GENERALIZED BLOCKED TRIANGULAR MATRIX RINGS ASSOCIATED WITH FINITE ABELIAN CENTRALIZER NEAR-RINGS

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For N any member of a large class of finite abelian right centralizer near-rings, the subring of the ring End(N) of endomorphisms of $(N, +)$ generated by the set of right multiplication maps on N is explicitly described as a generalized blocked triangular matrix ring, which in some cases turns out to be a structural matrix ring.

1. Introduction

Apart from structural considerations, various types of rings of matrices provide a rich source of examples and counterexamples. In fact, Rowen (see [**12**, p. 29]) states that rings of matrices "are undoubtedly the most widely studied class of non-communicative rings". Generalized matrix rings form one of the largest classes of matrix rings, and in their own right these rings have been studied extensively in various contexts. See, for example, [**2**, **5**, **6** and **7**]. This class contains the class of structural matrix rings, which in turn contains the complete matrix rings and the triangular matrix rings, as well as the (complete) blocked triangular matrix rings. See, for example, [**3**, **4**, **10**, **14** and **15**].

In [**13**] it was shown that the right ring associated with a finite simple abelian near-ring contains a set of matrix units and is therefore isomorphic to a complete matrix ring. Secondly, a Wedderburn-Artin type result was obtained for the factor ring $\mathcal{R}/\mathcal{I}$, in the sense that $\mathcal{R}/\mathcal{I}$ is isomorphic to a direct sum of complete matrix rings, where $\mathcal{R}$ is the right ring associated with a finite abelian centralizer near-ring and $\mathcal{I}$ is a certain nilpotent ideal of $\mathcal{R}$. However, in neither of the above cases was any specific isomorphism exhibited.

Throughout the sequel N will be a zero-symmetric right near-ring with identity, and we assume that N is abelian, i.e. $(N, +)$ is abelian. We consider a specific subring of the ring End(N) of endomorphisms of the abelian group $(N, +)$, namely the subring $\mathcal{R}$ generated by the set of right multiplication maps $\rho[n]: N \to N$, with $\rho[n](m) = mn$, $m, n \in N$, called the *right ring associated with N* in [**13**]. Note that every element of $\mathcal{R}$ can be written as a (finite) sum $\sum_u \rho[n_u]$ of right multiplication maps, with the n_u's elements of N.

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The purpose of this paper is to describe $\mathcal{R}$ explicitly as a generalized blocked triangular matrix ring in the case where N is any member of a large class of finite (abelian) centralizer near-rings. This will be done in a constructive way, i.e. an isomorphism map will be established. We will show that the mentioned generalized blocked triangular matrix ring can turn out to be a structural matrix ring.

2. The representation of the right ring associated with a finite simple abelian near-ring as a complete matrix ring

In this section we consider the setting in [13, Section 3]. For the ease of the reader we provide the pertinent results, definitions and notation.

A basic near-ring-theoretic result states that every finite simple zero-symmetric near-ring N (with 1) which is not a ring is isomorphic to a centralizer near-ring

$$M_A(G) := \{f : G \to G \mid f(0) = 0 \quad \text{and} \quad f(\alpha v) = \alpha f(v), \; \forall \alpha \in A, \forall v \in G\},$$

where G is a finite group (written additively) and A is a group of fixed point free automorphisms of G. Under the added assumption that N be an abelian near-ring, G is an abelian group.

Henceforth in this section G will be a finite abelian group, $A := \{\alpha_1, \ldots, \alpha_s\}$ will be a group of fixed point free automorphisms of G, and $N := M_A(G)$. Let $\{v_1, \ldots, v_t\}$ be a complete set of representatives of the nonzero A-orbits of G. For $i, j = 1, \ldots, t$, let $e_{ij} \in N$ be defined by $e_{ij}(v_j) = v_i$ and $e_{ij}(v_k) = 0$ if $k \neq j$, and let $E_{ij} := \rho[e_{ji}]$, the right multiplication map on N determined by e_{ji}. Then $\{E_{ij} : 1 \le i, j \le t\}$ is a set of matrix units in $\mathcal{R}$, i.e. $E_{11} + \cdots + E_{tt} = 1_{\mathcal{R}}$ and $E_{ij} E_{kl} = \delta_{jk}$ where δ_{jk} is the Kronecker delta. So much for preliminaries.

Let $\alpha \in A$, let $1 \le i \le t$, and define $e_{ii}^{\alpha} \in N$ by

$$e_{ii}^{\alpha}(v_i) = \alpha v_i \quad \text{and} \quad e_{ii}^{\alpha}(v_j) = 0 \text{ if } j \neq i.$$

Then $e_{ii}^{\alpha} e_{ii}^{\beta} = e_{ii}^{\beta\alpha}$, $\beta \in A$. In fact, $(e_{11}^{\alpha} + \cdots + e_{tt}^{\alpha})(e_{11}^{\beta} + \cdots + e_{tt}^{\beta}) = e_{11}^{\beta\alpha} + \cdots + e_{tt}^{\beta\alpha}$, and so

$$\rho[e_{11}^{\alpha} + \cdots + e_{tt}^{\alpha}] \cdot \rho[e_{11}^{\beta} + \cdots + e_{tt}^{\beta}] = \rho[e_{11}^{\alpha\beta} + \cdots + e_{tt}^{\alpha\beta}]. \tag{1}$$

Hence by [13, Lemma 1] the set

$$\mathcal{T} := \{a_1 \rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + a_s \rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}] : a_i \in \mathbb{Z}_k, \forall i\}$$

is a subring of $\mathcal{R}$, where k is the least common multiple of the orders of the elements in G. Let R denote the subring of $\text{End}(G)$ generated by A. Then R consists of endomorphisms of G of the form

$$a_1 \alpha_1 + \cdots + a_s \alpha_s,$$

where $a_i \in \mathbb{Z}_k, \forall i$.

Lemma 2.1. *With the above notation,* $\Phi\colon \mathcal{T} \to \mathcal{R}$, *defined by*

$$a_1\rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + a_s\rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}] \mapsto a_1\alpha_1 + \cdots + a_s\alpha_s,$$

is a ring isomorphism.

Proof. The technique used here is based on the technique used in the proof of [13, Theorem 2]. However, we provide the details. Suppose first that

$$a_1\rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + a_s\rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}] = b_1\rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}]$$
$$+ \cdots + b_s\rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}].$$

Then for every $n \in N$ and every v_i, $i = 1, \ldots, t$,

$$a_1 n\alpha_1(v_i) + \cdots + a_s n\alpha_s(v_i) = b_1 n\alpha_1(v_i) + \cdots + b_s n\alpha_s(v_i),$$

and so

$$(a_1\alpha_1 + \cdots + a_s\alpha_s)n(v_i) = (b_1\alpha_1 + \cdots + b_s\alpha_s)n(v_i).$$

Since, for a suitable choice of n, $n(v_i)$ can be any element of G, we have $a_1\alpha_1 + \cdots + a_s\alpha_s = b_1\alpha_1 + \cdots + b_s\alpha_s$, i.e. Φ is well defined. Furthermore, by (1) Φ is multiplicative, and it is certainly additive and onto. Lastly, suppose $a_1\alpha_1 + \cdots + a_s\alpha_s = 0$. Then $(a_1\alpha_1 + \cdots + a_s\alpha_s)n(v_i) = 0$ for every $n \in N$ and every v_i, $i = 1, \ldots, t$. Hence, reversing the above steps, we have

$$(a_1\rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + a_s\rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}])(n) = 0$$

for every $n \in N$, and so $a_1\rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + a_s\rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}] = 0$. Therefore Φ is 1-1. □

Let $|S|$ denote the cardinality of a set S. We are now in a position to exhibit an isomorphism map from $\mathcal{R}$ to $\mathbb{M}_t(R)$. See [13, Theorem 2].

Theorem 2.2. *Let* $N := M_A(G)$ *be a simple abelian near-ring, where* G *is a finite (abelian) group and* $A = \{\alpha_1, \ldots, \alpha_s\}$ *is a fixed point free group of automorphisms of* G. *Let* R *be the subring of* $\mathrm{End}(G)$ *generated by the automorphisms in* A. *If there are* t *nonzero* A-*orbits of* G, *then the right ring* $\mathcal{R}$ *associated with* N *is isomorphic to the complete matrix ring* $\mathbb{M}_t(R)$ *via*

$$\Psi\colon \sum_u \rho[n_u] \mapsto [x_{ij}],$$

with

$$x_{ij} := a_1^{(ij)}\alpha_1 + \cdots + a_s^{(ij)}\alpha_s$$

and

$$a_m^{(ij)} := |\{u : n_u(v_i) = \alpha_m v_j\}| \ (\text{mod } k), \quad m = 1, \ldots, s,$$

where $\{v_1, \ldots, v_t\}$ is a complete set of nonzero orbit representatives of G, and k is the least common multiple of the orders of the elements in G.

Proof. Since $\{E_{ij} : 1 \le i, j \le t\}$ is a set of matrix units in $\mathcal{R}$, it follows from [9, p. 52, Proposition 6] that every element $\sum_u \rho[n_u]$ of $\mathcal{R}$ can be written in one and only one way in the form

$$\sum_{i,j=1}^{t} b_{ij} E_{ij}, \tag{2}$$

where

$$b_{ij} := \sum_{l=1}^{t} E_{li}\left(\sum_u \rho[n_u]\right)E_{jl} = \sum_{l=1}^{t} \rho[e_{il}]\left(\sum_u \rho[n_u]\right)\rho[e_{lj}]$$

$$= \sum_{l=1}^{t} \sum_u \rho[e_{lj}n_u e_{il}]$$

$$= \sum_u \sum_{l=1}^{t} \rho[e_{lj}n_u e_{il}].$$

Note that

$$e_{lj}n_u e_{il} = \begin{cases} e_{ll}^{\alpha_q}, & \text{if } n_u(v_i) = \alpha_q v_j, \quad 1 \le q \le s; \\ 0, & \text{if } n_u(v_i) \notin Av_j. \end{cases}$$

Hence,

$$\sum_{l=1}^{t} \rho[e_{lj}n_u e_{il}] = \begin{cases} \rho[e_{11}^{\alpha_q} + \cdots + e_{tt}^{\alpha_q}], & \text{if } n_u(v_i) = \alpha_q v_j, \\ 0, & \text{if } n_u(v_i) \notin Av_j. \end{cases}$$

Therefore,

$$\sum_u \sum_{l=1}^{t} \rho[e_{lj}n_u e_{il}] = a_1^{(ij)}\rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + a_s^{(ij)}\rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}],$$

where $a_m^{(ij)} = |\{u: n_u(v_i) = \alpha_m v_j\}| \pmod{k}, \quad m = 1, \ldots, s.$ Furthermore, if $\sum_u \rho[n_u] = \sum_{u'} \rho[n'_{u'}]$, then the uniqueness of b_{ij} in (2) ensures that

$$\sum_u \sum_{l=1}^{t} \rho[e_{ij} n_u e_{il}] = \sum_{u'} \sum_{l=1}^{t} \rho[e_{ij} n'_{u'} e_{il}],$$

and so if

$$\sum_{u'} \sum_{l=1}^{t} \rho[e_{ij} n'_{u'} e_{il}] = b_1^{(ij)} \rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + b_s^{(ij)} \rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}],$$

with $b_m^{(ij)} = |\{u': n'_{u'}(v_i) = \alpha_m v_j\}| \pmod{k}, m = 1, \ldots, s$, then

$$a_1^{(ij)} \rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + a_s^{(ij)} \rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}]$$
$$= b_1^{(ij)} \rho[e_{11}^{\alpha_1} + \cdots + e_{tt}^{\alpha_1}] + \cdots + b_s^{(ij)} \rho[e_{11}^{\alpha_s} + \cdots + e_{tt}^{\alpha_s}],$$

which by Lemma 2.1 concludes the proof. $\square$

We illustrate the map in Theorem 2.2 in the following example.

Example 2.3. Let $G = (\mathbb{Z}_9, +)$, and let $A = \{id, \sigma\}$, where $\sigma(v) := -v$, $v \in \mathbb{Z}_9$. (See [13, Section 3, Example (b)].) Consider the elements $4\rho[n_1] + \rho[n_2]$ and $6\rho[n_1] + 7\rho[n_2] + 2\rho[n_3]$ of $\mathcal{R}$, where n_1, n_2 and n_3 are the elements of $N(= M_A(G))$ defined in the table below, and where 1, 2, 3 and 4 are the representatives of the four nonzero A-orbits $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$ and $\{4, 5\}$ respectively. Since we shall need n_1^2, $n_2 n_1$, $n_3 n_1$, $n_1 n_2$, n_2^2 and $n_3 n_2$, we include them in the table too.

	n_1	n_2	n_3	n_1^2	$n_2 n_1$	$n_3 n_1$	$n_1 n_2$	n_2^2	$n_3 n_2$
1	$\sigma(2)$	$id(2)$	$id(1)$	$id(2)$	$id(4)$	$id(3)$	$\sigma(2)$	$\sigma(4)$	$\sigma(3)$
2	$\sigma(2)$	$\sigma(4)$	$\sigma(3)$	$id(2)$	$id(4)$	$id(3)$	$\sigma(2)$	$\sigma(3)$	$id(1)$
3	$\sigma(1)$	$\sigma(3)$	$id(1)$	$id(2)$	$\sigma(2)$	$\sigma(1)$	$id(1)$	$id(3)$	$\sigma(1)$
4	$id(2)$	$id(3)$	$\sigma(1)$	$\sigma(2)$	$\sigma(4)$	$\sigma(3)$	$\sigma(1)$	$\sigma(3)$	$id(1)$

(The above notation means that, for example, $n_1(1) = \sigma(2) = 7$.)

The function $\lambda: R \to \mathbb{Z}_9$, defined by

$$a_i id + a_2 \sigma \mapsto a_1 - a_2 \pmod{9},$$

is a ring isomorphism. Let $\Lambda: \mathbb{M}_4(R) \to \mathbb{M}_4(\mathbb{Z}_9)$ be the induced isomorphism $[x_{ij}] \mapsto [\lambda(x_{ij})]$. Then by Theorem 2.2 we have

$$
\Lambda\Psi(4\rho[n_1] + \rho[n_2]) = \Lambda\left(\begin{bmatrix} 0 & i+4\sigma & 0 & 0 \\ 0 & 4\sigma & 0 & \sigma \\ 4\sigma & 0 & \sigma & 0 \\ 0 & 4i & i & 0 \end{bmatrix}\right)
$$

$$
= \begin{bmatrix} 0 & 6 & 0 & 0 \\ 0 & 5 & 0 & 8 \\ 5 & 0 & 8 & 0 \\ 0 & 4 & 1 & 0 \end{bmatrix}.
$$

Similarly,

$$
\Lambda\Psi(6\rho[n_1] + 7\rho[n_2] + 2\rho[n_3]) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 7 & 2 \\ 5 & 0 & 2 & 0 \\ 7 & 6 & 7 & 0 \end{bmatrix}.
$$

Also,

$$
\Lambda\Psi\big((4\rho[n_1] + \rho[n_2])(6\rho[n_1] + 7\rho[n_2] + 2\rho[n_3])\big)
$$
$$
= \Lambda\Psi(6\rho[n_1^2] + \rho[n_2 n_1] + 8\rho[n_3 n_1] + 6\rho[n_1 n_2] + 7\rho[n_2^2] + 2\rho[n_3 n_2])
$$
$$
= \begin{bmatrix} 0 & 0 & 6 & 3 \\ 2 & 0 & 1 & 1 \\ 5 & 5 & 7 & 0 \\ 5 & 3 & 3 & 8 \end{bmatrix},
$$

which equals the product of the two matrices in $\mathbb{M}_4(\mathbb{Z}_9)$ above. This is an illustration of the multiplicativity of Ψ and $\Lambda\Psi$.

3. The representation of the right ring associated with a finite abelian centralizer near-ring as a generalized blocked triangular matrix ring

In this section N will be a centralizer near-ring $M_A(G)$, where $(M_A(G), +)$ is a finite abelian group and A is a group of automorphisms of the finite group G. We note that G, as pointed out in [13, Section 4], need not be abelian.

By Betsch's Lemma (see, for example, [11, Proposition 9.199]) there is a $n \in N$ such that $n(v) = w$, with $v, w \in G$, if and only if stab(v) $\subseteq$ stab(w), where stab(v) denotes the stabilizer $\{\alpha \in A : \alpha v = v\}$ of v, which is a subgroup of A. Let $\mathcal{O}_1, \ldots, \mathcal{O}_t$ denote the nonzero A-orbits of G. Consider the relations $\sim, <$ and $\lesssim$ defined on the set $\{\mathcal{O}_1, \ldots, \mathcal{O}_t\}$ as follows:

$$\mathcal{O}_i \sim \mathcal{O}_j :\Leftrightarrow \exists v_i \in \mathcal{O}_i, v_j \in \mathcal{O}_j \text{ such that stab}(v_i) = \text{stab}(v_j),$$

$$\mathcal{O}_i < \mathcal{O}_j :\Leftrightarrow \exists v_i \in \mathcal{O}_i, v_j \in \mathcal{O}_j \text{ such that stab}(v_i) \supset \text{stab}(v_j),$$

$$\mathcal{O}_i \lesssim \mathcal{O}_j :\Leftrightarrow \mathcal{O}_i \sim \mathcal{O}_j \text{ or } \mathcal{O}_i < \mathcal{O}_j.$$

(The symbol $\subset$ denotes strict inclusion.)

We assume henceforth that it is possible to select the t A-orbit representatives $v_1, \ldots, v_t$ such that for $i, j = 1, \ldots, t$,

$$\mathcal{O}_i \lesssim \mathcal{O}_j \Leftrightarrow \text{stab}(v_i) \supseteq \text{stab}(v_j) \quad \text{and} \quad \text{stab}(v_i) \supset \text{stab}(v_j) \Rightarrow i > j. \qquad (3)$$

Therefore $\mathcal{O}_i \sim \mathcal{O}_j$ if and only if $\text{stab}(v_i) = \text{stab}(v_j)$. The equivalence relation $\sim$ on the $\mathcal{O}_i$'s induces an equivalence relation on the v_i's, which we denote by $\sim$ too, in the obvious way:

$$v_i \sim v_j :\Leftrightarrow \mathcal{O}_i \sim \mathcal{O}_j. \qquad (4)$$

Our examples will show that condition (3), which we call the *stabilizer condition*, is far from being restrictive. (In fact, we have no examples of (3) not being satisfied.)

Let p be the number of equivalence classes induced by the equivalence relation $\sim$ in (4) on the set $\{v_1, \ldots, v_t\}$, and let $v_{i_1}, \ldots, v_{i_p}$ be representatives of these p equivalence classes, i.e. v_{i_1} is an orbit representative in the first equivalence class, etc. Set $t_m := |[v_{i_m}]|$, i.e. the cardinality of the equivalence class $[v_{i_m}]$ containing v_{i_m}, $m = 1, \ldots, p$. Then $t_1 + \cdots + t_p = t$, and if $[v_{i_m}] = \{v_{i_{m,1}}, \ldots, v_{i_{m,t_m}}\}$, then by (3) and (4) we can arrange the ordering so that $\{i_{m,1}, \ldots, i_{m,t_m}\}$ is a set of consecutive natural numbers, and $\{i_{1,1} \ldots, i_{1,t_1}, \ldots, i_{p,1}, \ldots, i_{p,t_p}\} = \{1, \ldots, t\}$.

In [13, Section 4] it was shown that the subring $\mathcal{I}$ of $\mathcal{R}$ generated by the set $\{\rho[n]: n \in N$ with $\text{stab}(v) \subset \text{stab}(n(v)), \forall 0 \neq v \in G\}$ is a nilpotent ideal of $\mathcal{R}$, and that $\mathcal{R}/\mathcal{I}$ is isomorphic to a direct sum of complete matrix rings, with $\mathcal{R}$ the right ring associated with N. Consider the set $\{E_{11}, \ldots, E_{tt}\}$ of orthogonal idempotents in $\mathcal{R}$ (with sum $1_{\mathcal{R}}$) as in Section 2, and note that if $\text{stab}(v_{i_m}) \not\subseteq \text{stab}(v_{i_l})$, with $1 \leq l, m \leq p$, then, as every element of $E_{i_m i_m} \mathcal{R} E_{i_l i_l}$ has the form

$$\rho[e_{i_m i_m}]\left(\sum_u \rho[n_u]\right)\rho[e_{i_l i_l}] = \sum_u \rho[e_{i_l i_l} n_u e_{i_m i_m}],$$

it follows from (3) that $E_{i_m i_m}\mathcal{R}E_{i_l i_l} = \{0\}$. Furthermore, in this case

$$E_{i_{m,k} i_{m,k}}\mathcal{R}E_{i_{l,k'} i_{l,k'}} = \{0\}$$

for all k and k' such that $1 \leq k \leq t_m$, $1 \leq k' \leq t_l$, since $\text{stab}(v_{i_{m,k}}) = \text{stab}(v_{i_m}) \not\subseteq \text{stab}(v_{i_l}) = \text{stab}(v_{i_{l,k'}})$.

Let $y := t_1 + \cdots + t_{m-1}$ and $z := t_1 + \cdots + t_{p-1}$. We conclude from the preceding arguments and from arguments similar to those used in, for example [2, Proposition

2.1 and Theorem 2.3], [1] or [5], that we have proved the following lemma.

Lemma 3.1. $\mathcal{R}$ *is isomorphic to the* $t \times t$ *generalized blocked triangular matrix ring*

$$
\begin{bmatrix}
\begin{bmatrix} \mathcal{R}_{1,1} & \cdots & \mathcal{R}_{1,t_1} \\ \vdots & & \vdots \\ \mathcal{R}_{t_1,1} & \cdots & \mathcal{R}_{t_1,t_1} \end{bmatrix} & \cdots & \begin{bmatrix} \mathcal{R}_{1,y+1} & \cdots & \mathcal{R}_{1,y+t_m} \\ \vdots & & \vdots \\ \mathcal{R}_{t_1,y+1} & \cdots & \mathcal{R}_{t_1,y+t_m} \end{bmatrix} & \cdots & \begin{bmatrix} \mathcal{R}_{1,z+1} & \cdots & \mathcal{R}_{1,t} \\ \vdots & & \vdots \\ \mathcal{R}_{t_1,z+1} & \cdots & \mathcal{R}_{t_1,t} \end{bmatrix} \\
& \ddots & \vdots & & \vdots \\
& & \begin{bmatrix} \mathcal{R}_{y+1,y+1} & \cdots & \mathcal{R}_{y+1,y+t_m} \\ \vdots & & \vdots \\ \mathcal{R}_{y+t_m,y+1} & \cdots & \mathcal{R}_{y+t_m,y+t_m} \end{bmatrix} & \cdots & \begin{bmatrix} \mathcal{R}_{y+1,z+1} & \cdots & \mathcal{R}_{y+1,t} \\ \vdots & & \vdots \\ \mathcal{R}_{y+t_m,z+1} & \cdots & \mathcal{R}_{y+t_m,t} \end{bmatrix} \\
0 & & & \ddots & \vdots \\
& & & & \begin{bmatrix} \mathcal{R}_{z+1,z+1} & \cdots & \mathcal{R}_{z+1,t} \\ \vdots & & \vdots \\ \mathcal{R}_{t,z+1} & \cdots & \mathcal{R}_{t,t} \end{bmatrix}
\end{bmatrix}
$$
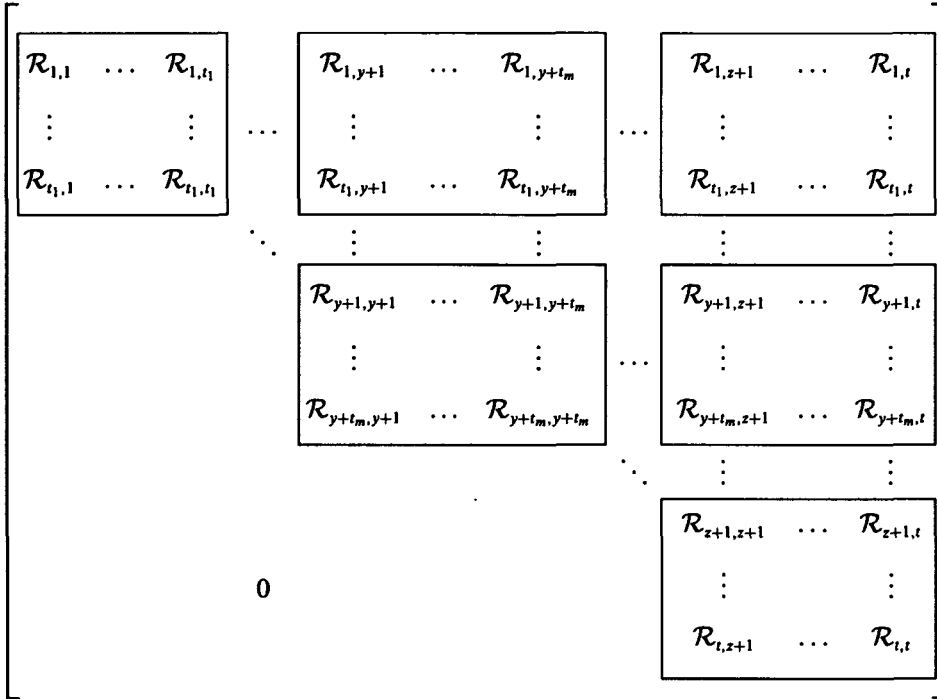
comprising all the matrices with (i,j)*-th entries in* $\mathcal{R}_{i,j} := E_{ii} \mathcal{R} E_{jj}$ *via*

$$ r \mapsto E_{ii} r E_{jj}, \quad r \in \mathcal{R}. $$

By [13, Lemma 3 and Theorem 3] every diagonal block in the generalized blocked triangular matrix ring above is isomorphic to a complete matrix ring. We now turn our attention to the $\mathcal{R}_{ij}$'s in the blocks in the non-symmetric part of the generalized blocked triangular matrix ring in Lemma 3.1.

Let $e_{ij}: G \to G$ be defined as in Section 2. Then $e_{ij} \in N$ if and only if stab$(v_j) \subseteq$ stab(v_i). For such e_{ij}'s we set $E_{ji} := \rho[e_{ij}]$, as in Section 2.

For the sake of simplicity of notation, let us assume that $v_1, v_2 \in [v_{i_1}]$ and $v_3, v_4 \in [v_{i_2}]$, with stab$(v_{i_1}) \subset$ stab(v_{i_2}). Then $e_{31}, e_{41}, e_{32}, e_{42} \in N$, and so E_{13}, E_{14}, E_{23}, $E_{24} \in \mathcal{R}$. (Note that, for example, $e_{13} \notin N$.) Therefore $\mathcal{R}_{1,3} \cong \mathcal{R}_{2,4}$ as $\mathcal{R}_{1,1}$-$\mathcal{R}_{4,4}$-bimodules, and since $\mathcal{R}_{1,1} \cong \mathcal{R}_{2,2}$ and $\mathcal{R}_{3,3} \cong \mathcal{R}_{4,4}$ as rings, it follows from, for example, [14, Lemma 2.3] that $\mathcal{R}_{1,3} \cong \mathcal{R}_{2,4} \cong \mathcal{R}_{1,4} \cong \mathcal{R}_{2,3}$ as $\mathcal{R}_{11}$-$\mathcal{R}_{33}$-bimodules. Consequently all the $\mathcal{R}_{i,j}$'s in a specific non-symmetric block are isomorphic as bimodules over the appropriate rings.

However, merely stated as in Lemma 3.1, and taking into consideration the preceding paragraph, one has not become any wiser as far as the structure of the $\mathcal{R}_{ij}$'s in the non-symmetric blocks is concerned. After all, for example, the ring $\mathbb{Z}_6$ contains the idempotents 3 and 4, and $3 + 4 = 1_{\mathbb{Z}_6}$, and so $\mathbb{Z}_6$ is isomorphic to the 2×2 generalized matrix ring $\begin{bmatrix} 3\mathbb{Z}_6 3 & 3\mathbb{Z}_6 4 \\ 4\mathbb{Z}_6 3 & 4\mathbb{Z}_6 4 \end{bmatrix}$. Therefore, only after studying the four components does one realize that this isomorphism is merely the fact that $\mathbb{Z}_6 \cong \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_3 \end{bmatrix} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

The purpose of this section is to find the structure of the components $\mathcal{R}_{i,j}$ in Lemma 3.1 in order to eventually provide (Theorem 3.2) an explicit representation of $\mathcal{R}$ as a $t \times t$ generalized blocked triangular matrix ring.

Henceforth we set

$$
\begin{aligned}
K_m &:= \text{stab}(v_{i_m}), \\
H_m &:= \{w \in G : \text{stab}(v_{i_m}) \subseteq \text{stab}(w)\},
\end{aligned}
\tag{5}
$$

$m = 1, \ldots, p$. Also, let k_m denote the least common multiple of the elements of the (abelian) subgroup H_m of G, as in [13, Section 4]. For the sake of simplicity of notation let us assume that

$$
i_m = m, m = 1, 2, 3, 4, \quad \text{and} \quad K_1 \subset K_2 \subset K_3 \subset K_4.
\tag{6}
$$

We consider $E_{11}\mathcal{R}E_{22}$.

By [13, Lemma 2] we have $E_{11}\mathcal{R}E_{22} = E_{11}\mathcal{I}E_{22}$, since we have assumed that $\text{stab}(v_1) \subset \text{stab}(v_2)$, where $\mathcal{I}$ is the ideal of $\mathcal{R}$ as described in the discussion preceding Lemma 3.1. A typical element of $E_{11}\mathcal{I}E_{22}$ has the form

$$
\rho[e_{11}]\left(\sum_u \rho[n_u]\right)\rho[e_{22}] = \sum_u \rho[e_{22}n_u e_{11}],
$$

with each nonzero $e_{22}n_u e_{11} \in N$ such that

$$
e_{22}n_u e_{11}(v_1) = \alpha v_2
\tag{7}
$$

where, by the definition of $\mathcal{I}$, the element α of A is such that $\text{stab}(v_1) \subset (\alpha v_2)$.

We claim, assuming that $K_1 \subset K_2$ (see (6)), that for $\alpha \in A$,

$$
\text{stab}(v_1) \subset \text{stab}(\alpha v_2) \Leftrightarrow \alpha|_{H_2} \in \text{End}(H_2, H_1),
\tag{8}
$$

where $\text{End}(H_2, H_1)$ denotes the group of endomorphisms from H_2 to H_1. (Note that $\text{End}(H_2, H_1)$ is abelian, since the H_i's are abelian. See [13, Section 4].) To this end,

suppose that $\mathrm{stab}(v_1) \subset \mathrm{stab}(\alpha v_2)$, and let $w \in H_2$. Then, by (5), (6) and, for example [11, Proposition 9.201], $\mathrm{stab}(\alpha w) = \alpha\,\mathrm{stab}(w)\alpha^{-1} \supseteq \alpha\,\mathrm{stab}(v_2)\alpha^{-1} = \mathrm{stab}(\alpha v_2) \supset \mathrm{stab}(v_1)$, and so $\alpha w \in H_1$. Conversely, if $\alpha|_{H_2} \in \mathrm{End}(H_2, H_1)$, then, as $v_2 \in H_2$, we have $\alpha v_2 \in H_1$. Thus $\mathrm{stab}(v_1) \subseteq \mathrm{stab}(\alpha v_2)$, and since $|\mathrm{stab}(v_1)| < |\mathrm{stab}(v_2)| = |\alpha\,\mathrm{stab}(v_2)\alpha^{-1}| = |\mathrm{stab}(\alpha v_2)|$, it follows that $\mathrm{stab}(v_1) \subset \mathrm{stab}(\alpha v_2)$, which establishes (8).

Note that every $\alpha \in N(K_2)$ satisfies (8), where $N(K_2)$ denotes the normalizer of K_2 in A, since for every $\alpha \in N(K_2)$ we have $\mathrm{stab}(\alpha v_2) = \alpha\,\mathrm{stab}(v_2)\alpha^{-1} = \mathrm{stab}(v_2) \supset \mathrm{stab}(v_1)$. Moreover, arguments similar to the preceding ones show that

$$\alpha \in N(k_2) \Leftrightarrow \alpha|_{H_2} \in \mathrm{Aut}(H_2).$$

These results tie in with the fact that $\mathrm{Aut}(H_2) \subseteq \mathrm{End}(H_2, H_1)$.

Furthermore, with $e_{22}n_u e_{11}(v_1) = \alpha v_2$ as in (7) and (8), $(\rho[e_{22}n_u e_{11}](n))(v_1) = \alpha n(v_2)$ for every $n \in M_A(G)$. Since $\mathrm{stab}(v_2) \subseteq \mathrm{stab}(n(v_2))$, it follows that $n(v_2) \in H_2$, and so the idea in the proof of [13, Lemma 1] shows that the additive order of $\rho[e_{22}n_u e_{11}]$ is k_2.

Next, let A_{12} be the set of left cosets $\bar{\alpha} := \alpha K_2$ of K_2 in A, with α as in (8), i.e.

$$A_{12} := \{\bar{\alpha} : \mathrm{stab}(v_1) \subset \mathrm{stab}(\alpha v_2)\}.$$

Then $A_2 \subseteq A_{12}$, with $A_2 = N(K_2)/K_2$ as in [13, Section 4]. Let $[A : N(K_2)]$ denote the index of $N(K_2)$ in A. We note that if $[A : N(K_2)] = 2$ and $\mathrm{stab}(v_1) \subset \mathrm{stab}(\alpha v_2)$ for some $\alpha \in A \setminus N(K_2)$, then $\mathrm{stab}(v_1) \subset \mathrm{stab}(\beta v_2)$ for all $\beta \in A \setminus N(K_2)$, since by, for example, [8, Corollary II.4.4 (iii)], the number of subgroups of A conjugate to K_2 equals $[A : N(K_2)]$, and so if $\beta \in A \setminus N(K_2)$, then $\mathrm{stab}(\beta v_2) = \beta\,\mathrm{stab}(v_2)\beta^{-1} \neq \mathrm{stab}(v_2)$, which implies that $\mathrm{stab}(\beta v_2) = \mathrm{stab}(\alpha v_2) \supset \mathrm{stab}(v_1)$.

In general, with $K_l \subset K_m$, $1 \le l \le m \le p$, we set

$$A_m := \{\bar{\alpha}_{m,1}, \ldots, \bar{\alpha}_{m,q_m}\} \quad \text{and} \quad A_{lm} := \{\bar{\alpha}_{m,1}, \ldots, \bar{\alpha}_{m,q_m}, \ldots, \bar{\alpha}_{m,q_m^{(l)}}\}. \tag{9}$$

We may view A_{lm}, in the light of (8), as a subset of $\mathrm{End}(H_m, H_l)$ by setting

$$\bar{\alpha}w := \alpha w, \tag{10}$$

$\bar{\alpha} \in A_{wl}$, $w \in H_2$. Since $\alpha v_m = \beta v_m$ if and only if $\alpha^{-1}\beta \in \mathrm{stab}(v_m)$ if and only if $\alpha K_m = \beta K_m$, it follows from (7) that if $n_{m,j} \in N$ is defined by

$$n_{m,j}(v_l) = \alpha_{m,j}v_m \quad \text{and} \quad n_{m,j}(v) = 0 \text{ if } v \notin Av_l, \tag{11}$$

$j = 1, \ldots, q_m^{(l)}$, then the foregoing results show that $E_{i_l i_l}\mathcal{R}E_{i_m i_m}$ is the subgroup

$$\{a_1\rho[n_{m,1}] + \cdots + a_{q_m^{(l)}}\rho[n_{m,q_m^{(l)}}] : a_j \in \mathbb{Z}_{k_m}, \forall j\}$$

of $(\mathrm{End}(N), +)$. Since k_m is the least common multiple of the elements of H_m, every element of $\mathrm{End}(H_m, H_l)$ has additive order which divides k_m. Therefore the subgroup of

End(H_m, H_l) generated by A_{lm} is

$$S_{lm} := \{a_1 \bar{\alpha}_{m,1} + \cdots + a_{q_m^{(l)}} \bar{\alpha}_{m,q_m^{(l)}} : a_j \in \mathbb{Z}_{k_m}, \forall j\}. \tag{12}$$

Then we claim that

$$E_{i_l i_l} \mathcal{R} E_{i_m i_m} \cong S_{lm} \tag{13}$$

as groups via

$$\Phi_{lm} : a_1 \rho[n_{m,1}] + \cdots + a_{q_m^{(l)}} \rho[n_{m,q_m^{(l)}}] \mapsto a_1 \bar{\alpha}_{m,1} + \cdots + a_{q_m^{(l)}} \bar{\alpha}_{m,q_m^{(l)}}.$$

We now verify (13). For the sake of simplicity of notation, and without loss of generality, we set $l = 1$ and $m = 2$, and we use (6). Suppose first that $a_1 \rho[n_{2,1}] + \cdots + a_{q_2^{(1)}} \rho[n_{2,q_2^{(1)}}] = b_1 \rho[n_{2,1}] + \cdots + b_{q_2^{(1)}} \rho[n_{2,q_2^{(1)}}]$. Then, as in the proof of [13, Theorem 2] it follows from (11) that, for every $n \in N$,

$$(a_1 \alpha_{2,1} + \cdots + a_{q_2^{(1)}} \alpha_{2,q_2^{(1)}}) n(v_2) = ((a_1 \rho[n_{2,1}] + \cdots + a_{q_2^{(1)}} \rho[n_{2,q_2^{(1)}}])(n))(v_1)$$

$$= ((b_1 \rho[n_{2,1}] + \cdots + b_{q_2^{(1)}} \rho[n_{2,q_2^{(1)}}])(n))(v_1)$$

$$= (b_1 \alpha_{2,1} + \cdots + b_{q_2^{(1)}} \alpha_{2,q_2^{(1)}}) n(v_2).$$

Now $n(v_2)$ is an arbitrary element w of H_2, and so by (10)

$$a_1 \bar{\alpha}_{2,1} + \cdots + a_{q_2^{(1)}} \bar{\alpha}_{2,q_2^{(1)}} = b_1 \bar{\alpha}_{2,1} + \cdots + b_{q_2^{(1)}} \bar{\alpha}_{2,q_2^{(1)}},$$

i.e. Φ_{12} is well defined. By (12) Φ is onto, and it is certainly additive. Lastly, if $a_1 \bar{\alpha}_{2,1} + \cdots + a_{q_2^{(1)}} \bar{\alpha}_{2,q_2^{(1)}} = 0$, then $(a_1 \bar{\alpha}_{2,1} + \cdots + a_{q_2^{(1)}} \bar{\alpha}_{2,q_2^{(1)}}) n(v_2) = 0$ for every $n \in N$, and reversing the above steps it follows that

$$((a_1 \rho[n_{2,1}] + \cdots + a_{q_2^{(1)}} \rho[n_{2,q_2^{(1)}}])(n))(v_1) = 0.$$

The second part of (11) now shows that $a_1 \rho[n_{2,1}] + \cdots + a_{q_2^{(1)}} \rho[n_{2,q_2^{(1)}}] = 0$, which establishes (13).

For $m = 1, \ldots, p$, set

$$R_m := \{a_1 \bar{\alpha}_{m,1} + \cdots + a_{q_m} \bar{\alpha}_{m,q_m} : a_j \in \mathbb{Z}_{k_m}, \forall j\}. \tag{14}$$

By (9) and (12) $R_m \subseteq S_{lm}$ as sets, and R_m is the subring of Aut(H_m) generated by the elements of A_m (see [13, Section 4]). Note also that if $K_{l'} \subset K_l \subset K_m$, with $1 \le l' < l < m \le p$, then, as sets, $S_{lm} \subseteq S_{l'm}$. Moreover, if K_m is a normal subgroup of A, then, as sets,

$$R_m = S_{lm} = S_{l'm}. \tag{15}$$

We now turn S_{lm} into an R_l-R_m-bimodule. From [13, Section 4] we know that

$$E_{i_m i_m} \mathcal{R} E_{i_m i_m} = \{b_1 \rho[n_{m,1}] + \cdots + b_{q_m} \rho[n_{m,q_m}] : b_j \in \mathbb{Z}_{k_m}, \forall j\}$$

and

$$\Phi_m : E_{i_m i_m} \mathcal{R} E_{i_m i_m} \to R_m,$$

defined by

$$b_1 \rho[n_{m,1}] + \cdots + b_{q_m} \rho[n_{m,q_m}] \mapsto b_1 \bar{\alpha}_{m,1} + \cdots + b_{q_m} \bar{\alpha}_{m,q_m}, \tag{16}$$

is a ring isomorphism. Together with the group isomorphism $\Phi_{lm} : E_{i_l i_l} \mathcal{R} E_{i_m i_m} \to S_{lm}$ in (13) we obtain S_{lm} naturally as a right R_m-module via

$$s_{lm} \cdot r_m = \Phi_{lm}\big(\Phi_{lm}^{-1}(s_{lm})\Phi_m^{-1}(r_m)\big), \tag{17}$$

$s_{lm} \in S_{lm}, r_m \in R_m$. Similarly, we obtain S_{lm} as a left R_l-module; in fact, S_{lm} is an R_l-R_m-bimodule.

In the light of Lemma 3.1 it should now be clear, again keeping (6) in mind, that

$$\phi_{13}^{(2)} : S_{12} \otimes_{R_2} S_{23} \to S_{13}, \quad \text{defined by} \quad (s_{12}, s_{23}) \mapsto \Phi_{13}\big(\Phi_{12}^{-1}(s_{12})\Phi_{23}^{-1}(s_{23})\big), \tag{18}$$

$s_{12} \in S_{12}, s_{23} \in S_{23}$, is an R_1-R_3-homomorphism, and that, with $\varphi_{14}^{(2)}$, $\varphi_{14}^{(3)}$ and $\varphi_{24}^{(3)}$ defined similarly, the diagram

$$
\begin{array}{ccc}
S_{12} \otimes_{R_2} S_{23} \otimes_{R_3} S_{34} & \xrightarrow{id_{12} \otimes \varphi_{24}^{(3)}} & S_{12} \otimes_{R_2} S_{24} \\
\downarrow{\scriptstyle \varphi_{13}^{(2)} \otimes id_{34}} & & \downarrow{\scriptstyle \varphi_{14}^{(2)}} \\
S_{13} \otimes_{R_3} S_{34} & \xrightarrow{\varphi_{14}^{(3)}} & S_{14}
\end{array}
$$

commutes, where id_{lm} denotes the identity function on S_{lm}.

For $1 \leq l < m \leq t$, we set

$$T_{lm} := \begin{cases} S_{lm}, & \text{if } K_l \subset K_m; \\ \{0\}, & \text{otherwise}; \end{cases}$$

Thus, armed with the foregoing results, we conclude from Theorem 2.2, Lemma 3.1 and, for example, [1] or [7], that we have proved our main result:

Theorem 3.2. *Let N be a centralizer near-ring $M_A(G)$, where $(M_A(G), +)$ is a finite abelian group and A is a group of automorphisms of the finite group G such that the stabilizer condition is satisfied. Then, with the above notation, the right ring $\mathcal{R}$ associated with N is isomorphic to the $t \times t$ generalized blocked triangular matrix ring*

$$
\begin{bmatrix}
\mathbb{M}_{t_1}(R_1) & \cdots & \mathbb{M}_{t_1 \times t_l}(T_{1l}) & \cdots & \mathbb{M}_{t_1 \times t_m}(T_{1m}) & \cdots & \mathbb{M}_{t_1 \times t_p}(T_{1p}) \\
& \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
& & \mathbb{M}_{t_l}(R_l) & \cdots & \mathbb{M}_{t_l \times t_m}(T_{lm}) & \cdots & \mathbb{M}_{t_l \times t_p}(T_{lp}) \\
& & & \ddots & \vdots & \ddots & \vdots \\
& & & & \mathbb{M}_{t_m}(R_m) & \cdots & \mathbb{M}_{t_m \times t_p}(T_{mp}) \\
& 0 & & & & \ddots & \vdots \\
& & & & & & \mathbb{M}_{t_p}(R_p)
\end{bmatrix}
$$

via

$$
\Psi : \sum_u \rho[n_u] \mapsto [x_{ij}],
$$

where

$$
x_{ij} := a_1^{(ij)} \bar{\alpha}_{m,1} + \cdots + a_{q_m^{(l)}}^{(ij)} \bar{\alpha}_{m, q_m^{(l)}},
$$

with

$$
a_1^{(ij)} := |\{u : n_u(v_i) = \alpha_{m,1} v_j\} vert \pmod{k_m}, \ldots, a_{q_m^{(l)}}^{(ij)} := |\{u : n_u(v_i) = \alpha_{m, q_m^{(l)}} v_j\}| \pmod{k_m},
$$

if $i \in \{i_{l,1}, \ldots, i_{l,t_l}\}, j \in \{i_{m,1}, \ldots, i_{m,t_m}\}, 1 \leq l \leq m \leq p$, and $q_m^{(m)} := q_m$.

By [14, Section 2] the class of structural matrix rings may be viewed as a subclass of the class of generalized blocked triangular matrix rings in the sense that only one base ring plays a role in the description of any particular structural matrix ring and that no substructures of that base ring play a role in the mentioned description. To be more precise, if $B := [b_{ij}]$ is a reflexive and transitive $n \times n$ Boolean matrix, i.e. $b_{ii} = 1$ for every i, and $b_{ij} = 1 = b_{jk}$ implies $b_{ik} = 1, 1 \leq i, j, k \leq n$, and if R is any ring, then the structural matrix ring $\mathbb{M}(B, R)$ is the subring of the complete matrix ring $\mathbb{M}_n(R)$ comprising all matrices with 0 in position (k, l) whenever $b_{kl} = 0$. Recent papers on structural matrix rings, studied in their own right, include [3], [4], [15] and [16].

From Theorem 3.2 we conclude that if the rings $R_m, m = 1, \ldots, p$, are isomorphic to one another, and if $R_m = S_{lm}$ (as sets) in the case $K_l \subset K_m, 1 \leq l < m \leq p$, which

occurs if, for example, K_m is a normal subgroup of A, then $\mathcal{R}$ is isomorphic to a $t \times t$ structural matrix ring over R_1. See Example 3.4.

Example 3.3. Let $G = \mathbb{Z}_8 \oplus \mathbb{Z}_8$, and let A be the group (under matrix multiplication) of real matrices generated by

$$\alpha := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \beta := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

acting on G via matrix multiplication, where we view the elements of $\mathbb{Z}_8 \oplus \mathbb{Z}_8$ as column matrices. Then (see, for example [8, p. 33, Exercise 4]) $A \cong \mathcal{D}_4$, the dihedral group of order 8, generated by α and β, with $\alpha^4 = id$ (the identity matrix), $\beta^2 = id$ and $(\alpha\beta)^2 = id$. Since $\alpha^i\beta = \beta\alpha^{4-i}$ for $0 \leq i \leq 3$, the elements of A can be represented as id, α, $\alpha^2, \alpha^3, \beta, \alpha\beta, \alpha^2\beta$ and $\alpha^3\beta$, with

$$\alpha^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \alpha^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\alpha\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \alpha^2\beta = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \alpha^3\beta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here $t = 14$, and we select the A-orbit representatives $v_1, \ldots, v_{14}$ of $\mathcal{O}_1, \ldots, \mathcal{O}_{14}$ as follows:

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, v_6 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, v_7 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

$$v_8 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, v_9 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, v_{10} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, v_{11} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, v_{12} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, v_{13} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, v_{14} = \begin{bmatrix} 4 \\ 4 \end{bmatrix},$$

Then

$$\mathcal{O}_{14} < \mathcal{O}_{13} < \mathcal{O}_9 \sim \mathcal{O}_8 \sim \mathcal{O}_7 \sim \mathcal{O}_6 \sim \mathcal{O}_5 \sim \mathcal{O}_4 < \mathcal{O}_3 \sim \mathcal{O}_2 \sim \mathcal{O}_1,$$
$$\mathcal{O}_{14} < \mathcal{O}_{12} \sim \mathcal{O}_{11} \sim \mathcal{O}_{10} < \mathcal{O}_3 \sim \mathcal{O}_2 \sim \mathcal{O}_1,$$

and it can be verified that (3) is satisfied. Thus $p = 5$, and the five equivalence classes induced by $\sim$ on $\{v_1, \ldots, v_{14}\}$ are $[v_{i_1}] := \{v_1, v_2, v_3\}$, $[v_{i_2}] := \{v_4, v_5, v_6, v_7, v_8, v_9\}$, $v_{i_3} := \{v_{10}, v_{11}, v_{12}\}$, $v_{i_4} := \{v_{13}\}$ and $v_{i_5} := \{v_{14}\}$. The stabilizers $K_m, m = 1, \ldots, 5$, are

$$K_1 = \{id\}, K_2 = \{id, \alpha\beta\}, K_3 = \{id, \alpha^2\beta\}, K_4 = \{i, \alpha^2, \alpha\beta, \alpha^3\beta\}, k_5 = A,$$

with

$$K_1 \subset K_2 \subset K_4 \subset K_5 \quad \text{and} \quad K_1 \subset K_3 \subset K_5.$$

Note that K_4 is the only non-trivial normal subgroup of A amongst the K_m's. By Theorem 3.2 $\mathcal{R}$ is isomorphic to the 14×14 generalized blocked triangular matrix ring

$$\begin{bmatrix} \mathbb{M}_3(R_1) & \mathbb{M}_{3\times 6}(S_{12}) & \mathbb{M}_{3\times 3}(S_{13}) & \mathbb{M}_{3\times 1}(R_4) & \mathbb{M}_{3\times 1}(R_5) \\ & \mathbb{M}_6(R_2) & 0 & \mathbb{M}_{6\times 1}(R_4) & \mathbb{M}_{6\times 1}(R_5) \\ & & \mathbb{M}_3(R_3) & 0 & \mathbb{M}_{3\times 1}(R_5) \\ & 0 & & R_4 & R_5 \\ & & & & R_5 \end{bmatrix},$$

where we use the set equality in (15), and where the zeros denote the zero matrices of the appropriate sizes.

Direct calculations show that $k_1 = k_2 = k_3 = 8$ and $k_4 = k_5 = 2$. Since $\alpha^2 = -id$, $\alpha^3\beta = -\alpha\beta$, $\alpha^2\beta = -\beta$ and $\alpha^3 = -\alpha$, it follows readily that the ring R_1 is isomorphic to the subring

$$\left\{ \begin{bmatrix} a & c \\ d & b \end{bmatrix} : a, b, c, d \in \mathbb{Z}_8, a - b, c - d \in 2\mathbb{Z}_8 \right\}$$

of $\mathbb{M}_2(\mathbb{Z}_8)$ via

$$\{a_1 id + a_2\alpha + a_3\alpha^2 + a_4\alpha^3 + a_5\beta + a_6\alpha\beta + a_7\alpha^2\beta + a_8\alpha^3\beta : a_i \in \mathbb{Z}_8, \forall i\}$$
$$\mapsto \begin{bmatrix} (a_1 - a_3) + (a_6 - a_8) & (a_5 - a_7) + (a_2 - a_4) \\ (a_5 - a_7) - (a_2 - a_4) & (a_1 - a_3) - (a_6 - a_8) \end{bmatrix}.$$

Next, $N(K_2)/K_2 = \{idK_2, \alpha^2 K_2\}$, and since $\alpha^2 = -id$, it follows that $R_2 \cong \mathbb{Z}_8$ as rings. Similarly, $R_3 \cong \mathbb{Z}_8$ and $R_5 \cong \mathbb{Z}_2$ as rings. However, $N(K_4)/K_4 = \{idK_4, \alpha K_4\}$ and so $R_4 \cong \mathbb{Z}_2[\mathbb{Z}_2]$, the group ring of (the additive group) $\mathbb{Z}_2$ over (the field) $\mathbb{Z}_2$. Similarly, $S_{12} \cong S_{13} \cong (\mathbb{Z}_8[\mathbb{Z}_2], +)$, and so

$$\mathcal{R} \cong \begin{bmatrix} \mathbb{M}_3(R_1) & \mathbb{M}_{3\times 6}(\mathbb{Z}_8[\mathbb{Z}_2]) & \mathbb{M}_{3\times 3}(\mathbb{Z}_8[\mathbb{Z}_2]) & \mathbb{M}_{3\times 1}(\mathbb{Z}_2[\mathbb{Z}_2]) & \mathbb{M}_{3\times 1}(\mathbb{Z}_2) \\ & \mathbb{M}_6(\mathbb{Z}_8) & 0 & \mathbb{M}_{6\times 1}(\mathbb{Z}_2[\mathbb{Z}_2]) & \mathbb{M}_{6\times 1}(\mathbb{Z}_2) \\ & & \mathbb{M}_3(\mathbb{Z}_8) & 0 & \mathbb{M}_{3\times 1}(\mathbb{Z}_2) \\ & 0 & & \mathbb{Z}_2[\mathbb{Z}_2] & \mathbb{Z}_2 \\ & & & & \mathbb{Z}_2 \end{bmatrix}.$$

Since we have already provided an illustration of Ψ (see Example 2.3) as far as the diagonal blocks, i.e. the complete matrix rings $\mathbb{M}_{t_m}(R_m)$, are concerned, we conclude this example by illustrating the multiplicativity of Ψ as regards the non-symmetric part of the above generalized blocked triangular matrix ring by using (17) and (18).

Let $n_u \in N, u = 1, \dots, 15$, be defined as in the following table:

	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
v_1	idv_1	αv_1	$\alpha^2 v_1$	$\alpha\beta v_4$	$\alpha^2\beta v_4$	βv_{13}	$\alpha\beta v_{13}$	$\alpha^2\beta v_{14}$	$\alpha^3\beta v_4$		$\alpha^3 v_4$		$\alpha^3 v_{13}$	βv_{14}	
v_4									$\alpha\beta v_4$	$\alpha^3\beta v_4$		$\alpha^2 v_4$	$\alpha^2\beta v_{13}$		$\alpha^3\beta v_{14}$
v_{13}														αv_{14}	
v_{14}															$\alpha\beta v_{14}$

(The table is read as follows: $n_9(v_1) = \alpha^3\beta v_4$, $n_9(v_4) = \alpha\beta v_4$, and n_9 is 0 on the other twelve nonzero A-orbits.)

Since $A_{12} = \{idK_2, \alpha K_2, \alpha^2 K_2, \alpha^3 K_2\}$, $A_{13} = \{idK_3, \alpha K_3, \alpha^2 K_3, \alpha_3 K_3\}$ and $A_4 = A_{14} = A_{24} = \{idK_4, \alpha K_4\}$, and since $\alpha^2 = -id$ and $\alpha^3 = -\alpha$, we use the following representation for S_{12}, S_{13} and $R_4 = S_{14} = S_{24}$:

$S_{12} = \{a_1\overline{id} + a_2\bar{\alpha}: a_1, a_2 \in \mathbb{Z}_8\}$ (where the cosets are the left cosets of K_2 in A),

$S_{13} = \{b_1\overline{id} + b_2\bar{\alpha}: b_1, b_2 \in \mathbb{Z}_8\}$ (left cosets of K_3 in A),

$R_4 = S_{14} = S_{24} = \{c_1\overline{id} + c_2\bar{\alpha}: c_1, c_2 \in \mathbb{Z}_2\}$ (left cosets of K_4 in A).

The other representations should now be clear.

Let $V, W \in \mathcal{R}$ be the following:

$$V := \sum_{i=1}^{8} a_i\rho[n_i] = 4\rho[n_1] + 5\rho[n_2] + \rho[n_3] + \rho[n_4] + 3\rho[n_5] + \rho[n_6] + 7\rho[n_7] + 5\rho[n_8];$$

$$W := \sum_{j=9}^{15} b_j\rho[n_j] = 2\rho[n_9] + \rho[n_{10}] + \rho[n_{11}] + 5\rho[n_{12}] + 3\rho[n_{13}] + 7\rho[n_{14}] + 5\rho[n_{15}].$$

(Here $a_1 = 4, \dots, a_8 = 5$, and $b_9 = 2, \dots, b_{15} = 5$.) Then by Theorem 3.2 the first row of the 14×14 matrix $\Psi(V)$ is

$$[3id + 5\alpha \quad 0 \quad 0 \quad idK_2 + 3\alpha K_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad idK_4 + \alpha K_4 \quad 1].$$

We abbreviate this representation for $\Psi(V)$ by

$$(3id + 5\alpha)\epsilon_{1,1} + (idK_2 + 3\alpha K_2)\epsilon_{1,4} + (idK_4 + \alpha K_4)\epsilon_{1,13} + \epsilon_{1,14},$$

using $\epsilon_{i,j}$ to denote the matrix with 1 in position (i, j) and zeros elsewhere. Similarly,

$$\Psi(W) = (6idK_2 + 7\alpha K_2)\epsilon_{1,4} + (\alpha K_4)\epsilon_{1,13} + \epsilon_{1,14} + (3idK_2)\epsilon_{4,4} + (\alpha K_4)\epsilon_{4,13}$$
$$+ \epsilon_{4,14} + \epsilon_{13,14} + \epsilon_{14,14}.$$

Now $VW = \sum_{i=1}^{8}\sum_{j=9}^{15} a_i b_j \rho[n_j n_i]$, and

$$\left(\sum_{i=1}^{8}\sum_{j=9}^{15} a_i b_j n_j n_i\right)(v_1) = (7id + 7\alpha + \alpha^2 + 7\alpha^2 + 28\beta + 2\alpha\beta + 14\alpha^3\beta)v_4$$

$$+ (24id + 3\alpha + 15\alpha^3)v_{13}$$
$$+ (25\alpha + 5\alpha + 15\alpha^3 + 77\beta + 35\alpha\beta + 7\alpha^2\beta + 7\alpha^3\beta)v_{14},$$

and $\sum_{i=1}^{8}\sum_{j=9}^{15} a_i b_j n_j n_i$ is 0 on the other thirteen orbits. Hence, again by Theorem 3.2,

$$\Psi(VW) = (2idK_2 + 4\alpha K_2)\epsilon_{1,4} + \epsilon_{1,14},$$

which equals the product of the matrices $\Psi(V)$ and $\Psi(W)$.

In the following example the generalized blocked triangular matrix ring in Theorem 3.2 turns out to be a structural matrix ring. (See the discussion following Theorem 3.2.)

Example 3.4. Let $A = \{id, \alpha, \beta, \epsilon, \gamma, \delta\}$, where the elements of A are the following elements of $\mathbb{M}_4(\mathbb{Z}_2)$: id is the 4×4 identity matrix and

$$\alpha := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \beta := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}(= \alpha^T),$$

$$\epsilon := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \gamma := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \delta := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Under matrix multiplication A is a group which is isomorphic to the symmetric group S_3; in fact, the multiplication table for A is the following:

	α	β	ϵ	γ	δ
α	id	γ	δ	β	ϵ
β	δ	id	γ	ϵ	α
ϵ	γ	δ	id	α	β
γ	ϵ	α	β	δ	id
δ	β	ϵ	α	id	γ

So under matrix multiplication A may be viewed as a group of automorphisms of $G := \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In this case there are seven nonzero A-orbits of G. Let the orbit representatives be as follows:

$$
v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_7 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.
$$

Then $\mathcal{O}_7 \sim \mathcal{O}_6 \sim \mathcal{O}_5 < \mathcal{O}_4 \sim \mathcal{O}_3 \sim \mathcal{O}_2 \sim \mathcal{O}_1$, and (3) is satisfied. Furthermore, $K_1 = \{id, \alpha\} \subset K_2 = A$, $N(K_1) = K_1$ and $k_1 = 2 = k_2$. Therefore $\mathcal{R}$ is isomorphic to the following 7×7 structural matrix ring over $\mathbb{Z}_2$:

$$
\begin{bmatrix} \mathbb{M}_4(\mathbb{Z}_2) & \mathbb{M}_{4\times3}(\mathbb{Z}_2) \\ 0 & \mathbb{M}_3(\mathbb{Z}_2) \end{bmatrix}.
$$

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