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ON PROPERTY $\mathscr{B}$ OF FAMILIES OF SETS

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A family $\mathscr{F}$ of sets is said to have property $\mathscr{B}$ if there exists a set B such that $B \cap F \neq \varnothing$ and $B \not\supseteq F$ for every $F \in \mathscr{F}$. Such a B will be called suitable with respect to $\mathscr{F}$. It is known (see [3]) that for each positive integer n there exists a family $\mathscr{F}$ of sets satisfying the following conditions:

(a) $|F|=n$ for each $F \in \mathscr{F}$
(b) $|F \cap G| \leq 1$ for $F, G \in \mathscr{F}, F \neq G$
(c) $\mathscr{F}$ does not have property $\mathscr{B}$.

The proof of this result uses probabilistic methods. A simple constructive proof is given in [2]. Let us call $\mathscr{F}$ n-critical if, in addition to (a), (b) and (c), it also satisfies:

(d) Every proper subfamily of $\mathscr{F}$ has property $\mathscr{B}$.

It can be deduced from results of Erdős and Hajnal ([3] Theorem 12.9) or Lovász ([4], pp. 65–67) that for every n, arbitrarily large n-critical families exist. The proofs of these results are quite complicated. In this note we establish the existence of arbitrarily large n-critical families by means of a simple construction. In addition, we answer a question which was raised in [1].

THEOREM. *If $n>1$ and there exists an n-critical family of size m, then there exists an n-critical family of size $nm+1$.*

Proof. Let $\mathscr{F}_i$, $i=1, 2, \ldots, n$, be n-critical families with $|\mathscr{F}_i|=m$. We suppose that $F \cap G=\varnothing$ if $F \in \mathscr{F}_i$, $G \in \mathscr{F}_k$ and $i \neq k$. For each j let $F_j \in \mathscr{F}_j$ and $a_j \in F_j$. Let $a \notin \bigcup_{i=1}^{n} (\bigcup_{F \in \mathscr{F}_i} F)$. Let $\mathscr{F}^*$ be the family consisting of the following sets:

(i) $\{a_1, a_2, \ldots, a_n\}$
(ii) $\{a\} \cup (F_j \sim \{a_j\})$ $j=1, 2, \ldots, n$.
(iii) The sets in $\bigcup_{j=1}^{n} \mathscr{F}_j$ excluding $F_1, F_2, \ldots, F_n$.

Note that $|\mathscr{F}^*|=nm+1$. We now show that $\mathscr{F}^*$ is n-critical. Condition (a) obviously holds and one can easily verify (b). It remains to verify (c) and (d).

To establish (c), suppose that $\mathscr{F}^*$ has property $\mathscr{B}$ and let B be suitable with respect to $\mathscr{F}^*$. Since $\mathscr{F}_j$ is n-critical, we must have $B \supseteq F_j$ or $B \cap F_j=\varnothing$ for each j. It cannot occur that $B \supseteq F_j$ for all j since this implies $B \supseteq \{a_1, a_2, \ldots, a_n\}$. Also, we cannot have $B \cap F_j=\varnothing$ for all j since this gives $B \cap \{a_1, a_2, \ldots, a_n\}=\varnothing$. Thus $B \supseteq F_j$ for $j=1, 2, \ldots, r$ and $B \cap F_j=\varnothing$ for $j=r+1, \ldots, n$, say. This implies, however, that if $a \in B$, $B \supseteq \{a\} \cup (F_1 \sim \{a_1\})$, while, if $a \notin B$, $B \cap (\{a\} \cup (F_{r+1} \sim \{a_{r+1}\}) = \varnothing$. This is a contradiction. Hence $\mathscr{F}^*$ does not have property $\mathscr{B}$ and (c) holds.

Finally, we must establish (d). This is slightly more involved. We have to show that every proper subfamily of $\mathscr{F}^*$ has property $\mathscr{B}$. Clearly it suffices to consider only those families $\mathscr{F}$ obtained from $\mathscr{F}^*$ by deleting a single set F. We consider three cases. In each case we exhibit a set B which is suitable with respect to $\mathscr{F} = \mathscr{F}^* \sim \{F\}$.

Case (i) $F = \{a_1, a_2, \ldots, a_n\}$

Let $B_j \subseteq \mathscr{F}_j$ be suitable with respect to $\mathscr{F}_j \sim \{F_j\}$. Then either $B_j \supseteq F_j$ or $B_j \cap F_j = \varnothing$, since otherwise $\mathscr{F}_j$ would have property $\mathscr{B}$. There is no loss of generality in assuming that $B_j \supseteq F_j$ since otherwise we may replace B_j by its complement in $\cup \mathscr{F}_j$. It is now easy to check that $B = \bigcup_{j=1}^n B_j$ is suitable with respect to $\mathscr{F}$.

Case (ii) $F = \{a\} \cup (F_i \sim \{a_i\})$ for some i.

Let B_j be suitable with respect to $\mathscr{F}_j \sim \{F_j\}$ and suppose as in case (i) that $B_j \supseteq F_j$. Let $\bar{B}_i$ denote the complement of B_i in $\cup \mathscr{F}_i$. Then $B = (\bigcup_{j \neq i} B_j) \cup \bar{B}_i$ is suitable with respect to $\mathscr{F}$.

Case (iii) $F \in \mathscr{F}_i \sim \{F_i\}$ for some i.

For $j \neq i$ let B_j be suitable with respect to $\mathscr{F}_j \sim \{F_j\}$ and suppose $B_j \supseteq F_j$. Let B_i be suitable with respect to $\mathscr{F}_i \sim \{F\}$, $B_i \supseteq F$. Then if $a_i \in B_i$, $B = \{a\} \cup (\cup_{j \neq i} \bar{B}_j) \cup B_i$ is suitable with respect to $\mathscr{F}$, while if $a_i \notin B_i$, $B = \bigcup_{j=1}^n B_j$ is suitable.

This completes the proof of the theorem.

In [1] the following question was considered. Let $n \geq 3$ and $N \geq 2n-1$. Denote by $m(N, n)$ the least integer for which there exists a family $\mathscr{F}$ of $m(N, n)$ sets satisfying (a), (c), (d) and the condition $|\cup \mathscr{F}| = N$. It was shown in [1] that there exist constants α_n and β_n such that $\alpha_n \leq m(N, n)/N \leq \beta_n$ and it was asked whether $\text{limit}_{N \to \infty}(m(N, n)/N)$ exists. This question can now be answered affirmatively as follows. For $j = 1, 2, \ldots, n$ let $N_j \geq 2n-1$ and let $\mathscr{F}_j$ be a family of sets satisfying (a), (c), (d) and the condition $|\cup \mathscr{F}_j| = N_j$. Let $\mathscr{F}^*$ be constructed as in the proof of the theorem. One can then show that $\mathscr{F}^*$ has properties (a), (c) and (d) and hence that

$$(1) \qquad m\left(1 + \sum_{j=1}^n N_j, n\right) \leq 1 + \sum_{j=1}^n m(N_j, n).$$

The proof parallels closely the proof of the theorem, so we do not present the details here. It follows easily from (1) and Fekete's Lemma [5] that $\text{limit}_{N \to \infty}(m(N, n)/N)$ exists.

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