J. Aust. Math. Soc. **92** (2012), 37–44
doi:10.1017/S1446788712000079

VARIATIONS AROUND A PROBLEM OF MAHLER AND MENDÈS FRANCE

YANN BUGEAUD

(Received 25 February 2011; accepted 16 January 2012)

Communicated by F. Pappalardi

To the memory of Alf van der Poorten

Abstract

We discuss the following general question and some of its extensions. Let $(\varepsilon_k)_{k\geq 1}$ be a sequence with values in $\{0, 1\}$, which is not ultimately periodic. Define $\xi := \sum_{k\geq 1} \varepsilon_k/2^k$ and $\xi' := \sum_{k\geq 1} \varepsilon_k/3^k$. Let $\mathcal{P}$ be a property valid for almost all real numbers. Is it true that at least one among ξ and ξ' satisfies $\mathcal{P}$?

2010 *Mathematics subject classification*: primary 11J82.

Keywords and phrases: transcendental numbers, algebraic numbers, rational approximation, numbers in different bases.

1. Introduction

The main motivation for this note comes from the following problem, which appeared at the end of a paper by Mendès France [15]. According to him (see the discussion in [4, p. 403]), it was proposed by Mahler; however, we were unable to find any mention of it in Mahler's works.

PROBLEM 1.1 (Mahler–Mendès France). For an arbitrary infinite sequence $(\varepsilon_k)_{k\geq 1}$, with values in $\{0, 1\}$, prove that the real numbers

$$\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{2^k} \quad \text{and} \quad \sum_{k=1}^{+\infty} \frac{\varepsilon_k}{3^k}$$

are both algebraic if and only if both are rational.

The resolution of this problem seems to be far beyond our current state of knowledge. Nonetheless, in this note, we discuss the following more general question. Throughout, 'almost all' always refers to Lebesgue measure.

PROBLEM 1.2. Let $\mathcal{P}$ be a property valid for almost all real numbers. Let b be an integer greater than 1. Let b_1 and b_2 be distinct integers, at least as great as b. Let $(\varepsilon_k)_{k\geq 1}$ be a sequence with values in $\{0, 1, \ldots, b-1\}$, which is not ultimately periodic. Is it true that at least one among the numbers

$$\xi_1 := \sum_{k\ge1} \frac{\varepsilon_k}{b_1^k} \quad \text{and} \quad \xi_2 := \sum_{k\ge1} \frac{\varepsilon_k}{b_2^k}$$

satisfies property $\mathcal{P}$?

If $\mathcal{P}$ is the property of 'being transcendental' and $b = b_1 = 2$ while $b_2 = 3$, then giving a positive answer to Problem 1.2 is equivalent to solving the Mahler–Mendès France problem. The aim of this note is to discuss Problem 1.2 for other properties $\mathcal{P}$, including 'not being a Liouville number' or 'not being badly approximable'.

Recall that the irrationality exponent of an irrational real number ξ, which we denote by $\mu(\xi)$, is the supremum of the set of real numbers μ for which the inequality $|\xi - p/q| < q^{-\mu}$ has infinitely many solutions in rational numbers p/q for which $q \ge 1$. A real number ξ is a Liouville number if and only if $\mu(\xi)$ is infinite. The irrationality exponent of every irrational real number is at least as great as 2. Recall also that an irrational real number ξ for which there exists a positive real number c such that $|\xi - p/q| \ge c/q^2$ for every pair (p, q) of integers for which $q \ge 1$ is called a badly approximable number.

For both properties mentioned above, the answer to Problem 1.2 is negative. Indeed, it is easy to check that

$$\sum_{k\ge1} \frac{1}{b^{k!}}$$

is a Liouville number for every integer b greater than 1. Furthermore, Shallit [19] has shown that

$$\sum_{k\ge1} \frac{1}{b^{2^k}}$$

is a badly approximable number for every integer $b \ge 2$. For this, he used a version of the folding lemma for continued fractions, which was first established by Mendès France [14] and then rediscovered by several authors [11, 12, 17, 18, 20, 21] (this list is not exhaustive).

Furthermore, it has been proved recently [7], again using the folding lemma, that

$$\mu\Big(\sum_{k\ge1} \frac{1}{b^{\lfloor c^k \rfloor}}\Big) = c,$$

for every real number $c \ge 2$ and every integer $b \ge 2$. Here and below, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

More generally, most of the recent Diophantine results on real numbers expressed as

$$\sum_{k\ge1} \frac{\varepsilon_k}{b^k}$$

depend on combinatorial properties of the sequence of digits $(\varepsilon_k)_{k\geq 1}$, but not on the integer base b (here, it is assumed that all ε_k are in $\{0, 1, \ldots, b-1\}$) (see, for example, [1, 2, 6, 8, 9]).

All these results motivate the following problems.

Problem 1.3. Let $b \geq 2$ be an integer greater than 1. Let b_1 and b_2 be distinct integers at least as great as b. Does there exist a sequence $(\varepsilon_k)_{k\geq 1}$ with values in $\{0, 1, \ldots, b-1\}$ such that

$$\mu\Big(\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}\Big) = +\infty \quad \text{and} \quad \mu\Big(\sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}\Big) < +\infty?$$

Since the set of Liouville numbers has zero Hausdorff dimension, metric arguments do not seem to help to solve Problem 1.3.

Problem 1.4. Let $b \geq 2$ be an integer greater than 1. Let b_1 and b_2 be distinct integers at least as great as b. Find an explicit sequence $(\varepsilon_k)_{k\geq 1}$ with values in $\{0, 1, \ldots, b-1\}$ such that

$$\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}$$

is badly approximable, while

$$\sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}$$

is not badly approximable.

With the notation of Problem 1.4, the Hausdorff dimension of the set of badly approximable real numbers of the form

$$\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k},$$

with ε_k in $\{0, 1, \ldots, b-1\}$ for $k \geq 1$, is equal to the Hausdorff dimension of the set of real numbers of this form, namely, $(\log b)/(\log b_1)$ (see, for example, [10]). This implies that sequences $(\varepsilon_k)_{k\geq 1}$ with the properties required in Problem 1.4 do exist when b_1 is less than b_2. The difficult point is to provide an explicit construction of such a sequence.

Problem 1.5. Let $b \geq 2$ be an integer greater than 1. Let b_1 and b_2 be distinct integers at least as great as b. Let μ_1 and μ_2 be real numbers at least as great as 2. Does there exist a sequence $(\varepsilon_k)_{k\geq 1}$ with values in $\{0, 1, \ldots, b-1\}$ such that

$$\mu\Big(\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}\Big) = \mu_1 \quad \text{and} \quad \mu\Big(\sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}\Big) = \mu_2?$$

Surprisingly, it does not even seem to be easy to construct a sequence $(\varepsilon_k)_{k\geq 1}$ with values in $\{0, 1, \ldots, b-1\}$ such that

$$\mu\Big(\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}\Big) \neq \mu\Big(\sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}\Big);$$

see Theorem 2.1 below for a contribution to this question.

https://doi.org/10.1017/S1446788712000079 Published online by Cambridge University Press

Problem 1.5 is difficult since, in most cases, knowing the b-ary expansion of a real number gives no information on its irrationality exponent (see [5]). However, if the sequence $(\varepsilon_k)_{k\geq 1}$ contains long repetitions which occur unexpectedly early, then, by truncating and completing by periodicity, one can construct very good rational approximants to $\sum_{k\geq 1} \varepsilon_k/b_1^k$ of the form $P(b_1)/(b_1^r(b_1^s - 1))$, where r and s are positive integers and $P(X)$ is an integral polynomial. However, it is not clear at all whether $P(b_1)/(b_1^r(b_1^s - 1))$ is written in reduced form. Furthermore, $P(b_2)/(b_2^r(b_2^s - 1))$ is then a good rational approximant to $\sum_{k\geq 1} \varepsilon_k/b_2^k$, but we also do not know whether it is written in reduced form. Such information is crucial when one wishes to determine the exact value of the irrationality exponent. Otherwise, we get only a lower bound for it. A related question has been discussed by Mahler [13].

We conclude this section with an extension of the Mahler–Mendès France problem.

Problem 1.6. Let $b \geq 2$ be an integer greater than 1. Let b_1 and b_2 be distinct integers at least as great as b. Let $(\varepsilon_k)_{k\geq 1}$ be a sequence with values in $\{0, 1, \dots, b-1\}$, which is not ultimately periodic. Are the real numbers

$$\xi_1 := \sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k} \quad \text{and} \quad \xi_2 := \sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}$$

algebraically independent?

Under strong additional assumptions on the sequence $(\varepsilon_k)_{k\geq 1}$, a positive answer to Problem 1.6 has been given using the so-called Mahler method (see, for example, Nishioka's monograph [16, Ch. 3]). In particular, the real numbers

$$\sum_{k\geq 1} \frac{1}{b^{2^k}},$$

where $b \geq 2$, are algebraically independent.

2. Our result

Our small contribution towards Problem 1.5 is the following result.

Theorem 2.1. *Let b and b_1 be integers such that $b_1 > b \geq 2$ and $b_1 \neq b^2$. Let a be a real number and let w be an integer such that $a \geq 3$ and $w \geq 3a$. For $k \geq 1$, set $n_k = \lfloor (aw)^k \rfloor$. Define the sequence of integers $(\varepsilon_k)_{k\geq 1}$ as follows. We set $\varepsilon_k = b$ if and only if there exist $h \geq 1$ and $m = 0, 1, \dots, w-1$ such that $k = n_h + 1 + m(2n_h + 1)$. We set $\varepsilon_k = 1$ if and only if there exist $h \geq 1$ and $m = 1, 2, \dots, w$ such that $k = m(2n_h + 1)$. Otherwise, we set $\varepsilon_k = 0$. Define*

$$\xi := \sum_{k\geq 1} \frac{\varepsilon_k}{(b^2)^k} \quad \textit{and} \quad \xi_1 := \sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}.$$

Then

$$\mu(\xi) = \frac{a(2w+1)}{a+2}$$

https://doi.org/10.1017/S1446788712000079 Published online by Cambridge University Press

and

$$\mu(\xi_1) = \frac{a(2w+1)}{2(a+1)}.$$

The conclusion of Theorem 2.1 also holds if the real number a satisfies $a > 2 + 1/w$ for a sufficiently large integer w.

Observe that for every sufficiently large real number μ, there are integers w and w_1 and real numbers a and a_1 such that $3 \le a \le w/3$ while $3 \le a_1 \le w_1/3$ and

$$\mu = \frac{a(2w+1)}{a+2} = \frac{a_1(2w_1+1)}{2(a_1+1)}.$$

The proof of Theorem 2.1 is elementary. The basic idea is to truncate the b^2-ary expansion of ξ (or the b_1-ary expansion of ξ_1) and then to complete by periodicity to construct good rational approximants to ξ (or to ξ_1). The denominators of these rationals, when written in their lowest form, are essentially of the form $b^r(b^s - 1)$ (or $b_1^r(b_1^s - 1)$ respectively), where r and s are positive integers.

Proof of Theorem 2.1. The key point is the observation that

$$\frac{b \cdot b^{2n} + 1}{b^{2(2n+1)} - 1} = \frac{1}{b^{2n+1} - 1},$$

while the fraction

$$\frac{b \cdot b_1^n + 1}{b_1^{2n+1} - 1}$$

is nearly in reduced form.

To be more precise, observe that

$$(b \cdot b_1^n + 1)(b_1^{n+1} - 1) = b(b_1^{2n+1} - 1) + (b_1 - b)b_1^n + b - 1,$$

thus $\gcd(b \cdot b_1^n + 1, b_1^{2n+1} - 1)$ divides $(b_1 - b)b_1^n + b - 1$. Since

$$(b_1 - b)(b \cdot b_1^n + 1) - b((b_1 - b)b_1^n + b - 1) = b_1 - b^2,$$

it follows that $\gcd(b \cdot b_1^n + 1, b_1^{2n+1} - 1)$ divides $b_1 - b^2$, hence this greatest common divisor is bounded independently of n.

Observe that

$$\xi = \sum_{k \ge 1} (b(b^2)^{-n_k - 1} + (b^2)^{-2n_k - 1})(1 + (b^2)^{-2n_k - 1} + \cdots + (b^2)^{-(w-1)(2n_k+1)})$$

and

$$\xi_1 = \sum_{k \ge 1} (b \cdot b_1^{-n_k - 1} + b_1^{-2n_k - 1})(1 + b_1^{-2n_k - 1} + \cdots + b_1^{-(w-1)(2n_k+1)}).$$

https://doi.org/10.1017/S1446788712000079 Published online by Cambridge University Press

To construct good rational approximants to ξ (or to ξ_1), we simply truncate the summation and complete by periodicity. When $K \geq 2$, define

$$\begin{aligned}\xi_K &:= \sum_{k=1}^{K-1} (b(b^2)^{-n_k-1} + (b^2)^{-2n_k-1})(1 + (b^2)^{-2n_k-1} + \cdots + (b^2)^{-(w-1)(2n_k+1)}) \\ &\quad + \frac{b(b^2)^{-n_K-1} + (b^2)^{-2n_K-1}}{1-(b^2)^{-2n_K-1}} \\ &= \frac{m_K}{(b^2)^{w(2n_{K-1}+1)}} + \frac{b(b^2)^{n_K}+1}{(b^2)^{2n_K+1}-1}\end{aligned}$$

and

$$\begin{aligned}\xi_{1,K} &:= \sum_{k=1}^{K-1} (b \cdot b_1^{-n_k-1} + b_1^{-2n_k-1})(1 + b_1^{-2n_k-1} + \cdots + b_1^{-(w-1)(2n_k+1)}) \\ &\quad + \frac{bb_1^{-n_K-1} + b_1^{-2n_K-1}}{1-b_1^{-2n_K-1}} \\ &= \frac{m_{1,K}}{(b_1)^{w(2n_{K-1}+1)}} + \frac{b(b_1)^{n_K}+1}{(b_1)^{2n_K+1}-1},\end{aligned}$$

for some integers m_K and $m_{1,K}$. It follows from the key point explained at the beginning of the proof that there exist integers p_K and $p_{1,K}$ such that

$$\xi_K = \frac{p_K}{(b^2)^{w(2n_{K-1}+1)}(b^{2n_K+1}-1)}$$

in lowest form and

$$\xi_{1,K} = \frac{p_{1,K}}{b_1^{w(2n_{K-1}+1)}(b_1^{2n_K+1}-1)},$$

and the greatest common divisor of $p_{1,K}$ and $b_1^{w(2n_{K-1}+1)}(b_1^{2n_K+1}-1)$ is bounded independently of K.

Since $a > 2 + 1/w$, the inequality $n_{K+1} + 1 > n_K + 1 + w(2n_K + 1)$ is satisfied if K is sufficiently large. If this is the case, then we may check that

$$b(b^2)^{-n_K-1-w(2n_K+1)} \leq |\xi - \xi_K| \leq 2b(b^2)^{-n_K-1-w(2n_K+1)} \tag{2.1}$$

and

$$bb_1^{-n_K-1-w(2n_K+1)} \leq |\xi_1 - \xi_{1,K}| \leq 2bb_1^{-n_K-1-w(2n_K+1)}. \tag{2.2}$$

Since we know the reduced form of the rational numbers ξ_K and $\xi_{1,K}$, up to a bounded numerical constant, it then follows from (2.1) and (2.2) that

$$|\xi - \xi_K| \asymp (\mathrm{den}(\xi_K))^{-2(n_K+1+w(2n_K+1))/(2w(2n_{K-1}+1)+2n_K+1)}$$

and

$$|\xi_1 - \xi_{1,K}| \asymp (\mathrm{den}(\xi_{1,K}))^{-(n_K+1+w(2n_K+1))/(w(2n_{K-1}+1)+2n_K+1)},$$

https://doi.org/10.1017/S1446788712000079 Published online by Cambridge University Press

where the notation $A_K \asymp B_K$ means that the ratio A_K/B_K is bounded from above and from below by positive constants independent of K.

This gives the lower bounds

$$\mu(\xi) \geq \frac{a(2w+1)}{a+2} \quad \text{and} \quad \mu(\xi_1) \geq \frac{a(2w+1)}{2(a+1)}. \tag{2.3}$$

It remains to show that the inequalities in (2.3) are indeed equalities. To do this, we use a classical lemma whose proof is based on the triangle inequality (see, for example, [3, Lemma 4.1]).

Lemma 2.2. *Let ξ be a real number such that there exist positive real numbers c_1, c_2, μ, θ and reduced rational numbers $(p_k/q_k)_{k\geq 1}$ such that*

$$\frac{c_1}{q_k^\mu} \leq \left|\xi - \frac{p_k}{q_k}\right| \leq \frac{c_2}{q_k^\mu}$$

and

$$q_k \leq q_{k+1} \leq q_k^\theta$$

for all $k \geq 1$. If $\theta \leq (\mu-1)^2$, then the irrationality exponent of ξ is equal to μ.

We check that

$$\lim_{K\to+\infty}\left|\frac{\log \operatorname{den}(\xi_{K+1})}{\log \operatorname{den}(\xi_K)} - \frac{2w(2n_K+1)+2n_{K+1}+1}{2w(2n_{K-1}+1)+2n_K+1}\right| = 0$$

and

$$\lim_{K\to+\infty}\left|\frac{\log \operatorname{den}(\xi_{1,K+1})}{\log \operatorname{den}(\xi_{1,K})} - \frac{w(2n_K+1)+2n_{K+1}+1}{w(2n_{K-1}+1)+2n_K+1}\right| = 0.$$

Consequently, by the definition of $(n_k)_{k\geq 1}$, the sequences

$$(\log \operatorname{den}(\xi_{K+1})/\log\ \operatorname{den}(\xi_K))_{K\geq 1} \quad \text{and} \quad (\log \operatorname{den}(\xi_{1,K+1})/\log\ \operatorname{den}(\xi_{1,K}))_{K\geq 1}$$

both tend to aw as K tends to infinity.

Since

$$aw \leq \left(\frac{a(2w+1)}{2(a+1)} - 1\right)^2$$

if $a \geq 3$ and $w \geq 3a$, the theorem follows from Lemma 2.2. □

References

[1] B. Adamczewski, 'On the expansion of some exponential periods in an integer base', *Math. Ann.* **346** (2010), 107–116.

[2] B. Adamczewski and Y. Bugeaud, 'On the complexity of algebraic numbers I. Expansions in integer bases', *Ann. of Math.* (2) **165** (2007), 547–565.

[3] B. Adamczewski and T. Rivoal, 'Irrationality measures for some automatic real numbers', *Math. Proc. Cambridge Philos. Soc.* **147** (2009), 659–678.

https://doi.org/10.1017/S1446788712000079 Published online by Cambridge University Press

[4] J.-P. Allouche and J. O. Shallit, *Automatic Sequences: Theory, Applications, Generalizations* (Cambridge University Press, Cambridge, 2003).
[5] M. Amou and Y. Bugeaud, 'Expansions in integer bases and exponents of Diophantine approximation', *J. Lond. Math. Soc.* **81** (2010), 297–316.
[6] Y. Bugeaud, 'On the rational approximation to the Thue–Morse–Mahler numbers', *Ann. Inst. Fourier (Grenoble)* **61** (2011), 1667–1678.
[7] Y. Bugeaud, 'Diophantine approximation and Cantor sets', *Math. Ann.* **341** (2008), 677–684.
[8] Y. Bugeaud and J.-H. Evertse, 'On two notions of complexity of algebraic numbers', *Acta Arith.* **133** (2008), 221–250.
[9] Y. Bugeaud, D. Krieger and J. Shallit, 'Morphic and automatic words: maximal blocks and diophantine approximation', *Acta Arith.* **149** (2011), 181–199.
[10] D. Kleinbock, E. Lindenstrauss and B. Weiss, 'On fractal measures and Diophantine approximation', *Selecta Math.* **10** (2004), 479–523.
[11] M. Kmošek, 'Rozwiniecie niektórych liczb niewymiernych na ułamki łańcuchowe', Master Thesis, Warsaw, 1979.
[12] G. Köhler, 'Some more predictable continued fractions', *Monatsh. Math.* **89** (1980), 95–100.
[13] K. Mahler, 'Some suggestions for further research', *Bull. Aust. Math. Soc.* **29** (1984), 101–108.
[14] M. Mendès France, 'Sur les fractions continues limitées', *Acta Arith.* **23** (1973), 207–215.
[15] M. Mendès France, 'Sur les décimales des nombres algébriques réels', in: *Seminar on Number Theory, 1979–1980*, Exp. No, 28 (Univ. Bordeaux I, Talence, 1980), p. 7.
[16] K. Nishioka, *Mahler Functions and Transcendence*, Lecture Notes in Mathematics, 1631 (Springer, Berlin, 1996).
[17] A. Pethő, 'Simple continued fractions for the Fredholm numbers', *J. Number Theory* **14** (1982), 232–236.
[18] P. Riyapan, V. Laohakosol and T. Chaichana, 'Two types of explicit continued fractions', *Period. Math. Hungar.* **52** (2006), 51–72.
[19] J. O. Shallit, 'Simple continued fractions for some irrational numbers', *J. Number Theory* **11** (1979), 209–217.
[20] J. O. Shallit, 'Simple continued fractions for some irrational numbers, II', *J. Number Theory* **14** (1982), 228–231.
[21] A. van der Poorten and J. O. Shallit, 'Folded continued fractions', *J. Number Theory* **40** (1992), 237–250.

YANN BUGEAUD, Université de Strasbourg, Mathématiques, 7, rue René Descartes, 67084 Strasbourg, France
e-mail: bugeaud@math.unistra.fr