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Coretraction-fibrations are retractions

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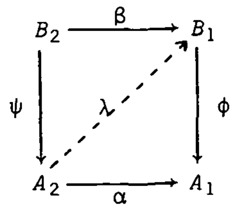
We prove that if C is an abelian category and M is the class of all coretractions, then the class of M-fibrations is the class of all retractions. As a corollary we prove that the class of all retractions is contained in the class of M-fibrations for any M.

1. Introduction

Let C be a fixed abelian category. For a morphism $f : A \to B$ in C, let us write K_f, C_f and I_f for the kernel of f, the cokernel of f and the image of f, respectively. In Theorem 3.3 we prove the result asserted in the title of this paper. Using this theorem and a result of Ringel [5], we show that the class of retractions is the smallest possible class of fibrations. In Theorem 3.6 we characterize the p-fibrations of Hilton [3] in the language of Bauer and Dugundji [1].

2. Obstructions to liftings in commutative squares

We recall some definitions and results from [4], using the same notation. The commutative square



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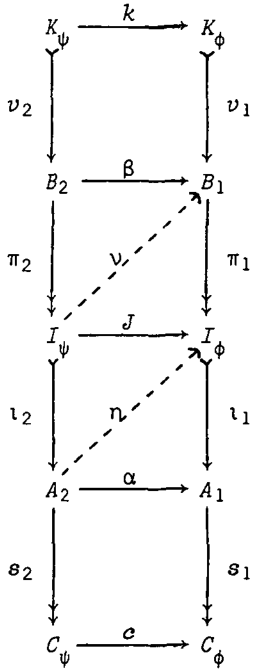
is said to have a lifting if there is a morphism $\lambda : A_2 \to B_1$ with $\phi\lambda = \alpha$ and $\lambda\psi = \beta$. This square induces the following diagram:



$\iota_1\pi_1 = \phi$

$\iota_2\pi_2 = \psi$

A lifting ν exists iff $k = 0$, and a lifting η exists iff $c = 0$.

Suppose $k = 0$ and $c = 0$. Then $J = \pi_1\nu = \eta\iota_2$ and the diagram may be rewritten as

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$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
0 & & K_\phi \\
\downarrow & & \downarrow{\scriptstyle \nu_1} \\
I_\psi & \xrightarrow{\nu} & B_1 \\
{\scriptstyle \iota_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
A_2 & \xrightarrow{\eta} & I_\phi \\
{\scriptstyle s_2}\downarrow & & \downarrow \\
C_\psi & & 0 \\
\downarrow & & \\
0 & &
\end{array}$$

The middle commutative square with ι_2 a monomorphism and π_1 an epimorphism is termed a co-special square.

Assume the commutative square

$$\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle d} \\
B & \xrightarrow{g} & Z
\end{array}$$

is co-special and consider the following diagram

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$$\begin{array}{ccc} & & X = K_d \\ & & \big\downarrow c \\ A & \xrightarrow{\ f\ } & Y \\ a\big\downarrow & & \big\downarrow d \\ B & \xrightarrow{\ g\ } & Z \\ \big\downarrow & & \\ C_a = C & & \end{array}$$

The two short exact sequences above give

$$\begin{array}{rccccccccc} E'g : & 0 & \longrightarrow & X & \xrightarrow{\ u\ } & P & \xrightarrow{\ v\ } & B & \longrightarrow & 0 \\ & & & \big\| & & y\big\downarrow & & g\big\downarrow & & \\ E' : & 0 & \longrightarrow & X & \xrightarrow{\ c\ } & Y & \xrightarrow{\ d\ } & Z & \longrightarrow & 0 \end{array}$$

$$\begin{array}{rccccccccc} E'' : & 0 & \longrightarrow & A & \xrightarrow{\ a\ } & B & \xrightarrow{\ b\ } & C & \longrightarrow & 0 \\ & & & f\big\downarrow & & z\big\downarrow & & \big\| & & \\ fE'' : & 0 & \longrightarrow & Y & \xrightarrow{\ q\ } & Q & \xrightarrow{\ r\ } & C & \longrightarrow & 0 , \end{array}$$

where P is the pullback of g via d and Q is the pushout of f via a. Then $\langle g, -d \rangle : B \oplus Y \to Z$ is an epimorphism with kernel $\{v, y\}$ and $\{a, f\} : A \to B \oplus Y$ is a monomorphism with cokernel $\langle z, -q \rangle$. $\langle g, -d \rangle \{a, f\} = 0$ implies that there exists unique morphisms e and w as shown below,

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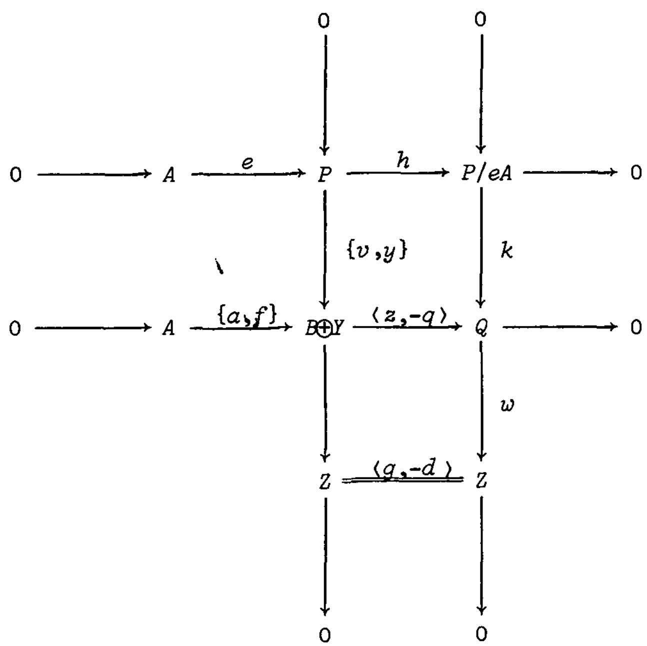

where P = kernel $\langle g, -d\rangle$ and eA = image $\{a, f\}$.
H = kernel $\langle g, -d\rangle$ | image $\{a, f\} = P/eA$ is termed the homology of the given square. It follows that in the above diagram all the squares commute and all the rows and columns are exact.

Consider the following diagram

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$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
G & : & 0 & \longrightarrow & X & \xrightarrow{hu} & H & \xrightarrow{rk} & C & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle k} & & & & \\
fE'' & : & 0 & \longrightarrow & Y & \xrightarrow{q} & Q & \xrightarrow{r} & C & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle w} & & & & \\
 & & & & Z & = & Z & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

Again all the squares commute and all the rows and columns are exact. The following lemma and theorem are proved in Pressman [4]:

LEMMA 2.1. *The short exact sequence* G *splits if and only if there is a lifting for the given co-special square.*

THEOREM 2.2. *The commutative square*

$$
\begin{array}{ccc}
B_2 & \xrightarrow{\beta} & B_1 \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} \\
A_2 & \xrightarrow{\alpha} & A_1
\end{array}
$$

has a lifting if and only if

(i) $k : K_\psi \to K_\phi$ *is zero,*

(ii) $c : C_\psi \to C_\phi$ *is zero, and*

(iii) $H \approx K_\phi \oplus C_\psi$ *, where* H *is the homology of the square.*

We shall also require the following

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LEMMA 2.3. *If* E' *and* E'' *split, then so does* G .

Proof. If E'' splits, then so does fE'' . Let $\bar{q}$ be a left inverse of q in fE'' and let $\bar{c}$ be a left inverse of c in E' . Then $\bar{c}\bar{q}k : H \to X$ and

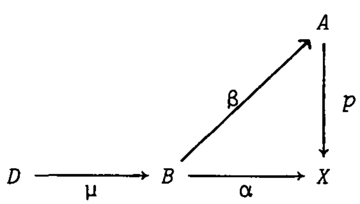
$$\begin{aligned}(-\bar{c}\bar{q}k)(hu) &= -\left(\bar{c}\bar{q}(zv-qy)u\right)\\ &= -\bar{c}\bar{q}zvu + \bar{c}\bar{q}qyu\\ &= 0 \oplus \bar{c}yu\\ &= 1_X .\end{aligned}$$

Thus G splits.

3. Fibrations

We shall now consider some ideas introduced in [1]. There, Bauer and Dugundji defined a concept of fibration so that each class M of morphisms in C determines a concept of fibration in C .

DEFINITION. A morphism $p : A \to X$ in C is called an M-fibration if for each diagram



with $p\beta\mu = \alpha\mu$ and $\mu \in M$, there is a morphism $\beta' : B \to A$ in C with $p\beta' = \alpha$ and $\beta'\mu = \beta\mu$.

The following results may be deduced from results in [1].

THEOREM 3.1. *Let* M *be a fixed class of morphisms in* C . *Then:*

(i) *the composition of two* M*-fibrations is an* M*-fibration;*

(ii) *the pullback of an* M*-fibration via any morphism is an* M*-fibration;*

(iii) *the product of two* M*-fibrations is an* M*-fibration;*

(iv) *all isomorphisms are* M*-fibrations;*

(v) *all trivial morphisms* $p : A \to *$ *are* M*-fibrations.*

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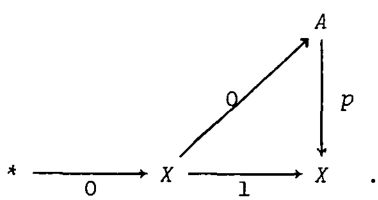
We denote the class of all M-fibrations by $\{M\text{-Fib}\}$ for each class M of morphisms in C. Clearly if $M \subset N$, we have $\{N\text{-Fib}\} \subset \{M\text{-Fib}\}$.

LEMMA 3.2. *Let M be any one of the following classes of morphisms in C: identities, isomorphisms, retractions, strong epimorphisms, epimorphisms. Then $\{M\text{-Fib}\}$ consists of all the morphisms in C.*

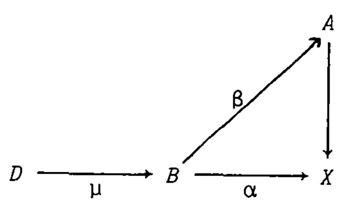
THEOREM 3.3. *Let M be the class of all coretractions. Then $\{M\text{-Fib}\}$ is the class of all retractions.*

Proof. Let $p : A \to X$ be an M-fibration and consider the diagram



Then $p00 = 0 = 10$ and $0 \in M$. Thus there exists a $j : X \to A$ with $pj = 1_X$. Thus p is a retraction.

Conversely let $p : A \to X$ be a retraction and consider the diagram



with $p\beta\mu = \alpha\mu$ and μ a coretraction. This diagram determines the following diagram

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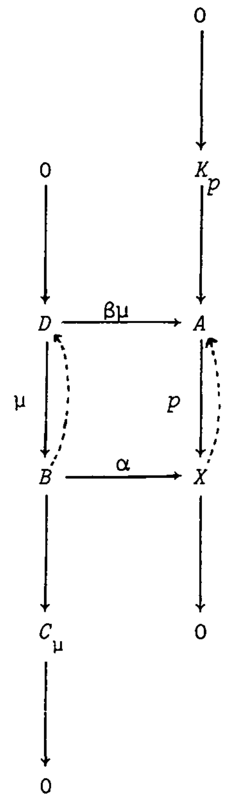

in which the square commutes and both columns are exact. Furthermore, since μ is a coretraction and p a retraction, both columns split. By Lemma 2.3 and Theorem 2.2 the square has a lifting. There exists a morphism $\beta' : B \to A$ with $p\beta' = \alpha$ and $\beta'\mu = \beta\mu$. Thus p is an M-fibration and the theorem is proved.

COROLLARY 3.4. *Let M be any arbitrary class of morphisms in C . Then every retraction is an M-fibration.*

Proof. According to Ringel [5], pages 222-223, $\{M\text{-Fib}\} = \{L \cap Q\text{-Fib}\}$ where Q is the class of all pushouts in M via arbitrary morphisms, and L is the class of all coretractions of C . Hence since $L \cap Q \subset L$, we have $\{L\text{-Fib}\} \subset \{M\text{-Fib}\}$. This is true in arbitrary categories. Hence in our category we have $\{\text{retractions}\} \subset \{M\text{-Fib}\}$, since in the above theorem $\{L\text{-Fib}\} = \{\text{retractions}\}$.

REMARK. If M is the class of all morphisms in C , then $\{M\text{-Fib}\}$

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is the smallest class of fibrations. By the above corollary, we have $\{M\text{-Fib}\} = \{\text{retractions}\}$. Thus the class of all retractions is the smallest class of fibrations.

We now restrict ourselves to an abelian category C with sufficient projectives. In [3] the following definition is made. We follow the terminology and notation used there.

DEFINITION. A morphism $p : A \to X$ is called a p-fibration if for all B , and for all $\alpha, \alpha' : B \to X$ with $\alpha \simeq_p \alpha'$, and for all $\beta : B \to A$ with $p\beta = \alpha$, there exists a $\beta' : B \to A$ with $p\beta' = \alpha'$ and $\beta' \simeq_p \beta$.

LEMMA 3.5. *For* $p : A \to X$ *the following are equivalent:*

(i) p *is a* p*-fibration;*

(ii) *for all projective objects* P *and morphisms* $\alpha : P \to X$, *there exists a morphism* $\beta : P \to A$ *with* $p\beta = \alpha$;

(iii) p *is an epimorphism.*

If we write $\mu : D \to B$ as $\mu = \iota_\mu \pi_\mu$, that is, in its canonical factorization

$$D \xrightarrow{\pi_\mu} I_\mu \xrightarrow{\iota_\mu} B$$

then in

$$D \xrightarrow{\mu} B \underset{g}{\overset{f}{\rightrightarrows}} X$$

we have $f\mu = g\mu$ iff $f\iota_\mu = g\iota_\mu$.

Let $M = \{\mu : D \to B$ such that B/I_μ is projective$\}$. Then we have the following result.

THEOREM 3.6. $\{M\text{-Fib}\} = \{p\text{-fibrations}\}$.

Proof. Let $p : A \to X$ be an M-fibration and consider the following diagram

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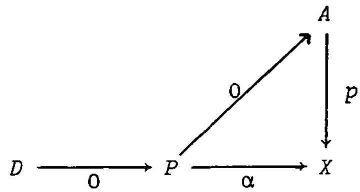
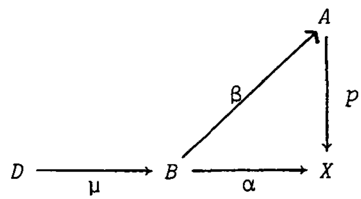
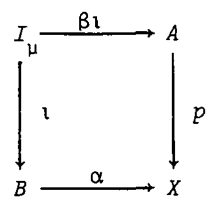


with P projective. Thus there exists a $\beta : P \to A$ with $p\beta = \alpha$. Thus p is a p-fibration.

Conversely let $p : A \to X$ be a p-fibration and consider the diagram



with $p\beta\mu = \alpha\mu$ and $\mu \in M$. Thus $p\beta\iota_\mu = \alpha\iota_\mu$ and this gives the following commutative square



This is a cospecial square and $B/I_\mu = C_\mu = C_\iota$ is projective. Thus the short exact sequence

$$0 \to K_p \to H \to C_\iota \to 0 ,$$

which is defined in §2, splits. So by Lemma 2.1, there is a lifting for the square. Thus there exists a $\beta' : B \to A$ with $p\alpha = \beta'$ and $\beta'\iota = \beta\iota$. Thus $p\alpha = \beta'$ and $\beta'\mu = \beta\mu$, and hence p is an M-fibration.

COROLLARY 3.7. *Let* M *be the class of all coretractions with projective cokernel, then* $\{M\text{-Fib}\} = \{p\text{-fibrations}\}$.

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