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PURE SIMPLE AND INDECOMPOSABLE RINGS

BY
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1. **Introduction.** P. M. Cohn [7] calls a submodule P of the left A-module M pure iff $0 \to E \otimes P \to E \otimes M$ is exact for all right modules E. This concept has been studied in [11] and [12]. We will call a non-zero module pure simple iff its only pure submodules are 0 and itself, and the ring A left pure simple iff it is pure simple as a left A-module. We relate these concepts to the PP and PF rings of Hattori [13], and give several new characterizations of these rings. In order to establish these, we use the following known result: the Jacobson radical of any module is the sum of all its small submodules.

Parts of this paper are contained in the author's doctoral thesis [10] at McGill University.

Throughout this paper A will be an associative ring with l, but not necessarily commutative. All modules are unitary, $\otimes$ means $\otimes_A$, "fg" means finitely generated, and "fp" finitely presented.

2. **Pure Left Ideals.** Before proceeding to the main theorem of this section, we make a number of important definitions which will be used here and later.

A subset S of A is idempotent iff $S^2 = S$, where S^2 is the collection of all finite sums of elements of the form ss' with s and s' in S.

An element $a \in A$ will be called a left zero divisor iff there exists $0 \neq b \in A$ so that $ab = 0$. This is equivalent to saying that the homomorphism $f_a: A \to A$ (as left A-modules) defined by $f_a(b) = ab$ is not mono. Similar comments apply for right zero divisors. If we set $r(a) = (b \in A \mid ab = 0)$, the right annihilator of a, then a is a left zero divisor iff $r(a) \neq 0$. The same comments apply to right zero divisors, with $l(a) = (b \in A \mid ba = 0)$, the left annihilator of a. We note thar $r(a)$ is a right ideal of A and $l(a)$ is a left ideal of A.

A ring without non-zero left zero divisors (which is equivalent to the absence of non-zero right zero divisors) will be called an integral domain; thus A is an integral domain iff $l(a) = 0$ iff $r(a) = 0$ for all $0 \neq a \in A$.

A left A-module $M \neq 0$ will be called simple (resp. pure simple, indecomposable) iff 0 and M are the only submodules (resp. pure submodules, direct summands) of M. The ring A will be called left simple (resp. left pure simple, left indecomposable) iff it is simple (resp. pure simple, indecomposable) as a left module; and it will be called simple (resp. pure simple, indecomposable) iff it is both left and right simple (resp. pure simple, indecomposable). Clearly every simple module or ring is pure

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simple, and every pure simple module or ring is indecomposable. Note that a left simple ring is a division ring.

THEOREM 1. *For any left ideal P of A the following conditions are equivalent:*

(1) *A/P is flat*

(1′) *P is pure in A*

(2) *$KP=K\cap P$ for all right ideals K*

(2′) *$KP=K\cap P$ for all fg right ideals K*

(2″) *$KP=P\cap P$ for all principal right ideals K*

(3) *$aP=aA\cap P$ for all a in A*

(4) *For each p in P, there exists an a in $r(p)=(x\in A \mid px=0)$, such that $\hat{a}=\hat{l}$ (where $\hat{a}$ is the image of a in A/P). Furthermore $a\neq 0$ unless $P=A$.*

Proof. Since A is flat, it suffices by Theorem 1 of [12] to show (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(3) $\Rightarrow$ (4): $p\in P\Rightarrow p\in pA\cap P=pP\Rightarrow p=pp'$ for some p' in P. And $a=l-p'$ is in $r(p)$ with $\hat{a}=\hat{l}$ since p' is in P. Clearly $a\neq 0$ unless $P=A$.

(4) $\Rightarrow$ (1): To prove that A/P is flat, it suffices to show that Tor $(A/K, A/P)=0$ for any right ideal K ([4] Prop. 1, p. 55). Now Tor $(A/K, A/P)=(K\cap P)/KP$ by Cartan-Eilenberg ([5] p. 126). If $k\in K\cap P$, there exists $a\in A$ such that $ka=0$ and $\hat{a}=\hat{l}$. Hence $l-a=p\in P$. Therefore $kp=k(l-a)=k$ and $K\cap P=KP$. Hence Tor $(A/K, A/P)=0$.

COROLLARY 1. (1) *If $P\neq A$ is a pure left ideal of A, then all its elements are left zero divisors.*

(2) *Every integral domain is pure simple, and hence, indecomposable.*

Proof. (1) By (4) of the theorem, $r(p)\neq 0$ for each $p\in P$ since $P\neq A$.

(2) is obvious.

COROLLARY 2. (1) *If P is a pure left ideal of A, then for each $p\in P$, there exists a sequence p_i in P, $i=1, 2, \ldots$* such that *$p=p\,p_1\,p_2\ldots p_n$ for all $n=1, 2\ldots$*

(2) 0 *is the only left T-nilpotent pure left ideal of A.*

Remark. Bass [1] calls a (left) ideal I of A left T-nilpotent iff for every sequence x_n, $n=0, 1, 2$ of elements of I, there exists an integer m such that $x_0\,x_1\,x_2\ldots x_m=0$.

Proof. (1) The sequence can be constructed inductively using the method which was used in proving (3) $\Rightarrow$ (4) in the theorem.

(2) For any sequence p_i in P, there exists an n such that $p_1\ldots p_n=0$.

COROLLARY 3. (1) *Every pure left ideal P is idempotent.*

(2) *Let P be a left ideal. If $K\cap P$ is idempotent for all principal right ideals K, then P is pure in A.*

Proof. (1) Let $P'=PA\geq P$. Then $P^2=P'P=P'\cap P=P$.

(2) $K\cap P=(K\cap P)^2=(K\cap P)(K\cap P)\leq KP$. Hence $K\cap P=KP$ for all principal right ideals K, and P is pure in A.

COROLLARY 4. *If P is a principal left ideal, say Ab, then Part* (3) *of the theorem becomes* (3′): $aAb = aA \cap Ab$ *for all a in A.*

3. **Small Submodules.** A submodule S of E is small in E iff for every submodule F of E such that $S+F=E$, we have $F=E$. Clearly any submodule of a small submodule is small.

LEMMA 1. *Any finite sum of small submodules of E is small in E.*

Proof. Use induction on the number of small submodules.

PROPOSITION 1. *For any x in E, any left A-module, Ax is small in E iff x is in the sum of all small submodules of E.*

Proof. $\Rightarrow$: is clear since x is in Ax.

$\Leftarrow$: We must have $x=\sum x_i$ (finite sum) with x_i in S_i, a small submodule of E. Each Ax_i is a submodule of S_i, and hence small in E. Therefore $S=\sum Ax_i$ is small in E, since the sum is finite. Since x is in S, Ax is a submodule of S and therefore small in E.

THEOREM 2. *For any module E, $J(E)$ (the Jacobson radical of E) is the sum of all small submodules of E.*

Remark. The statement of this theorem is due to Sandomierski and Kasch (see [15] Exercise 7, p. 62).

As I have not seen a proof in the literature I add the following:

Proof. By Proposition 1, it suffices to show that Ax is small in E iff x is in $J(E)$, or equivalently: x is not in $J(E)$ iff Ax is not small in E. We shall show the latter statement.

$\Rightarrow$: If x is not in $J(E)$, then there exists some maximal submodule M of E such that x is not in M. Therefore $Ax+M=E$, and Ax cannot be small since $M \neq E$.

$\Leftarrow$: We will call a submodule F of E proper iff $F \neq E$. The collection $\mathscr{C}$ of all proper submodules F of E such that $Ax+F=E$ is nonempty since Ax is not small. Since each F of $\mathscr{C}$ is a proper submodule, we have $x \notin F$ for each F. It is also clear that any proper submodule of E which contains a member of $\mathscr{C}$, is itself a member of $\mathscr{C}$. Therefore if we order $\mathscr{C}$ by set inclusion, the union F of any chain F_i of members of $\mathscr{C}$ is a proper submodule, since x is not in F_i for all i. Hence F is a member of $\mathscr{C}$, since it contains each F_i. Therefore by Zorn's lemma we can choose a maximal element M of $\mathscr{C}$. We claim that M is a maximal submodule of E. Since M is in $\mathscr{C}$, x is not in M and therefore M is proper. Any proper submodule of E containing M is a member of $\mathscr{C}$, and therefore equal to M by the maximality of M in $\mathscr{C}$. Therefore M is a maximal submodule of E. Since x is not in M, x is not in $J(E)$.

COROLLARY 1. *If $J(E)$ is small in E, then it is the largest small submodule of E.*

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Proof. Obvious.

COROLLARY 2. (a) *If E is fg, then $J(E)$ is small in E.*

(b) *$J(A)$ is small in A and therefore all left ideals and all right ideals contained in $J(A)$ are small.*

(c) *If A is a local ring, then all ideals $I \neq A$ (left, right and two-sided) are small.*

Proof. (a) Suppose $J(E)+F=E$. If $F\neq E$ then F is contained in some maximal submodule M of E, since E is fg ([3] Prop. 4, p. 30).

Therefore $J(E)+F$ is contained in $M\neq E$. This contradiction shows that we must have $F=E$, i.e. $J(E)$ small in E.

(b) A is fg.

(c) All ideals are contained in $J(A)$, which is small.

Remark. (i) Corollary 2 is untrue for E non-fg. For example, Q the abelian group ($=Z$-module) of rationals has no maximal subgroups and therefore $J(Q)=Q$.

(ii) Mares [16] has shown that $J(E)$ is small in E if E is semiperfect.

We now come to one of the main theorems of this section.

THEOREM 3. *If P is a projective module then 0 is the only small pure submodule of P.*

Proof. If K is a small pure submodule of P then K is a small pure submodule of a free module $F=P\oplus Q$, with base x_i, say. For any $x\in K$ we have $x=\sum a_i x_i=\sum a_i k_i$ with $k_i\in K$ since K is pure in F. Then $k_i=\sum a_{ij}x_j$ with $a_{ij}\in N=J(A)=$ Jacobson radical of A since $K\subseteq NF=J(F)$ by the smallness of K. Thus $a_j=\sum a_i a_{ij}$ and $I=IN$ where I is the right ideal (finitely) generated by the a_i. By Nakayma's lemma ([3] p. 68) $I=0$ and therefore $K=0$.

COROLLARY. (1) *If $J(P)$ is a small submodule of the projective module P, then 0 is the only pure submodule of P contained in $J(P)$.*

(2) *$J(A)$ contains no pure left ideals and no pure right ideals of A other than 0.*

A module is called regular projective [12], *iff it is a projective module, all of where submodules are pure.*

(3) *If P is a regular projective module, then 0 is the only small submodule, and therefore $NP=J(P)=0$.*

(4) *A flat module has a projective cover iff it is projective. Thus over a regular ring the modules with projective covers are the projective ones.*

(5) *A is (left) perfect iff every flat left A-module has a projective cover.*

(6) *The following are equivalent:*

(a) *every finitely generated flat module is projective*

(b) *every finitely generated flat module has a projective cover*

(c) *every finitely generated flat module is finitely presented.*

THEOREM 4. *Any local ring A is pure simple and hence indecomposable.*

Proof. If $P\neq A$ is any pure left or right ideal of A, then P is contained in the

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radical of A. Therefore P is pure small in A and hence $P=0$ by Corollary 1 of Theorem 3. Thus A is both left and right pure simple.

COROLLARY. *Any regular local ring A is a skewfield.*

Proof. Since A is regular, its radical is pure in A. But the radical is always small in A, and therefore must be 0 by Theorem 3, and A is a skewfield. (See also Theorem 2 of [12].)

THEOREM 5. *Let P be a regular projective left A-module. Then*

(1) *If A is left indecomposable, then P is free.*

(2) *If A is left pure simple, then either $P=0$ or A is left simple (i.e. A has no left ideals other than 0 and A).*

Proof. (1) By Theorem 10 of [12], $P=\oplus J_i$ where the J_i are left ideals which are direct summands of A: therefore $J_i=0$ or A.

(2) Continuing from (1), if $J_i=A$ for some i, then A is regular since the J_i are all regular. But A is pure simple, and therefore A must be left simple. If A is not left simple, then we must have $J_i=0$ for all i, and therefore $P=0$.

COROLLARY. *If A is an integral domain which is not a field, 0 is the only regular projective A-module.*

Proof. A is pure simple, but not simple.

4. **PP and PF Rings.** Following Hattori [13], we will call a ring A left PP (resp. left PF) iff every principal left ideal of A is projective (resp. flat), and PP (resp. PF) iff it is both left and right. PP (resp. PF). We recall that the ring A is left (semi-) hereditary iff every (fg) left ideal is projective. See Cartan-Eilenberg ([5] p. 13).

PROPOSITION 2. (1) *Every left PP ring is left PF; every PP ring is PF.*

(2) *Every left semihereditary ring is left PP.*

(3) *Every regular ring is PP.*

(4) *If wgl $A\leq 1$, then A is PF.*

(5) *If lgl $A\leq 1$, then A is left PP.*

where wgl = weak global dimension and lgl = left global dimension (i.e. homological).

Proof. (1) Every projective left ideal is flat.

(2) Every fg left ideal is projective by definition.

(3) Every fg left and every fg right ideal is a direct summand, and therefore projective.

(4) Every left and every right ideal is flat.

(5) Every left ideal is projective since A is left hereditary.

We now characterize both left PP and left PF rings.

THEOREM 6. *A is left PP (resp. left PF) iff $l(a)=(b\in A \mid ba=0)$ is a direct summand of (resp. pure in) A for all a in A.*

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Proof. For any a in A, we have an exact sequence of left A-modules $0 \to l(a) \to A \to Aa \to 0$. And A is left PP (resp. left PF) iff Aa is projective (resp. flat) for all a in A iff $l(a)$ is a direct summand of (resp. pure in) A for all a in A (see Theorem 1).

THEOREM 7. *Every integral domain is PP and hence PF. Conversely A is an integral domain if either* (1) *A is left pure simple and left PF, or* (2) *A is left indecomposable and left PP.*

Proof. If A is an integral domain, then for all $0 \neq a \in A$ we have $l(a) = r(a) = 0$, which is a direct summand of A. Hence by Theorem 6, A is PP and therefore PF.

Conversely if A is left PF (resp. left PP) then $l(a)$ is pure in (resp. a direct summand of) A. Since A is left pure simple (resp. left indecomposable), $l(a) = 0$ or A. But $l(a) = A \Rightarrow a = 0$. Therefore $l(a) = 0$ for all $0 \neq a \in A$, and A is an integral domain.

Combining Theorem 7 with Corollary 1 of Theorem 1, we have immediately:

COROLLARY 1. *For any ring A the following conditions are equivalent:*

(1) *A is an integral domain.*

(2) *A is pure simple and PF.*

(3) *A is indecomposable and PP.*

(4) *A is left pure simple and left PF.*

(5) *A is left indecomposable and left PP.*

Remark. There are two additional equivalent conditions, obtained by replacing "left" by "right".

COROLLARY 2. *For a local ring A, the following conditions are equivalent:*

(1) *A has no zero divisors.*

(2) *A is PF.*

(3) *A is PP.*

Proof. Any local ring is pure simple and hence indecomposable (Theorem 4).

COROLLARY 3. *If A is a commutative local ring then A is an integral domain iff A is PP iff A is PF.*

Proof. Obvious using Corollary 2. Following Bourbaki ([4] Ex. 12, p. 63) we will call the ring A left coherent iff every fg left ideal of A is fp. Chase [6] has shown that A is left coherent iff every product of flat right A-modules is flat. It is easy to see that every left neotherian ring is left coherent. We call a ring A is left neat iff its left singular ideal is 0. (See Bourbaki [4] and Johnson [14]).

THEOREM 8. (1) *Every left PP ring is left neat, and therefore its complete ring of quotients* (*on the left side*) *is regular.*

(2) *Every left coherent left PF ring is left PP, and therefore has all the properties given in* (1).

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Proof. (1) For any element a in A we have an exact sequence $0 \to l(a) \to A \to Aa \to 0$ which is split exact since Aa is projective. If a is in $K({}_AA)$, the left Johnson singular ideal of A, then $l(a)$ is large in A and therefore $l(a) = A$ since $l(a)$ is a direct summand of A. But $l(a) = A$ implies $a = 0$. Therefore $K({}_AA) = 0$ and A is left neat. If A is left neat then its complete ring of quotients (on the left side) is regular. (See Lambek [15] p. 106.)

(2) For any a in A, the left ideal Aa is a fg and therefore a fp flat module, hence projective by Corollary 2 of Theorem 2 of [11]. Consequently A is left PP.

Remark. Cf. the characterizations given for commutative PP and PF in the next section.

5. **Localization.** We use the same notation and conventions for localization that we used in [12]. In particular, A is always commutative. $\mathcal{M} = \mathcal{M}(A)$ denotes the collection of maximal ideals of A.

THEOREM 9. *Let A have any one of the following properties:*
(1) *wgl $A \leq 0$ (i.e. A is regular).*
(2) *A is semihereditary.*
(3) *wgl $A \leq 1$.*
(4) *A is PP.*
(5) *A is PF.*
(6) *A is semiprime.*
Then the ring A_S has the same property, for any mult. set S.

Proof. (1) and (3): For any mult. set S, wgl $A_S \leq$ wgl A (see Cartan-Eilenberg [5] p. 123 and p. 142), and these parts are immediate.

(2), (4), (5): Any fg (resp. principal) ideal of A_S has the form I_S where I is a fg (resp. principal) ideal of A. If I is projective (resp. flat) then I_S is projective (resp. flat). See ([2] Cor. p. 120 and Prop. 7, p. 90) for the projective case, and ([4] Prop. 13, p. 115) for the flat case.

(6): Given in [4] Prop. 17, p. 97.

THEOREM 10. *Let K be the total quotient ring of A in the sense of Bourbaki* ([4] *Example* 7, *p.* 77).

(1) *wgl $A \leq 0$ (i.e. A is regular) iff A_m is a field for all m in $\mathcal{M}$.*

(2) *A is semihereditary iff K is a regular ring and A_m is a valuation domain for all m in $\mathcal{M}$.*

(3) *wgl $A \leq 1$ iff A_m is a valuation domain for all m in $\mathcal{M}$.*

(4) *A is PP iff K is a regular ring, and A_m is a local domain for all m in $\mathcal{M}$.*

(5) *A is PF iff A_m is a domain for all m in $\mathcal{M}$.*

(6) *A is semiprime iff A_m is semiprime for all m in $\mathcal{M}$.*

Remark. The first four parts are due to Endo [8] and [9]. We have restated them,

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sometimes in slightly different form, in order to emphasize the relationships between them.

Proof. In view of the remark, we shall only prove (5) and (6):

(5) $\Rightarrow$: By Theorem 9, A_m is a local PF ring and therefore an integral domain by Corollary 3 of Theorem 7.

$\Leftarrow$: If I is any principal ideal of A then for all m in $\mathscr{M}$, I_m is a principal ideal of A_m, and therefore A_m-flat since A_m is a (local) domain, hence PP. Therefore I is A-flat ([4] Cor. p. 116), and A is PF.

(6) By Theorem 9 we need only show:

$\Leftarrow$: Let N be the nilradical of A. Then for all m in $\mathscr{M}$, N_m is the nilradical of A_m ([4], Prop. 17, p. 97). Since each A_m is semiprime $N_m=0$ for all m in $\mathscr{M}$ and $N=0$ ([4], Cor. 2, p. 112).

Corollary. *Every commutative PF ring is semiprime and therefore neat.*

Proof. Use (5) and (6) of the theorem and the fact that every (local) domain is semiprime. A commutative ring is semiprime iff it is neat. See Lambek ([15] p. 108).

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