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Compact composition operators on model spaces*

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Abstract. **Let $\varphi : B_d \to \mathbb{D}$, $d \geq 1$, be a holomorphic function, where B_d denotes the open unit ball of $\mathbb{C}^d$ and $\mathbb{D} = B_1$. Let $\Theta : \mathbb{D} \to \mathbb{D}$ be an inner function and let K^p_Θ denote the corresponding model space. For $p > 1$, we characterize the compact composition operators $C_\varphi : K^p_\Theta \to H^p(B_d)$, where $H^p(B_d)$ denotes the Hardy space.**

1 Introduction

Let B_d denote the open unit ball of $\mathbb{C}^d$, $d \geq 1$, and let ∂B_d denote the unit sphere. Let $\sigma = \sigma_d$ denote the normalized Lebesgue measure on the sphere ∂B_d. We also use symbols $\mathbb{D}$ and $\mathbb{T}$ for the unit disc B_1 and the unit circle ∂B_1, respectively.

For $d \geq 1$, let $\mathcal{H}ol(B_d)$ denote the space of holomorphic functions in B_d. For $0 < p < \infty$, the classical Hardy space $H^p = H^p(B_d)$ consists of those $f \in \mathcal{H}ol(B_d)$ for which

$$\|f\|^p_{H^p} = \sup_{0<r<1} \int_{\partial B_d} |f(r\zeta)|^p \, d\sigma_d(\zeta) < \infty.$$

As usual, we identify the Hardy space $H^p(B_d)$, $p > 0$, and the space $H^p(\partial B_d)$ of the corresponding boundary values.

It is well known that the composition operator $C_\varphi : f \mapsto f \circ \varphi$ sends $H^p(\mathbb{D})$ into $H^p(B_d)$, $p > 0$. Indeed, let $f \in H^p(\mathbb{D})$. Then $|f|^p \leq h$ for an appropriate harmonic function h on $\mathbb{D}$. So, $|f \circ \varphi|^p \leq h \circ \varphi$, hence $f \circ \varphi \in H^p(B_d)$, as required.

Since C_φ maps $H^2(\mathbb{D})$ into $H^2(B_d)$, it is natural to ask for a characterization of those φ for which $C_\varphi : H^2(\mathbb{D}) \to H^2(B_d)$ is a compact operator. Two-sided estimates for the essential norm of the operator $C_\varphi : H^2(\mathbb{D}) \to H^2(B_d)$, $d \geq 1$, were obtained by B.R. Choe [4] in terms of the corresponding pull-back measure. A more explicit approach based on the Nevanlinna counting function was proposed in [10] for $d = 1$; see also [1] for an extension to the case $d \geq 1$.

***Definition* 1.1** A holomorphic function $\Theta : \mathbb{D} \to \mathbb{D}$ is called *inner* if $|\Theta(\zeta)| = 1$ for σ_1-a.e. $\zeta \in \mathbb{T}$.

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In the above definition, $\Theta(\zeta)$ stands, as usual, for $\lim_{r\to 1-}\Theta(r\zeta)$. Recall that the corresponding limit is known to exist σ_1-a.e. Also, by the above definition, unimodular constants are not inner functions.

Given an inner function Θ on $\mathbb{D}$, the classical model space K_Θ is defined as

$$K_\Theta = H^2(\mathbb{D}) \ominus \Theta H^2(\mathbb{D}).$$

In this paper, firstly, we characterize those φ for which $C_\varphi : K_\Theta \to H^2(B_d)$ is a compact operator. For $d = 1$, such characterizations were earlier obtained in [9]. Secondly, in Theorem 4.3, we solve the analogous problem for $C_\varphi : K^p_\Theta \to H^p(B_d)$, $p > 1$, where $K^p_\Theta := H^p \cap \Theta\overline{H^p}$. Note that $K^2_\Theta = K_\Theta$.

Organization of the paper

Auxiliary results, including Cohn's inequality and Stanton's formula, are presented in Section 2. Compact composition operators $C_\varphi : K_\Theta \to H^2(B_d)$ are characterized in Section 3. Real interpolation of Banach spaces is used in Section 4 to prove that the compactness of $C_\varphi : K^p_\Theta \to H^p(B_d)$ does not depend on p for $1 < p < \infty$.

2 Auxiliary results

2.1 Littlewood–Paley identity and related results

Given an $f \in H^2(\mathbb{D})$, the Littlewood–Paley identity states that

$$\|f\|^2_{H^2(\mathbb{D})} = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(w)|^2 \log\frac{1}{|w|}\, dA(w), \tag{2.1}$$

where A denotes the normalized area measure on $\mathbb{D}$.

2.1.1 Cohn's inequality

Let $\Theta : \mathbb{D} \to \mathbb{D}$ be an inner function. If $f \in K_\Theta$, then the lower estimate in (2.1) is improvable in the sense of the following theorem.

***Theorem* 2.1** *([5]) Let Θ be an inner function. There exists $p \in (0, 1)$ such that*

$$\|f\|^2_{H^2(\mathbb{D})} \ge |f(0)|^2 + C_p \int_{\mathbb{D}} \frac{|f'(w)|^2}{(1-|\Theta(w)|)^p} \log\frac{1}{|w|}\, dA(w) \tag{2.2}$$

for all $f \in K_\Theta$.

2.1.2 Stanton's formula

To study the composition operator generated by a holomorphic self-map ϕ of the unit disk, J. H. Shapiro [10] used for $f \circ \phi$ an analog of (2.1). This analog is based on the Nevanlinna counting function N_ϕ defined as

$$N_\phi(w) = \sum_{z\in\mathbb{D}:\, \phi(z)=w} \log\frac{1}{|z|}, \quad w \in \mathbb{D} \setminus \{\phi(0)\},$$

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where each pre-image is counted according to its multiplicity. The key technical result in Shapiro's argument is the following Stanton formula.

***Theorem* 2.2** *([10]) Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then*

$$\|f \circ \phi\|^2_{H^2(\mathbb{D})} = |f(\phi(0))|^2 + 2 \int_{\mathbb{D}} |f'(w)|^2 N_\phi(w)\, dA(w). \tag{2.3}$$

Given an $f \in \mathcal{H}ol(B_d)$ and a point $\zeta \in \partial B_d$, the slice-function $f_\zeta \in \mathcal{H}ol(\mathbb{D})$ is defined by $f_\zeta(\lambda) = f(\lambda\zeta)$, $\lambda \in \mathbb{D}$.

***Corollary* 2.3** *Let $\varphi : B_d \to \mathbb{D}$, $d \geq 1$, be a holomorphic function. Then*

$$\|f \circ \varphi\|^2_{H^2(B_d)} = |f(\varphi(0))|^2 + 2 \int_{\mathbb{D}} |f'(w)|^2 \left(\int_{\partial B_d} N_{\varphi_\zeta}(w)\, d\sigma_d(\zeta) \right) dA(w). \tag{2.4}$$

Proof Let $\zeta \in \partial B_d$. Applying Theorem 2.2 with $\phi = \varphi_\zeta$, and integrating with respect to the normalized Lebesgue measure σ_d on ∂B_d, we obtain (2.4). ∎

2.2 Subharmonicity inequality for the Nevanlinna counting function

***Proposition* 2.4** *([10, Section 4.6]) Let $w \in \mathbb{D}$ and $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Suppose that Δ is a disk centered at w and such that $\phi(0) \notin \Delta$. Then*

$$N_\phi(w) \leq \frac{1}{A(\Delta)} \int_\Delta N_\phi(z)\, dA(z). \tag{2.5}$$

***Corollary* 2.5** *Let $w \in \mathbb{D}$ and $\varphi : B_d \to \mathbb{D}$ be a holomorphic function. Suppose that Δ is a disk centered at w and such that $\varphi(0) \notin \Delta$. Then*

$$\int_{\partial B_d} N_{\varphi_\zeta}(w)\, d\sigma_d(\zeta) \leq \frac{1}{A(\Delta)} \int_\Delta \left(\int_{\partial B_d} N_{\varphi_\zeta}(z)\, d\sigma_d(\zeta) \right) dA(z). \tag{2.6}$$

Proof Let $\zeta \in \partial B_d$. To obtain (2.6), we apply Proposition 2.4 with $\phi = \varphi_\zeta$, integrate with respect σ_d, and use Fubini's Theorem. ∎

2.3 Reproducing kernels for K_Θ

Recall that the reproducing kernel $k_w(z)$ for K_Θ is given by

$$k_w(z) = \frac{1 - \Theta(z)\overline{\Theta}(w)}{1 - z\overline{w}}, \quad \|k_w\|^2 = \frac{1 - |\Theta(w)|^2}{1 - |w|^2}.$$

Let $D_\varepsilon(w) = \{z \in \mathbb{D} : |z - w| < \varepsilon|1 - z\overline{w}|\}$, that is, let $D_\varepsilon(w)$ denote the pseudohyperbolic ε-disk centered at $w \in \mathbb{D}$.

***Lemma* 2.6** *([9, Lemma 1]) Let $\{w_n\} \subset \mathbb{D}$ be such that $|w_n| \to 1$ and*

$$|\Theta(w_n)| < a \tag{2.7}$$

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for some $a \in (0,1)$. Then

(i) $k_{w_n}/\|k_{w_n}\| \xrightarrow{w^*} 0$ *as* $n \to \infty$;
(ii) *there exist* $\varepsilon > 0$, $C > 0$, *and* $n_0 \in \mathbb{N}$ *such that*

$$|k'_{w_n}(z)| \ge \frac{C}{(1-|w_n|^2)^2}, \quad z \in D_\varepsilon(w_n),$$

for all $n \ge n_0$.

3 Compact composition operators on K_Θ

***Theorem* 3.1** *Let* $\varphi : B_d \to \mathbb{D}$, $d \ge 1$, *be a holomorphic function, and let* $\Theta : \mathbb{D} \to \mathbb{D}$ *be an inner function. Then the following properties are equivalent.*

(i) *One has*

$$\int_{\partial B_d} N_{\varphi_\zeta}(w) \frac{1-|\Theta(w)|}{1-|w|}\, d\sigma_d(\zeta) \to 0 \quad \textit{as } |w| \to 1-. \tag{3.1}$$

(ii) $C_\varphi : K_\Theta \to H^2(B_d)$ *is a compact operator.*

Proof The principal arguments below are modelled after [10].
(i) $\Rightarrow$ (ii) For $n \in \mathbb{N}$, let

$$K_{\Theta,n}(\mathbb{D}) = \{f \in K_\Theta(\mathbb{D})\colon f \text{ has zero of order } n \text{ at the origin}\}.$$

Let $P_n : K_\Theta(\mathbb{D}) \to K_{\Theta,n}(\mathbb{D})$ denote the orthogonal projector. We claim that

$$\|C_\varphi P_n\|_{K_\Theta(\mathbb{D}) \to H^2(B_d)} \to 0 \tag{3.2}$$

as $n \to \infty$. Then C_φ is compact, since it is approximable by the finite-rank operators $C_\varphi(I - P_n)$.

To verify (3.2), fix $\varepsilon > 0$ and $f \in K_\Theta(\mathbb{D})$, $\|f\| \le 1$. Let $g_n = P_n f$, then $\|g_n\| \le 1$.

Firstly, let $p \in (0,1)$ be that provided by Theorem 2.1. By (3.1),

$$\int_{\partial B_d} N^p_{\varphi_\zeta}(w) \frac{(1-|\Theta(w)|)^p}{(1-|w|)^p}\, d\sigma_d(\zeta) \to 0 \quad \text{as } |w| \to 1-. \tag{3.3}$$

For $r \in (0,1)$, put

$$\mathfrak{N}_{p,r} = \sup_{r<|w|<1} \int_{\partial B_d} N_{\varphi_\zeta}(w) \frac{(1-|\Theta(w)|)^p}{1-|w|}\, d\sigma_d(\zeta).$$

By Littlewood's inequality [8],

$$N_{\varphi_\zeta}(w) \le \log\left|\frac{1-\overline{w}\varphi(0)}{\varphi(0)-w}\right|, \quad w \in \mathbb{D} \setminus \{\varphi(0)\}.$$

Thus, for $\frac{1+|\varphi(0)|}{2} < |w| < 1$, we obtain $N_{\varphi_\zeta}(w) \le C(1-|w|)$, hence,

$$N^{1-p}_{\varphi_\zeta}(w) \le C(1-|w|)^{1-p}, \quad \frac{1+|\varphi(0)|}{2} < |w| < 1.$$

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Therefore,

$$\mathfrak{N}_{p,r} \longrightarrow 0 \quad \text{as } r \longrightarrow 1- \tag{3.4}$$

by (3.3). Now, using (3.4) and applying (2.2) to $g_n \in K_\Theta$, $\|g_n\| \leq 1$, choose R so close to 1 that

$$\int_{\mathbb{D}\setminus R\mathbb{D}} \frac{|g_n'(w)|^2}{(1-|\Theta(w)|)^p} \mathfrak{N}_{p,R}(1-|w|)\,dA(w) < C\|g_n\|\mathfrak{N}_{p,R} < \varepsilon \tag{3.5}$$

for all $n \in \mathbb{N}$.

Secondly,

$$\max_{|w|<R} |g_n'(w)| \longrightarrow 0$$

as $n \longrightarrow \infty$. Thus,

$$\int_{R\mathbb{D}} |g_n'(w)|^2 \int_{\partial B_d} N_{\varphi_\zeta}(w)\,d\sigma_d(\zeta)\,dA(w) < \varepsilon \tag{3.6}$$

for all sufficiently large n.

By (2.4),

$$\begin{aligned}\frac{1}{2}\|C_\varphi g_n\|^2_{H^2(B_d)} &= \int_{\mathbb{D}} |g_n'(w)|^2 \int_{\partial B_d} N_{\varphi_\zeta}(w)\,d\sigma_d(\zeta)\,dA(w) + |g_n(\varphi(0))|^2\\ &= \int_{\mathbb{D}\setminus R\mathbb{D}} + \int_{R\mathbb{D}} + |g_n(\varphi(0))|^2.\end{aligned}$$

Observe that $|g_n(\varphi(0))|^2 \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, combining (3.5) and (3.6), we conclude that

$$\|C_\varphi g_n\|_{H^2(B_d)} \longrightarrow 0,$$

as required.

(ii) $\Longrightarrow$ (i) Assume that (i) does not hold. Then there exists a sequence $\{w_n\}_{n=1}^\infty \subset \mathbb{D}$ such that $|w_n| \longrightarrow 1$ and

$$\int_{\partial B_d} N_{\varphi_\zeta}(w_n)\frac{1-|\Theta(w_n)|}{1-|w_n|}\,d\sigma_d(\zeta) \geq c > 0. \tag{3.7}$$

Sequentially applying (2.4), Lemma 2.6(ii) and Corollary 2.5, we obtain

$$\begin{aligned}\|C_\varphi k_{w_n}(z)\|^2/\|k_{w_n}\|^2 &\geq \frac{1}{\|k_{w_n}\|^2}\int_{\mathbb{D}} |k_{w_n}'(z)|^2 \left(\int_{\partial B_d} N_{\varphi_\zeta}(z)\,d\sigma_d(\zeta)\right) dA(z)\\ &\geq \int_{D(w_n,\varepsilon)} \frac{C}{(1-|w_n|^2)^3}\left(\int_{\partial B_d} N_{\varphi_\zeta}(z)\,d\sigma_d(\zeta)\right) dA(z)\\ &\geq \frac{C_\varepsilon}{1-|w_n|^2}\int_{\partial B_d} N_{\varphi_\zeta}(w_n)\,d\sigma_d(\zeta).\end{aligned} \tag{3.8}$$

Now, recall that C_φ is a compact operator by (ii), thus, $\|C_\varphi k_{w_n}(z)\|/\|k_{w_n}\| \longrightarrow 0$ by Lemma 2.6(i). Therefore, (3.8) contradicts (3.7). The proof of the theorem is finished. ■

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4 Compact composition operators on K^p_Θ

For $0 < p < \infty$ and an inner function Θ, let

$$K^p_\Theta = K^p_\Theta(\mathbb{D}) \stackrel{\text{def}}{=} H^p(\mathbb{D}) \cap \Theta\overline{H^p}(\mathbb{D}).$$

It is well known and easy to see that $K^2_\Theta = K_\Theta$.

By definition, an inner function $\Theta : \mathbb{D} \to \mathbb{D}$ is called one-component if the set $\{z \in \mathbb{D} : |\Theta(z)| < r\}$ is connected for some $r \in (0, 1)$. The present section is motivated by the following assertion.

***Proposition* 4.1** *([2, 9]) Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function, $p > 1$ and let Θ be a one-component inner function. Then $C_\phi : K^p_\Theta \to H^p(\mathbb{D})$ is a compact operator if and only if*

$$N_\phi(w)\frac{1 - |\Theta(w)|}{1 - |w|} \to 0 \quad \text{as } |w| \to 1- .$$

Proof As indicated in [9, Section 4], the results of Baranov [2] on compact Carleson embeddings of the model spaces K^p_Θ guarantee that for a *one-component* inner function Θ, the compactness of the composition operator $C_\varphi : K^p_\Theta(\mathbb{D}) \to H^p(\mathbb{D})$ does not depend on $p \in (1, \infty)$. Finally, for $p = 2$, it suffices to apply Theorem 1 from [9] or Theorem 3.1. ∎

In Theorem 4.3 below, we show that the direct analog of Proposition 4.1 holds for an arbitrary inner function Θ. The arguments are based on the real interpolation method for Banach spaces, so we first recall related basic facts.

Let (A_0, A_1) be a compatible couple of Banach spaces. Given $0 < \theta < 1$ and $1 \le q \le \infty$, the real interpolation method provides $(A_0, A_1)_{\theta,q}$, an interpolation space between A_0 and A_1 (see, e.g., [3, Chapter 3] for details).

We need the following one-sided compactness theorem for the real interpolation method.

***Theorem* 4.2** *([6]) Let (A_0, A_1) and (B_0, B_1) be compatible couples of Banach spaces. Assume that $T : A_j \to B_j$, $j = 0, 1$, is a bounded linear operator such that $T : A_0 \to B_0$ is compact. Then $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is a compact operator for all admissible θ and q.*

***Theorem* 4.3** *Let $p > 1$ and let Θ be an inner function. Then $C_\varphi : K^p_\Theta \to H^p(B_d)$ is a compact operator if and only if property (3.1) holds.*

Proof By Theorem 3.1, it suffices to show that the compactness of $C_\varphi : K^p_\Theta(\mathbb{D}) \to H^p(B_d)$ does not depend on $p \in (1, \infty)$. To prove this property, assume that $p_0 \in (1, \infty)$ and $C_\varphi : K^{p_0}_\Theta(\mathbb{D}) \to H^{p_0}(B_d)$ is a compact operator. We are planning to prove that $C_\varphi : K^p_\Theta(\mathbb{D}) \to H^p(B_d)$ is also compact for all $p \in (1, \infty) \setminus \{p_0\}$. In fact, for definiteness and without loss of generality, we may assume that $p > p_0$.

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Fix p and p_1 such that $p_1 > p > p_0$. Define $\theta \in (0,1)$ by the following identity:

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{4.1}$$

On the one hand, (4.1) guarantees that $(K_\Theta^{p_0}, K_\Theta^{p_1})_{\theta,p} = K_\Theta^p$ (see, e.g., [7] for more general results even in the bidisk $\mathbb{D}^2$). On the other hand, it is known that $(H^{p_0}(B_d), H^{p_1}(B_d))_{\theta,p} = H^p(B_d)$, since $p_1 > p_0 > 1$. Indeed, application of the classical Riesz projection reduces interpolation between $H^{p_0}(B_d)$ and $H^{p_1}(B_d)$ to that between L^{p_0} and L^{p_1}.

Now, observe that $C_\varphi : K_\Theta^{p_1}(\mathbb{D}) \to H^{p_1}(B_d)$ is a bounded operator. Indeed, as indicated in the introduction, $C_\varphi : H^q(\mathbb{D}) \to H^q(B_d)$ is bounded for all $0 < q < \infty$. Thus, applying Theorem 4.2 for $q = p$ and the pairs $(K_\Theta^{p_0}, K_\Theta^{p_1})$ and $(H^{p_0}(B_d), H^{p_1}(B_d))$, we conclude that $C_\varphi : K_\Theta^p(\mathbb{D}) \to H^p(B_d)$ is compact, as required. ■

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