A CONVOLUTION SEMIGROUP OF MODULAR FUNCTIONS

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(Received 8 December 1965)

1. Introduction

Let S be a compact topological semigroup, and let $\mathfrak{S}$ be the collection of all normalized non-negative Borel measures on S. It is well-known that $\mathfrak{S}$, under convolution and the topology induced by the weak-star topology on the dual of the Banach space $C(S)$ of all complex valued continuous functions on S, forms a compact topological semigroup which is known as the convolution semigroup of measures (see for instance, Glicksberg [3], Collins [1], Schwarz [5] and the author [4]). Professor A. D. Wallace asked if the process of forming the convolution semigroup of measures might be generalized to a more general class of set functions, the so-called "modular functions." The purpose of the present note is to settle this question in the affirmative under a slight restriction. Before we are able to state the Wallace problem precisely, some preliminaries are necessary.

2. Preliminaries

Let S in this note be always a compact topological semigroup, and let $\mathscr{F}$ be the family of all closed subsets of S. By a ***modular function*** m on S is meant a real-valued set function m defined on $\mathscr{F}$ such that

$$m(A \cup B)+m(A \cap B) = m(A)+m(B),$$

for any A and B in $\mathscr{F}$. A modular function is said to be normalized if $m(S) = 1$ and $m(\square) = 0$, where $\square$ denotes the empty set.

DEFINITION 1. A modular function m on S is ***regular*** if and only if, for any F in $\mathscr{F}$ and any $\varepsilon > 0$, there is an open subset V of S containing F such that $0 \leqq m(E)-m(F) < \varepsilon$ for any E in $\mathscr{F}$ such that $F \subset E \subset V^-$, where $^-$ denotes the topological closure operator.

Let us write $S\#$ for the family of all normalized regular modular functions on S. We are now in position to restate Wallace's problem more clearly: is it possible to define a "convolution" and a topology for $S\#$ in such a way that $S\#$ becomes a compact topological semigroup?

We are able to answer this for the "isotonic" modular functions. A modular function is *isotone* if and only if, $A \supset B$ in $\mathscr{F}$ implies $m(A) \geqq m(B)$.

Let us agree, henceforth, that $S^{\#}$ *is the collection of all normalized regular isotonic modular functions on* S. An example of such a function is a normalized regular Borel measure on S restricted to $\mathscr{F}$. It is rather peculiar that the modular functions obtained in this way turn out to be all of $S^{\#}$, and furthermore there is a *unique* extension of an element of $S^{\#}$ to an element of $\tilde{S}$.

We need some additional symbolism. Let $\mathscr{B}$ denote the family of all Borel sets in S, and let $\mathscr{V}$ be the collection of all open sets in S. We write

$$m_0(V) = 1 - m(S \backslash V), \tag{1}$$

for each m in $S^{\#}$ and for all V in $\mathscr{V}$. We then define a transformation ^{t} on $S^{\#}$ by

$$m^t(B) = \inf \{m_0(V) : B \subset V \, \varepsilon \mathscr{V}\} \tag{2}$$

for each m in $S^{\#}$ and for all Borel sets B in $\mathscr{B}$.

3. A convolution semigroup of modular functions

THEOREM A. *The transformation* ^{t} *takes* $S^{\#}$ *in one-to-one fashion onto* $\tilde{S}$. *Indeed, we have the following relations:*

$$m^t|\mathscr{F} = m \text{ and } (\mu|\mathscr{F})^t = \mu$$

for every m *in* $S^{\#}$ *and for every* μ *in* $\tilde{S}$.

We divide the proof of this theorem into the following steps.

LEMMA 1. *If* U, V *are in* $\mathscr{V}$ *and if* F *is in* $\mathscr{F}$ *such that* $U \subset F \subset V$. *Then*

$$m_0(U) \leqq m(F) \leqq m_0(V) \tag{3}$$

for any m *in* $S^{\#}$.

PROOF: This is straightforward from (1).

LEMMA 2. *The set function* m_0 *is normalized, isotonic, countably additive on* $\mathscr{V}$ *for each* m *in* $S^{\#}$.

PROOF. It is clear that $m_0(\square) = 0$, $m_0(S) = 1$ and that

$$U = U^0 \supset V^0 = V$$

implies $m_0(U) \geqq m_0(V)$. Furthermore, m_0 is finitely additive; for if U and V are any disjoint open sets, then

$$(4)\quad \begin{aligned} m_0(U\cup V) &= 1-m((S\backslash U)\cap(S\backslash V)) \\ &= 1-[m(S\backslash U)+m(S\backslash V)-m((S\backslash U)\cup(S\backslash V))] \\ &= [1-m(S\backslash U)]+[1-m(S\backslash V)] \\ &= m_0(U)+m_0(V); \end{aligned}$$

for, $S = (S\backslash U)\cup(S\backslash V)$ and m is a normalized modular function.

Now we show that for any sequence $\{V_i: i \geqq 1\}$ of open sets V_i,

$$(5)\quad \sum_{i\geqq 1} m_0(V_i) \geqq m_0(\bigcup_{i\geqq 1} V_i).$$

For any $\varepsilon > 0$, since $S\backslash(\bigcup_{i\geqq 1} V_i) = \bigcap_{i\geqq 1}(S\backslash V_i) \in \mathscr{F}$ and by regularity of m, there is an open set W containing $\bigcap_{i\geqq 1}(S\backslash V_i)$ such that

$$(6)\quad m(\bigcap_{i\geqq 1}(S\backslash V_i))+\varepsilon \geqq m(W^-).$$

Compactness of S, then, yields a positive integer n such that

$$(7)\quad W \supset \bigcap_{n\geqq i\geqq 1}(S\backslash V_i).$$

It follows then from (1), (3), (4), (6), and (7) that

$$\sum_{n\geqq i\geqq 1} m_0(V_i)+\varepsilon \geqq m_0(\bigcup_{i\geqq 1} V_i)$$

and hence

$$\sum_{i\geqq 1} m_0(V_i)+\varepsilon \geqq m_0(\bigcup_{i\geqq 1} V_i).$$

Since ε was arbitrary, (5) is thus proved.

Finally, with an additional assumption that $\{V_i : i \geqq 1\}$ is disjoint, we have to show that

$$(8)\quad \sum_{i\geqq 1} m_0(V_i) \leqq m_0(\bigcup_{i\geqq 1} V_i).$$

From (3) and (4), we have

$$\sum_{n\geqq i\geqq 1} m_0(V_i) = m_0(\bigcup_{n\geqq i\geqq 1} V_i) \leqq m_0(\bigcup_{i\geqq 1} V_i)$$

for every positive integer n. Therefore (8) holds, and hence the lemma is proved.

LEMMA 3. *If $m \in S^\#$ then m^t is a countably additive measure on $\mathscr{B}$.*

PROOF. From (2) we have $m^t(\Box) = 0$. Let $\{B_i : i \geqq 1\}$ be a sequence of Borel sets such that $\bigcup\{B_i : i \geqq 1\}$ is also a Borel set. Then for any $\varepsilon > 0$ and for each positive integer i, there is an open set $V_i \supset B_i$ such that

$$m_0(V_i) \leqq m^t(B_i) + \frac{\varepsilon}{2^i}.$$

https://doi.org/10.1017/S1446788700004821 Published online by Cambridge University Press

Therefore, from (4) and (5), we have

$$m^t(\bigcup_{i\geqq 1} B_i) \leqq m_0(\bigcup_{i\geqq 1} V_i) \leqq \sum_{i\geqq 1} m_0(V_i) \leqq \sum_{i\geqq 1} m^t(B_i)+\varepsilon,$$

and thus,

$$m^t(\bigcup_{i\geqq 1} B_i) \leqq \sum_{i\geqq 1} m^t(B_i). \tag{9}$$

These together with the isotony of m^t show that m^t is an outer measure on the σ-algebra $\mathscr{B}$.

Since m is regular, by some standard computations, all sets in $\mathscr{F}$ are m^t-sets (= outer measurable sets). Now a celebrated theorem of Carathéodory (see for instance [2, p. 134]), tells us that all Borel sets are m^t-sets upon which m^t is countably additive.

Lemma 4. *If* $m \in S\#$ *then* m^t *is regular.*

Proof. This follows from the fact that

$$\begin{aligned} m^t(F) &= \inf\{m_0(V) : F \subset V \in \mathscr{V}\} \\ &= \inf\{m^t(V) : F \subset V \in \mathscr{V}\}. \end{aligned}$$

Lemma 5. *If* $m \in S\#$ *then* $m^t|\mathscr{F} = m$.

Proof. Since m is regular, for any F in $\mathscr{F}$ and any $\varepsilon > 0$ there is an open set V such that

$$F \subset V \text{ and } m(V^-) \leqq m(F)+\varepsilon.$$

Using (2) and (3) several times we arrive at

$$m(F) \leqq m^t(F) \leqq m_0(V) \leqq m(V^-) \leqq m(F)+\varepsilon$$

and see $m(F) = m^t(F)$ for all F in $\mathscr{F}$.

Lemma 6. *If* $\mu \in \tilde{S}$ *then* $(\mu|\mathscr{F})^t = \mu$.

Proof. It is fairly clear that $\mu|\mathscr{F}$ belongs to $S\#$. Let us denote $(\mu|\mathscr{F})^t$ by ν. Then by Lemma 5,

$$\nu(F) = \mu(F)$$

for every closed set F, and hence

$$\nu(V) = \mu(V)$$

for each open set V. We have then, by the regularity of μ and by (2),

$$\nu(B) = \mu(B)$$

for every Borel set B.

The proof of Theorem A is now clear from Lemmas 1-6. We are now ready to state our main theorem.

https://doi.org/10.1017/S1446788700004821 Published online by Cambridge University Press

THEOREM B. *Let S be a compact semigroup. Then the set $S^{\#}$ of all normalized regular isotonic modular functions may be introduced a convolution and a topology in such a way that $S^{\#}$ is topologically isomorphic to $\tilde{S}$.*

PROOF. We define, by virtue of Theorem A, the convolution $*$ on $S^{\#}$ naturally by

$$m * n = (m^t \cdot n^t)|\mathscr{F}$$

for all m, n in $S^{\#}$. Where $\cdot$ in the right hand side means the convolution of measures in the usual sense. Topologize $S^{\#}$ in such a way that a subset $\sum$ is open if and only if $\sum^t = \{\sigma^t : \sigma \in \sum\}$ is open in $\tilde{S}$. Then the mapping $h : S^{\#} \to \tilde{S}$ defined by $h(m) = m^t$ for all m in $S^{\#}$ is an isomorphism as well as a homeomorphism.

The author is grateful to Professor A. D. Wallace for this problem.

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