + 47$. It is easy to show that

$$\psi(r) \ge \psi\left(1 - \frac{\sqrt{6}}{10}\right) = 6 - \frac{\sqrt{6}}{2} > 0 \tag{5.25}$$

for all $r \in (0, 1)$. Thus, it follows from (5.23)–(5.25) that

$$f'(r) \le -\frac{1}{5}(1 - r)\left[22(1 + \alpha) + (1 - \alpha)\psi\left(1 - \frac{\sqrt{2}}{10}\right)\right] < 0, \tag{5.26}$$

for all $t \in [\frac{1}{5}, 1]$. Therefore, from inequalities (5.13), (5.14) and (5.26), we know there exists a unique $\xi_2 \in (0, 1)$ such that $f(\xi_2) = 0$.

□

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