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LINEAR MAPS PRESERVING TENSOR PRODUCTS OF RANK-ONE HERMITIAN MATRICES

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Abstract

For a positive integer $n \geq 2$, let M_n be the set of $n \times n$ complex matrices and H_n the set of Hermitian matrices in M_n. We characterize injective linear maps $\phi : H_{m_1 \cdots m_l} \to H_n$ satisfying

$$\text{rank}(A_1 \otimes \cdots \otimes A_l) = 1 \Longrightarrow \text{rank}(\phi(A_1 \otimes \cdots \otimes A_l)) = 1$$

for all $A_k \in H_{m_k}$, $k = 1, \ldots, l$, where $l, m_1, \ldots, m_l \geq 2$ are positive integers. The necessity of the injectivity assumption is shown. Moreover, the connection of the problem to quantum information science is mentioned.

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1. Introduction and the main theorem

Let M_n be the set of $n \times n$ complex matrices and H_n the set of Hermitian matrices in M_n. Suppose that $m, n \geq 2$ are positive integers and suppose that $A \in M_m$ and $B \in M_n$ are the states of two quantum systems. Then their tensor (Kronecker) product $A \otimes B \in M_m \otimes M_n$ will be the state in the (bipartite) joint system. In many applied and pure studies, one considers the tensor product of matrices (see for example [1, 4, 8, 13]). Most noticeably, the tensor product is often used in quantum information science [10]. In quantum physics, quantum states of a system with n physical states are represented as density matrices, that is, positive semi-definite matrices with trace one. If $A \in M_m$ and $B \in M_n$ are two quantum states in two quantum systems, then their tensor product $A \otimes B$ describes the joint state in the bipartite system, in which the general states are density matrices in $M_m \otimes M_n \equiv M_{mn}$. More generally, one may also consider tensor states and general states in a multipartite system $M_{n_1} \otimes \cdots \otimes M_{n_l} \equiv M_{m_1 \cdots m_l}$, where $l > 2$ is a positive integer.

Let us point out that it is relatively easy to extract information from matrices in tensor product form. For instance, if $A \in M_m$ has eigenvalues (respectively, singular values) $a_1, \ldots, a_m$ and $B \in M_n$ has eigenvalues (respectively, singular values) $b_1, \ldots, b_n$, then the eigenvalues (respectively, the singular values) of $A \otimes B$ have the form $a_i b_j$ with $1 \le i \le m$ and $1 \le j \le n$. Thus, it is interesting to get information on the tensor space M_{mn} by examining the properties of the small collection of matrices in tensor form $A \otimes B$. In particular, if we consider a linear map $\phi : M_{mn} \to M_{mn}$ and if one knows the images $\phi(A \otimes B)$ for $A \in M_m$ and $B \in M_n$, then the map ϕ can be completely characterized since every $C \in M_{mn}$ is a linear combination of matrices in tensor form $A \otimes B$. Nevertheless, the challenge is to use the limited information of the linear map ϕ on matrices in tensor form to determine the structure of ϕ.

Recently, there has been considerable interest in studying linear preserver problems on tensor spaces arising in quantum information science (see for example [5, 6, 11, 12] and references therein). In [6], Huang *et al.* and the third named author characterized linear maps on $H_m \otimes H_n \equiv H_{mn}$ preserving the spectrum $\sigma(A \otimes B)$ and the spectral radius $r(A \otimes B)$ for $A \in H_m$ and $B \in H_n$. Furthermore, in [11], Friedland *et al.* considered the linear group of automorphisms of Hermitian matrices which preserve the set of separable states. Recall that density matrices in H_{mn} that can be written as a convex combination of product states are separable states. It is easy to see that $S \in H_{mn}$ is separable if and only if it is a convex combination of $P_1 \otimes P_2$, where $P_1 \in H_m$ and $P_2 \in H_n$ are rank-one orthogonal projections. In [11], it was shown that a linear map sending the set of separable states onto itself has a very nice structure. Namely, it has the form

$$A \otimes B \mapsto \tau_1(A) \otimes \tau_2(B)$$

or $m = n$ and

$$A \otimes B \mapsto \tau_2(B) \otimes \tau_1(A),$$

where τ_k, $k = 1, 2$, has the form $X \mapsto U_k X U_k^*$ or $X \mapsto U_k X^t U_k^*$ for some unitary $U_1 \in M_m$ and $U_2 \in M_n$. Here, Y^t denotes the transpose of a matrix Y and Y^* denotes the conjugate transpose of Y. For more information on linear preserver problems, one may see [7] and references therein. We also refer the reader to [2, 9, 15], where some new results on the topic can be found.

The purpose of this paper is to study injective linear maps $\phi : H_{mn} \to H_{mn}$ satisfying

$$\operatorname{rank}(A \otimes B) = 1 \Longrightarrow \operatorname{rank}(\phi(A \otimes B)) = 1$$

for all $A \in H_m$, $B \in H_n$. We will show that such a map has the form

$$A \otimes B = \lambda T(\tau_1(A) \otimes \tau_2(B))T^{-1}$$

for some invertible matrix $T \in M_m \otimes M_n \equiv M_{mn}$, where $\lambda \in \{-1, 1\}$ and τ_k is the identity map or the transposition map $X \mapsto X^t$ for $k = 1, 2$. More generally, we consider linear maps on multipartite systems $H_{m_1} \otimes \cdots \otimes H_{m_l} \equiv H_{m_1 \cdots m_l}$, $l \ge 2$.

Before writing our main theorem, let us introduce some basic definitions and fix the notation. First of all, throughout the paper, $l, n, m_1, \ldots, m_l \ge 2$ will be positive integers with $n \geq m_1 \cdots m_l$ and $r = n - m_1 \cdots m_l$. For an integer $k \geq 1$, I_k denotes the $k \times k$ identity matrix, 0_k the $k \times k$ zero matrix, and $E_{ij}^{(k)}$, $1 \leq i, j \leq k$, the $k \times k$ matrix, all of whose entries are equal to zero except for the (i, j)th entry, which is equal to one. The set of rank-one matrices in H_k will be denoted by H_k^1. If k_1, k_2 are positive integers, then $M_{k_1 \times k_2}$ denotes the set of all $k_1 \times k_2$ complex matrices. As usual, we use the notation $\mathrm{Diag}(a_1, \ldots, a_k)$ to denote the $k \times k$ diagonal matrix with diagonal entries $a_1, \ldots, a_k$.

For the sake of readability, we will usually write 0 instead of 0_k for the $k \times k$ zero matrix. Similarly, for positive integers k_1, k_2, we will denote (where dimensions of the matrices are obvious) the $k_1 \times k_2$ zero matrix simply by 0.

We call a linear map π on $H_{m_1 \cdots m_l}$ canonical if

$$\pi(A_1 \otimes \cdots \otimes A_l) = \tau_1(A_1) \otimes \cdots \otimes \tau_l(A_l)$$

for all $A_k \in H_{m_k}$, $k = 1, \ldots, l$, where $\tau_k : H_{m_k} \to H_{m_k}$, $k = 1, \ldots, l$, is either the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t$. In this case, we write $\pi = \tau_1 \otimes \cdots \otimes \tau_l$.

Our main result reads as follows.

Main Theorem. *Let $\phi : H_{m_1 \cdots m_l} \to H_n$ be an injective linear map. Then, for any $A_1 \otimes \cdots \otimes A_l \in H_{m_1 \cdots m_l}$,*

$$\mathrm{rank}(A_1 \otimes \cdots \otimes A_l) = 1 \Longrightarrow \mathrm{rank}(\phi(A_1 \otimes \cdots \otimes A_l)) = 1 \tag{1.1}$$

if and only if there exist an invertible matrix $T \in M_n$, $\lambda \in \{-1, 1\}$, and a canonical map π on $H_{m_1 \cdots m_l}$ such that

$$\phi(A_1 \otimes \cdots \otimes A_l) = \lambda T(\pi(A_1 \otimes \cdots \otimes A_l) \oplus 0_r)T^* \tag{1.2}$$

for all $A_1 \otimes \cdots \otimes A_l \in H_{m_1 \cdots m_l}$.

Let us point out that n in the above theorem must be greater or equal to $m_1 \cdots m_l$ since ϕ is assumed to be injective. Moreover, the next two examples will show that without the injectivity assumption our main theorem does not hold in general.

Example 1.1. Let $R \in H_n^1$ be any rank-one Hermitian matrix and let $\varphi : H_{m_1 \cdots m_l} \to \mathbb{C}$ be any linear map such that $\varphi(A_1 \otimes \cdots \otimes A_l) \neq 0$ for all $A_k \in H_{m_k}^1$, $k = 1, \ldots, l$. Then a map $\phi : H_{m_1 \cdots m_l} \to H_n$ defined by

$$\phi(A_1 \otimes \cdots \otimes A_l) = \varphi(A_1 \otimes \cdots \otimes A_l)R, \qquad A_1 \otimes \cdots \otimes A_l \in H_{m_1 \cdots m_l},$$

is linear and it satisfies the condition (1.1). On the other hand, ϕ is not injective and it is not of the form (1.2).

In the next example, $l = 2$, $m_1 = m_2 = 2$, and $n = m_1 m_2 = 4$.

Example 1.2. Let $T = \begin{bmatrix} I_2 & E_{12}^{(2)} \\ 0 & E_{11}^{(2)} \end{bmatrix}$ and let $\phi : H_4 \to H_4$ be a map defined by

$$\phi(A \otimes B) = T(A \otimes B)T^*, \quad A, B \in H_2.$$

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Clearly, ϕ is linear and it satisfies the condition (1.1). Namely, if we write $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$T(A \otimes B)T^* = \begin{bmatrix} I_2 & E_{12}^{(2)} \\ 0 & E_{11}^{(2)} \end{bmatrix} \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ E_{21}^{(2)} & E_{11}^{(2)} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}B + a_{21}E_{12}^{(2)}B & a_{12}B + a_{22}E_{12}^{(2)}B \\ a_{21}E_{11}^{(2)}B & a_{22}E_{11}^{(2)}B \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ E_{21}^{(2)} & E_{11}^{(2)} \end{bmatrix}$$

and, hence, $\phi(A \otimes B)$ is equal to the 4×4 matrix of the form

$$\begin{bmatrix} a_{11}B + a_{21}E_{12}^{(2)}B + (a_{12}B + a_{22}E_{12}^{(2)}B)E_{21}^{(2)} & (a_{12}B + a_{22}E_{11}^{(2)}B)E_{11}^{(2)} \\ a_{21}E_{11}^{(2)}B + a_{22}E_{11}^{(2)}BE_{21}^{(2)} & a_{22}E_{11}^{(2)}BE_{11}^{(2)} \end{bmatrix}.$$

Suppose that there exist rank-one matrices $A, B \in H_2^1$ such that the rank of $\phi(A \otimes B) = T(A \otimes B)T^*$ is not one. Then $T(A \otimes B)T^*$ must be the zero matrix. In particular, $a_{22}E_{11}^{(2)}BE_{11}^{(2)} = a_{22}b_{11} = 0$. More precisely, either $a_{22} = 0$ or $b_{11} = 0$.

If $a_{22} = 0$, then $a_{12} = 0$ since $\operatorname{rank}(A) = 1$. Hence, $A = a_{11}E_{11}^{(2)}$ with $a_{11} \neq 0$ and, consequently,

$$0 = T(A \otimes B)T^* = \begin{bmatrix} a_{11}B & 0 \\ 0 & 0 \end{bmatrix}.$$

But then $B = 0$, which is a contradiction.

On the other hand, if $b_{11} = 0$, then $B = b_{22}E_{22}^{(2)}$ with $b_{22} \neq 0$ since $\operatorname{rank}(B) = 1$. Thus, $BE_{11}^{(2)} = 0$, $E_{11}^{(2)}B = 0$, $E_{12}^{(2)}B = b_{22}E_{12}^{(2)}$, $BE_{21}^{(2)} = b_{22}E_{21}^{(2)}$, and $E_{12}^{(2)}BE_{21}^{(2)} = b_{22}E_{11}^{(2)}$. This implies that

$$0 = T(A \otimes B)T^* = \begin{bmatrix} b_{22}\begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}.$$

But then $A = 0$, which is a contradiction. So, we showed that ϕ satisfies the condition (1.1). On the other hand, ϕ is not injective (since T is not invertible) and it is not of the form (1.2).

2. Preliminary results

Before proving the main theorem of this paper we introduce some additional results which will be used in the sequel. The first one is a direct consequence of [3, Theorem 2].

Proposition 2.1. *Let m, n be positive integers with $2 \leq m \leq n$ and $\phi : H_m \to H_n$ an injective linear map. Then, for any $A \in H_m$,*

$$\operatorname{rank}(A) = 1 \Longrightarrow \operatorname{rank}(\phi(A)) = 1$$

if and only if there exist an invertible matrix $T \in M_n$ and $\lambda \in \{-1, 1\}$ such that either

$$\phi(A) = \lambda T(A \oplus 0_{n-m})T^*$$

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for all $A \in H_m$ *or*

$$\phi(A) = \lambda T(A^t \oplus 0_{n-m})T^*$$

for all $A \in H_m$.

Remark 2.2. Let $\tau : H_m \to H_m$ be the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t$ and let $\phi : H_m \to H_n$ be a linear rank-one preserver of the form

$$\phi(A) = T(\tau(A) \oplus 0_{n-m})T^*, \quad A \in H_m,$$

where $T \in M_n$ is a fixed matrix. Then ϕ must be injective. Namely, if ϕ is not injective and if τ is the identity map, then, by [3, Theorem 2], there exists a nonzero vector $\beta \in \mathbb{C}^n$ such that

$$\phi(A) = \varphi(A)\beta\beta^*, \quad A \in H_m,$$

where $\varphi : H_m \to \mathbb{C}$ is a linear map satisfying $\varphi(A) \neq 0$ for all rank-one matrices $A \in H_m$. Hence,

$$T \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} T^* = \varphi(A)\beta\beta^*$$

for all $A \in H_m$. Since $\beta \neq 0$, we may find an invertible matrix $P \in M_n$ such that $\beta\beta^* = \lambda P E_{11}^{(n)} P^*$ for some nonzero scalar λ. Writing $\varphi(A)$ instead of $\lambda\varphi(A)$ and T instead of $P^{-1}T$,

$$T \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} T^* = \varphi(A)E_{11}^{(n)}.$$

Let $P = [p_{ij}]$. By choosing $A = E_{11}^{(m)}, E_{22}^{(m)}$ in the above equality, we conclude that $p_{i1} = p_{i2} = 0$ for $i \geq 2$. Moreover, $\varphi(E_{11}^{(m)}) = |p_{11}|^2$ and $\varphi(E_{22}^{(m)}) = |p_{12}|^2$. Now let ε be any nonzero scalar. Then, according to the above observations,

$$\varphi(\varepsilon E_{11}^{(m)} + \varepsilon^{-1}E_{22}^{(m)} + E_{12}^{(m)} + E_{21}^{(m)}) = \varepsilon^{-1}|\varepsilon p_{11} + p_{12}|^2.$$

Taking $\varepsilon = -p_{11}^{-1}p_{12}$, we get $\varphi(\varepsilon E_{11}^{(m)} + \varepsilon^{-1}E_{22}^{(m)} + E_{12}^{(m)} + E_{21}^{(m)}) = 0$, which contradicts the assumption that ϕ is a rank-one preserver. In the same way we show that ϕ is injective in the case when τ is the transposition map.

We continue with a series of simple lemmas. The proof of the first lemma will be omitted, since it is similar to the proof of Lemma 2.5 in [16].

Lemma 2.3. *Let* π_1, π_2 *be canonical maps on* $H_{m_1 \cdots m_k}$ *and let* $T \in M_{m_1 \cdots m_l}$ *be an invertible matrix. If*

$$T\pi_1(A_1 \otimes \cdots \otimes A_l)T^* = \pi_2(A_1 \otimes \cdots \otimes A_l)$$

for all $A_k \in H_{m_k}$, $k = 1, \ldots, l$, *then* $\pi_1 = \pi_2$ *and* $T = \lambda I_{m_1 \cdots m_l}$ *for some nonzero scalar* $\lambda \in \mathbb{C}$.

Lemma 2.4. *Let* $m \geq 2$ *be a positive integer and let* $A, B \in H_m$ *be Hermitian matrices with* $\operatorname{rank}(A) = 1$. *If* $\operatorname{rank}(\varepsilon A + B) \leq 1$ *for any real scalar* $\varepsilon \neq 0$, *then* $B = \lambda A$ *for some* $\lambda \in \mathbb{R}$.

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Proof. Without loss of generality, we may assume that $A = E_{11}^{(m)}$. If we write $B = \begin{bmatrix} b_{11} & \beta \\ \beta^* & B_{m-1} \end{bmatrix}$, where $B_{m-1} \in H_{m-1}$, $b_{11} \in \mathbb{R}$, and $\beta \in \mathbb{C}^{m-1}$, then, according to our assumptions,

$$\operatorname{rank}(\varepsilon A + B) = \operatorname{rank} \begin{bmatrix} \varepsilon + b_{11} & \beta \\ \beta^* & B_{m-1} \end{bmatrix} \le 1$$

for all nonzero scalars $\varepsilon \in \mathbb{R}$. Therefore, if ε is any nonzero real number not equal to $-b_{11}$, then

$$\operatorname{rank} \begin{bmatrix} \lambda + b_{11} & \beta \\ 0 & B_{n-1} - (\varepsilon + b_{11})^{-1}\beta^*\beta \end{bmatrix} \le 1$$

and, thus,

$$B_{n-1} - (\varepsilon + b_{11})^{-1}\beta^*\beta = 0.$$

Since this is true for all real scalars $\varepsilon \neq 0, -b_{11}$, it follows that $B_{m-1} = 0$ and $\beta^*\beta = 0$. This means that $\beta = 0$ and, consequently, $B = b_{11}E_{11}^{(m)} = b_{11}A$. □

Lemma 2.5. *Let r, s, t be positive integers with $t \ge s$ and let $d \in \mathbb{R}$ be a real number. Suppose that matrices $A \in H_s^1$, $A_1 = E_{11}^{(r)} \otimes A, A_2, A_3 \in H_{rs}$, $B, C \in M_{rs\times t}$, $D \in H_t^1$ satisfy*

$$\operatorname{rank} \begin{bmatrix} \varepsilon A_1 + A_2 + \varepsilon^{-1}A_3 & B + \varepsilon^{-1}C \\ B^* + \varepsilon^{-1}C^* & (d + \varepsilon^{-1})D \end{bmatrix} = 1$$

for all nonzero scalars $\varepsilon \in \mathbb{R}$. Then $d = 0$ and there exists an invertible matrix $T \in M_{t\times t}$ such that

$$B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} T \quad \text{and} \quad D = T^* \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} T.$$

Moreover, if $B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, then $D = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$.

Proof. According to the assumptions, we can write

$$A = \lambda_1 U E_{11}^{(s)} U^* \quad \text{and} \quad D = \lambda_2 V E_{11}^{(t)} V^*$$

for some invertible matrices $U \in M_s$, $V \in M_t$, and $\lambda_1, \lambda_2 \in \{-1, 1\}$. Let $\tilde{A}_2, \tilde{A}_3 \in H_{rs}$, $\tilde{B}, \tilde{C} \in M_{rs\times t}$ be matrices such that

$$\begin{aligned} A_2 &= (U \oplus I_{rs-s})\tilde{A}_2(U \oplus I_{rs-s})^*, \\ A_3 &= (U \oplus I_{rs-s})\tilde{A}_3(U \oplus I_{rs-s})^*, \\ B &= (U \oplus I_{rs-s})\tilde{B}V^*, \\ C &= (U \oplus I_{rs-s})\tilde{C}V^*. \end{aligned}$$

Then

$$\operatorname{rank} \begin{bmatrix} \varepsilon\lambda_1 E_{11}^{(rs)} + \tilde{A}_2 + \varepsilon^{-1}\tilde{A}_3 & \tilde{B} + \varepsilon^{-1}\tilde{C} \\ \tilde{B}^* + \varepsilon^{-1}\tilde{C}^* & (d + \varepsilon^{-1})\lambda_2 E_{11}^{(t)} \end{bmatrix} = 1$$

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for all nonzero scalars $\varepsilon \in \mathbb{R}$. Furthermore, if we write $\tilde{A}_2 = \left[\begin{smallmatrix} a_2 & \alpha_2 \\ \alpha_2^* & R \end{smallmatrix}\right]$, $\tilde{A}_3 = \left[\begin{smallmatrix} a_3 & \alpha_3 \\ \alpha_3^* & S \end{smallmatrix}\right]$, $\tilde{B} = \left[\begin{smallmatrix} b & \beta \\ \gamma & B_1 \end{smallmatrix}\right]$, $\tilde{C} = \left[\begin{smallmatrix} c & \delta \\ \eta & C_1 \end{smallmatrix}\right]$, then

$$\operatorname{rank}\begin{bmatrix} \varepsilon\lambda_1 + a_2 + \varepsilon^{-1}a_3 & \alpha_2 + \varepsilon^{-1}\alpha_3 & b + \varepsilon^{-1}c & \beta + \varepsilon^{-1}\delta \\ \alpha_2^* + \varepsilon^{-1}\alpha_3^* & R + \varepsilon^{-1}S & \gamma + \varepsilon^{-1}\eta & B_1 + \varepsilon^{-1}C_1 \\ \bar{b} + \varepsilon^{-1}\bar{c} & \gamma^* + \varepsilon^{-1}\eta^* & (d + \varepsilon^{-1})\lambda_2 & 0 \\ \beta^* + \varepsilon^{-1}\delta^* & B_1^* + \varepsilon^{-1}C_1^* & 0 & 0 \end{bmatrix} = 1$$

for all nonzero scalars $\varepsilon \in \mathbb{R}$. Thus, $B_1 + \varepsilon^{-1}C_1 = 0, \beta + \varepsilon^{-1}\delta = 0$ and, since this is true for all nonzero scalars $\varepsilon \in \mathbb{R}$, we have $B_1 = 0$ and $\beta = 0$.

Next, according to the above observations, the determinant of the submatrix $\left[\begin{smallmatrix} \varepsilon\lambda_1 + a_2 + \varepsilon^{-1}a_3 & b + \varepsilon^{-1}c \\ \bar{b} + \varepsilon^{-1}\bar{c} & (d + \varepsilon^{-1})\lambda_2 \end{smallmatrix}\right]$ must be zero. More precisely, for all nonzero scalars $\varepsilon \in \mathbb{R}$,

$$(d + \varepsilon^{-1})\lambda_2(\varepsilon\lambda_1 + a_2 + \varepsilon^{-1}a_3) = (b + \varepsilon^{-1}c)(\bar{b} + \varepsilon^{-1}\bar{c}).$$

This implies that $\lambda_1 = \lambda_2$, $d = 0$, and $|b| = 1$. Now assume that $\lambda_1 = \lambda_2 = 1$. Since, for all nonzero scalars $\varepsilon \in \mathbb{R}$,

$$\operatorname{rank}\begin{bmatrix} \alpha_2^* + \varepsilon^{-1}\alpha_3^* & \gamma + \varepsilon^{-1}\eta \\ \bar{b} + \varepsilon^{-1}\bar{c} & \varepsilon^{-1} \end{bmatrix} \leq 1,$$

we conclude that $\gamma = 0$. Thus,

$$B = \begin{bmatrix} U & 0 \\ 0 & I_{rs-s} \end{bmatrix}\begin{bmatrix} bE_{11}^{(s)} & 0 \\ 0 & 0 \end{bmatrix} V^* = \begin{bmatrix} UE_{11}^{(s)}U^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} bU^{-*} & 0 \\ 0 & I_{t-s} \end{bmatrix} V^*.$$

If we write

$$T := \begin{bmatrix} bU^{-*} & 0 \\ 0 & I_{t-s} \end{bmatrix} V^*,$$

then $B = \left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] T$ and

$$\begin{aligned} D = VE_{11}^{(t)}V^* &= V\begin{bmatrix} E_{11}^{(s)} & 0 \\ 0 & 0 \end{bmatrix} V^* \\ &= V\begin{bmatrix} \bar{b}U^{-1} & 0 \\ 0 & I_{t-s} \end{bmatrix}\begin{bmatrix} UE_{11}^{(s)}U^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} bU^{-*} & 0 \\ 0 & I_{t-s} \end{bmatrix} V^* \\ &= T^*\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} T. \end{aligned}$$

Finally, if $B = \left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and, if we write $T = \left[\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right]$, where $T_1 \in M_s$, then $AT_1 = A$ and $AT_2 = 0$. Therefore,

$$D = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}. \qquad \square$$

The next result is immediately obtained from [14, Lemma 2.2].

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LEMMA 2.6. *Let m, i, j be positive integers with $m \geq 2$ and $1 \leq i < j \leq m$. If $A \in H_m$ satisfies*

$$\mathrm{rank}(A + kE_{ii}^{(m)} + k^{-1}E_{jj}^{(m)}) = 1$$

for all positive integers k, then $A = \lambda E_{ij}^{(m)} + \bar{\lambda}E_{ji}^{(m)}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

The proof of our main result will heavily rely on the following lemma. Since its proof is quite long and technical, we will present it at the end of the paper (see Section 4).

LEMMA 2.7. *Let $t \geq m_1 \cdots m_l$ be a positive integer, $f : H_{m_1 \cdots m_l} \to M_{(m_1 \cdots m_l)\times t}$ an arbitrary map, and π a canonical map on $H_{m_1 \cdots m_l}$. Suppose that a linear map $L : H_{m_1 \cdots m_l} \to M_{(m_1 \cdots m_l)\times t}$ satisfies*

$$L(A_1 \otimes \cdots \otimes A_l) = \pi(A_1 \otimes \cdots \otimes A_l) f(A_1 \otimes \cdots \otimes A_l)$$

for all $A_k \in H_{m_k}^1$, $k = 1, \ldots, l$. Then there exists $Q \in M_{(m_1 \cdots m_l)\times t}$ such that

$$L(A_1 \otimes \cdots \otimes A_l) = \pi(A_1 \otimes \cdots \otimes A_l)Q$$

for all $A_k \in H_{m_k}$, $k = 1, \ldots, l$. Moreover, if $L(A) \neq 0$ whenever $0 \neq A \in H_{m_1 \cdots m_l}$, then $\mathrm{rank}(Q) = m_1 \cdots m_l$.

3. Proof of the main theorem

Since the sufficiency part of the main theorem is clear, we consider only the necessity part. So, throughout this section, we assume that $\phi : H_{m_1 \cdots m_l} \to H_n$ is an injective linear map satisfying (1.1).

Considering the map $X \mapsto \phi(E_{11}^{(m_1)} \otimes X)$, $X \in H_{m_2 \cdots m_l}$, and using Proposition 2.1, we conclude that

$$\phi(E_{11}^{(m_1)} \otimes X) = \lambda T((E_{11}^{(m_1)} \otimes \pi(X)) \oplus 0_r)T^*, \quad X \in H_{m_2 \cdots m_l}$$

for some invertible matrix $T \in M_n$, where $\lambda \in \{-1, 1\}$ and π is a canonical map on $H_{m_2 \cdots m_l}$. Composing ϕ with the map $Y \mapsto T^{-1}Y(T^*)^{-1}$ and, if necessary, with the map $Y \mapsto -Y$, we may assume that

$$\phi(E_{11}^{(m_1)} \otimes X) = (E_{11}^{(m_1)} \otimes \pi(X)) \oplus 0_r, \quad X \in H_{m_2 \cdots m_l}. \tag{3.1}$$

Let $1 \leq s \leq m_1$ be a positive integer. Using induction on s, we have to show that for all $A_s \in H_s$ and all $X \in H_{m_2 \cdots m_l}$,

$$\phi((A_s \oplus 0_{m_1 - s}) \otimes X) = (\tau_1(A_s \oplus 0_{m_1 - s}) \otimes \pi(X)) \oplus 0_r,$$

where τ_1 is either the identity map on H_{m_1} for all $1 \leq s \leq m_1$ or the transposition map on H_{m_1} for all $1 \leq s \leq m_1$.

By (3.1), the above statement holds true for $s = 1$. Now suppose that $s > 1$ and that

$$\phi((A_{s-1} \oplus 0_{m_1 - s + 1}) \otimes X) = (\tau_1(A_{s-1} \oplus 0_{m_1 - s + 1}) \otimes \pi(X)) \oplus 0_r \tag{3.2}$$

for all $A_{s-1} \in H_{s-1}$ and all $X \in H_{m_2 \cdots m_l}$.

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Set

$$\phi(E_{ss}^{(m_1)} \otimes X) = \begin{bmatrix} A_s(X) & B_s(X) \\ B_s(X)^* & D_s(X) \end{bmatrix}, \quad X \in H_{m_2\cdots m_l}$$

and

$$\phi((E_{1s}^{(m_1)} + E_{s1}^{(m_1)}) \otimes X) = \begin{bmatrix} A_{1s}(X) & B_{1s}(X) \\ B_{1s}(X)^* & D_{1s}(X) \end{bmatrix}, \quad X \in H_{m_2\cdots m_l},$$

where A_s and A_{1s} are maps from $H_{m_2\cdots m_l}$ to $H_{(s-1)m_2\cdots m_l}$. Then, of course, D_s maps from $H_{m_2\cdots m_l}$ to $H_{(n-(s-1)m_2\cdots m_l)}$. Let

$$\Delta_l := \{A_1 \otimes \cdots \otimes A_l \ : \ A_j \in H_{m_j}^1\}, \quad \Gamma_l := \{A_2 \otimes \cdots \otimes A_l \ : \ A_j \in H_{m_j}^1\}.$$

By injectivity of ϕ, $D_s(X) \neq 0$ for every $X \in \Gamma_l$. Indeed, if $D_s(X_0) = 0$ for some $X_0 \in \Gamma_l$, then $B_s(X_0) = 0$ and, thus,

$$\phi(E_{ss} \otimes X_0) = A_s(X_0) \oplus 0_{(n-(s-1)m_2\cdots m_l)}.$$

On the other hand, there exist $A_{s-1} \in H_{s-1}$ and $Y \in H_{m_2\cdots m_l}$ such that $\tau_1(A_{s-1} \oplus 0_{m_1-s+1}) \otimes \pi(Y) = A_s(X_0) \oplus 0_{(m_1-s+1)m_2\cdots m_l}$. Therefore, by (3.2),

$$\phi((A_{s-1} \oplus 0_{m_1-s+1}) \otimes Y) = A_s(X_0) \oplus 0_{(n-(s-1)m_2\cdots m_l)},$$

which contradicts the injectivity of ϕ. Moreover, if $X \in \Gamma_l$, then we have $\mathrm{rank}(D_s(X)) \leq \phi(E_{ss} \otimes X) = 1$ and, hence, $\mathrm{rank}(D_s(X)) = 1$. In other words, D_s maps tensor products of rank-one matrices to rank-one matrices.

Let $X \in \Gamma_l$ be an arbitrary matrix and let $\varepsilon \in \mathbb{R}$ be any nonzero scalar. Since $(\varepsilon E_{11}^{(m_1)} + \varepsilon^{-1} E_{ss}^{(m_1)} + E_{1s}^{(m_1)} + E_{s1}^{(m_1)}) \otimes X \in \Delta_l$,

$$\mathrm{rank}\phi((\varepsilon E_{11}^{(m_1)} + \varepsilon^{-1} E_{ss}^{(m_1)} + E_{1s}^{(m_1)} + E_{s1}^{(m_1)}) \otimes X) = 1.$$

Thus, by the induction hypothesis (3.2),

$$\mathrm{rank}\begin{bmatrix} \varepsilon E_{11}^{(s-1)} \otimes \pi(X) + A_{1s}(X) + \varepsilon^{-1} A_s(X) & B_{1s}(X) + \varepsilon^{-1} B_s(X) \\ B_{1s}(X)^* + \varepsilon^{-1} B_s(X)^* & D_{1s}(X) + \varepsilon^{-1} D_s(X) \end{bmatrix} = 1$$

and, hence,

$$\mathrm{rank}(D_{1s}(X) + \varepsilon^{-1} D_s(X)) \leq 1.$$

We already know that $\mathrm{rank}(D_s(X)) = 1$. Thus, by Lemma 2.4, there exists a scalar $d \in \mathbb{R}$ such that $D_{1s}(X) = dD_s(X)$. This yields that

$$\mathrm{rank}\begin{bmatrix} \varepsilon E_{11}^{(s-1)} \otimes \pi(X) + A_{1s}(X) + \varepsilon^{-1} A_s(X) & B_{1s}(X) + \varepsilon^{-1} B_s(X) \\ B_{1s}(X)^* + \varepsilon^{-1} B_s(X)^* & (d + \varepsilon^{-1}) D_s(X) \end{bmatrix} = 1. \tag{3.3}$$

Applying Lemma 2.5, we conclude that $d = 0$ and that there exists a map $f : \Gamma_l \to M_{(m_2\cdots m_l)\times(n-m_2\cdots m_l)}$ such that

$$B_{1s}(X) = \begin{bmatrix} \pi(X)f(X) \\ 0 \end{bmatrix}, \quad X \in \Gamma_l.$$

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Now, using Lemma 2.7,

$$B_{1s}(X) = \begin{bmatrix} \pi(X)Q \\ 0 \end{bmatrix}, \quad X \in H_{m_2\cdots m_l},$$

for some $Q \in M_{(m_2\cdots m_l)\times(n-m_2\cdots m_l)}$.

Furthermore, if $B_{1s}(X_0) = 0$ for some nonzero $X_0 \in H_{m_2\cdots m_l}$, then

$$\phi((E_{1s}^{(m_1)} + E_{s1}^{(m_1)}) \otimes X_0) = \begin{bmatrix} A_{1s}(X_0) & 0 \\ 0 & 0 \end{bmatrix},$$

which contradicts the injectivity of ϕ (see the arguments above for D_s). Applying the second part of Lemma 2.7, we obtain that $\operatorname{rank}(Q) = m_2 \cdots m_l$. Hence, $Q = [I_{m_2\cdots m_l}\ 0]R$ for some invertible matrix $R \in M_{n-(s-1)m_2\cdots m_l}$.

Now, without loss of generality, we may compose ϕ with the map $Y \mapsto (I_{(s-1)m_2\cdots m_l} \oplus R^{-1})^* Y (I_{(s-1)m_2\cdots m_l} \oplus R^{-1})$ (this does not change the induction hypothesis (3.2)). Next, we rewrite $B_s(X)(I_{(s-1)m_2\cdots m_l} \oplus R^{-1})$ as $B_s(X)$. Note that then $B_{1s}(X) = \left[\begin{smallmatrix} \pi(X) & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Using (3.3) and Lemma 2.5, we see that $D_s(X) = \left[\begin{smallmatrix} \pi(X) & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Hence,

$$\phi(E_{ss}^{(m_1)} \otimes X) = \begin{bmatrix} A_s(X) & B_s(X) \\ B_s(X)^* & \pi(X) \oplus 0 \end{bmatrix}, \quad X \in H_{m_2\cdots m_l}.$$

On the other hand, considering the map $X \mapsto \phi(E_{ss}^{(m_1)} \otimes X)$, $X \in H_{m_2\cdots m_l}$, and using Proposition 2.1, we conclude that

$$\phi(E_{ss}^{(m_1)} \otimes X) = \lambda T((E_{ss}^{(m_1)} \otimes \pi'(X)) \oplus 0_r)T^*, \quad X \in H_{m_2\cdots m_l},$$

for some invertible matrix $T \in M_n$, where $\lambda \in \{-1, 1\}$ and π' is a canonical map on $H_{m_2\cdots m_l}$. This yields that

$$\lambda T \begin{bmatrix} 0 & 0 \\ 0 & \pi'(X) \oplus 0 \end{bmatrix} T^* = \begin{bmatrix} A_s(X) & B_s(X) \\ B_s(X)^* & \pi(X) \oplus 0 \end{bmatrix}, \quad X \in H_{m_2\cdots m_l}.$$

Set

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

with $T_{11} \in M_{(s-1)m_2\cdots m_l}$, $T_{22} \in M_{m_2\cdots m_l}$, and $T_{33} \in M_{n-sm_2\cdots m_l}$. Then

$$\lambda \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \pi'(X) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}^* = \begin{bmatrix} A_s(X) & B_s(X) \\ B_s(X)^* & \pi(X) \oplus 0 \end{bmatrix}$$

and, thus,

$$\lambda \begin{bmatrix} T_{12}\pi'(X)T_{12}^* & T_{12}\pi'(X)T_{22}^* & T_{12}\pi'(X)T_{32}^* \\ T_{22}\pi'(X)T_{12}^* & T_{22}\pi'(X)T_{22}^* & T_{22}\pi'(X)T_{32}^* \\ T_{32}\pi'(X)T_{12}^* & T_{32}\pi'(X)T_{22}^* & T_{32}\pi'(X)T_{32}^* \end{bmatrix} = \begin{bmatrix} A_s(X) & B_s(X) \\ B_s(X)^* & \pi(X) \oplus 0 \end{bmatrix}.$$

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Choosing $X = I_{m_2\cdots m_l}$, we see that $\lambda = 1$, $T_{32} = 0$, and that T_{22} must be invertible. This implies that $T_{22}\pi'(X)T_{22}^* = \pi(X)$ and, by Lemma 2.3, $\pi = \pi'$. Moreover, there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $T_{22} = \mu I_{m_2\cdots m_l}$. If we write

$$\widetilde{T} = \begin{bmatrix} I_{(s-1)m_2\cdots m_l} & \bar{\mu}T_{12} & 0 \\ 0 & I_{m_2\cdots m_l} & 0 \\ 0 & 0 & I_{n-sm_2\cdots m_l} \end{bmatrix},$$

then

$$\phi(E_{ss} \otimes X) = \begin{bmatrix} T_{12}\pi(X)T_{12}^* & \bar{\mu}T_{12}\pi(X) & 0 \\ \mu\pi(X)T_{12}^* & \pi(X) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \widetilde{T} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \pi(X) & 0 \\ 0 & 0 & 0 \end{bmatrix} \widetilde{T}^*.$$

Hence, without loss of generality, we may assume that

$$\phi(E_{ss}^{(m_1)} \otimes X) = (E_{ss}^{(m_1)} \otimes \pi(X)) \oplus 0_r, \quad X \in H_{m_2\cdots m_l}. \tag{3.4}$$

To prove the induction, we have to show that for any $1 \le i < s$ and any $X \in H_{m_2\cdots m_l}$,

$$\phi((E_{is}^{(s)} + E_{si}^{(s)}) \otimes X) = ((E_{is}^{(s)} + E_{si}^{(s)}) \otimes \pi(X)) \oplus 0_r$$

and

$$\phi((\sqrt{-1}E_{is}^{(s)} - \sqrt{-1}E_{si}^{(s)}) \otimes X) = ((\sqrt{-1}E_{is}^{(s)} - \sqrt{-1}E_{si}^{(s)}) \otimes \pi(X)) \oplus 0_r,$$

since matrices $E_{is}^{(s)} + E_{si}^{(s)}$, $\sqrt{-1}E_{is}^{(s)} - \sqrt{-1}E_{si}^{(s)}$, $E_{jj}^{(s)}$, $1 \le i, j \le s$, form the base of H_s. To see this, let $X \in \Gamma_l$ be any matrix and let $1 \le i < s$ be any integer. Denote

$$T(X) := \operatorname{Diag}(T_1(X), \dots, T_s(X), I_{n-sm_2\cdots m_l}),$$

where $T_k(X) = I_{m_2\cdots m_l}$ for $1 \le k < s$, $k \ne i$, and $T_i(X) = T_s(X) \in M_{m_2\cdots m_l}$ is an invertible matrix such that $T_s(X)\pi(X)T_s^*(X) = E_{11}^{(m_2\cdots m_l)}$. By (3.2) and (3.4),

$$T(X)\phi(E_{ii}^{(m_1)} \otimes X)T(X)^* = (E_{ii}^{(m_1)} \otimes E_{11}^{(m_2\cdots m_l)}) \oplus 0_r$$

and

$$T(X)\phi(E_{ss}^{(m_1)} \otimes X)T(X)^* = (E_{ss}^{(m_1)} \otimes E_{11}^{(m_2\cdots m_l)}) \oplus 0_r.$$

Using Lemma 2.6, there exists $\lambda_X \in \mathbb{C}$ with $|\lambda_X| = 1$ such that

$$T(X)\phi((E_{is}^{(m_1)} + E_{si}^{(m_1)}) \otimes X)T(X)^* = (\lambda_X E_{is}^{(m_1)} \otimes E_{11}^{(m_2\cdots m_l)} + \bar{\lambda}_X E_{si}^{(m_1)} \otimes E_{11}^{(m_2\cdots m_l)}) \oplus 0_r.$$

Let

$$L(X) := \lambda_X T_i(X)^{-1} E_{11}^{(m_2\cdots m_l)} (T_s(X)^*)^{-1}.$$

Then

$$\phi((E_{is}^{(m_1)} + E_{si}^{(m_1)}) \otimes X) = (E_{is}^{(m_1)} \otimes L(X) + E_{si}^{(m_1)} \otimes L(X)^*) \oplus 0_r.$$

Note also that

$$L(X) = \lambda_X T_i(X)^{-1} E_{11}^{(m_2\cdots m_l)} (T_s(X)^*)^{-1} = \lambda_X \pi(X).$$

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If we define a map $f : \Gamma_l \to M_{m_2\cdots m_l}$ by $f(X) = \lambda_X I_{m_2\cdots m_l}$, $X \in \Gamma_l$, then $\lambda_X \pi(X) = f(X)\pi(X)$ and, using Lemma 2.7,

$$\lambda_X \pi(X) = \pi(X) f(X) = \pi(X) Q$$

for all $X \in H_{m_2\cdots m_l}$. Choosing $X = I_{m_2\cdots m_l}$, we see that $Q = \lambda_{I_{m_2\cdots m_l}} I_{m_2\cdots m_l}$. Thus, for every $X \in H_{m_2\cdots m_l}$,

$$\phi((E_{is}^{(m_1)} + E_{si}^{(m_1)}) \otimes X) = (\lambda_{is} E_{is}^{(m_1)} + \bar{\lambda}_{is} E_{si}^{(m_1)}) \otimes \pi(X),$$

where $\lambda_{is} = \lambda_{I_{m_2\cdots m_l}}$.

Similarly,

$$\phi((\sqrt{-1}E_{is}^{(m_1)} - \sqrt{-1}E_{si}^{(m_1)}) \otimes X) = (\mu_{is}\sqrt{-1}E_{is}^{(m_1)} - \bar{\mu}_{is}\sqrt{-1}E_{si}^{(m_1)}) \otimes \pi(X)$$

for some $\mu_{is} \in \mathbb{C}$ with $|\mu_{is}| = 1$.

Since the rank of the matrix

$$\phi((E_{ii}^{(m_1)} + (1 + \sqrt{-1})E_{is}^{(m_1)} + (1 - \sqrt{-1})E_{si}^{(m_1)} + 2E_{ss}^{(m_1)}) \otimes E_{11}^{(m_2\cdots m_l)})$$

is one, we have $(\lambda_{is} + \mu_{is}\sqrt{-1})(\bar{\lambda}_{is} - \bar{\mu}_{is}\sqrt{-1}) = 2$ and, hence, either $\lambda_{is} = \mu_{is}$ or $\lambda_{is} = -\mu_{is}$. More precisely, there exists $\varepsilon_{is} \in \{-1, 1\}$ such that $\mu_{is} = \varepsilon_{is}\lambda_{is}$.

Case 1. Let $s = 2$. Then there exists T such that

$$\phi((A_2 \oplus 0_{m_1-2}) \otimes X) = ((T\tau_1(A_2)T^* \oplus 0_{m_1-2}) \otimes \pi(X)) \oplus 0_r.$$

Now, if $\varepsilon_{12} = 1$, then $\mu_{is} = \lambda_{is}$ and τ_1 is the identity map, that is,

$$\phi((A_2 \oplus 0_{m_1-2}) \otimes X) = ((TA_2T^* \oplus 0_{m_1-2}) \otimes \pi(X)) \oplus 0_r.$$

On the other hand, if $\varepsilon_{12} = -1$, then $\mu_{is} = -\lambda_{is}$ and τ_1 is the transposition map, that is,

$$\phi((A_2 \oplus 0_{m_1-2}) \otimes X) = ((TA_2^tT^* \oplus 0_{m_1-2}) \otimes \pi(X)) \oplus 0_r.$$

Case 2. Let $s > 2$ and let us assume for a moment that $\tau_1 = \mathrm{id}$. Since the matrices $(E_{ii}^{(m_1)} + E_{ij}^{(m_1)} + E_{is}^{(m_1)} + E_{ji}^{(m_1)} + E_{jj}^{(m_1)} + E_{js}^{(m_1)} + E_{si}^{(m_1)} + E_{sj}^{(m_1)} + E_{ss}^{(m_1)}) \otimes E_{11}^{(m_2\cdots m_l)}$ and $(E_{ii}^{(m_1)} + \sqrt{-1}E_{ij}^{(m_1)} + E_{is}^{(m_1)} - \sqrt{-1}E_{ji}^{(m_1)} + E_{jj}^{(m_1)} - \sqrt{-1}E_{js}^{(m_1)} + E_{si}^{(m_1)} + \sqrt{-1}E_{sj}^{(m_1)} + E_{ss}^{(m_1)}) \otimes E_{11}^{(m_2\cdots m_l)}$ belong to the set Γ_l, we have, for $1 \le i < j < s$,

$$\operatorname{rank}\begin{bmatrix} 1 & 1 & \lambda_{is} \\ 1 & 1 & \lambda_{js} \\ \bar{\lambda}_{is} & \bar{\lambda}_{js} & 1 \end{bmatrix} = 1$$

and

$$\operatorname{rank}\begin{bmatrix} 1 & \sqrt{-1} & \lambda_{is} \\ -\sqrt{-1} & 1 & -\varepsilon_{js}\lambda_{js}\sqrt{-1} \\ \bar{\lambda}_{is} & \varepsilon_{js}\bar{\lambda}_{js}\sqrt{-1} & 1 \end{bmatrix} = 1.$$

Thus, $\lambda_{js} = \lambda_{is} = 1$ and $\varepsilon_{js} = 1$ and, consequently,

$$\phi((A_s \oplus 0_{m_1-s}) \otimes X) = ((A_s \oplus 0_{m_1-s}) \otimes \pi(X)) \oplus 0_r.$$

The proof is completed.

4. Proof of Lemma 2.7

We divide the proof into several lemmas.

Lemma 4.1. *Let t, m be positive integers with $t \geq m$, let $f : H_m \to M_{m\times t}$ be a map, and let τ be the identity map or the transposition map on H_m. Suppose that a linear map $L : H_m \to M_{m\times t}$ satisfies*

$$L(A) = \tau(A)f(A)$$

for all $A \in H_m^1$. Then

$$L(A) = \tau(A)Q, \quad A \in H_m,$$

where $Q = \sum_{i=1}^m E_{ii}^{(m)} f(E_{ii}^{(m)})$.

Proof. Without loss of generality, we may assume that τ is the identity map since the proof in the other case is similar.

First, it is clear that

$$L(E_{ii}^{(m)}) = E_{ii}^{(m)} f(E_{ii}^{(m)}) = E_{ii}^{(m)} \sum_{i=1}^{m} E_{ii}^{(m)} f(E_{ii}^{(m)}) = E_{ii}^{(m)} Q$$

for all $1 \leq i \leq m$. Now let $1 \leq i < j \leq m$ and let us denote

$$A(\lambda) = \lambda E_{ii}^{(m)} + E_{ij}^{(m)} + E_{ji}^{(m)} + \lambda^{-1} E_{jj}^{(m)}, \quad 0 \neq \lambda \in \mathbb{R}.$$

If we write $f(A(\lambda)) = [a_{rs}(\lambda)]$ and $Q = [q_{rs}]$, then

$$\begin{aligned} L(A(\lambda)) &= (\lambda E_{ii}^{(m)} + E_{ij}^{(m)} + E_{ji}^{(m)} + \lambda^{-1} E_{jj}^{(m)}) f(A(\lambda)) \\ &= \begin{bmatrix} 0 & \cdots & 0 \\ \lambda a_{i1}(\lambda) + a_{j1}(\lambda) & \cdots & \lambda a_{it}(\lambda) + a_{jt}(\lambda) \\ 0 & \cdots & 0 \\ a_{i1}(\lambda) + \lambda^{-1} a_{j1}(\lambda) & \cdots & a_{it}(\lambda) + \lambda^{-1} a_{jt}(\lambda) \\ 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} L(A(\lambda)) &= L(\lambda E_{ii}^{(m)} + E_{ij}^{(m)} + E_{ji}^{(m)} + \lambda^{-1} E_{jj}^{(m)}) \\ &= L(E_{ii}^{(m)} + E_{ij}^{(m)} + E_{ji}^{(m)} + E_{jj}^{(m)}) + (\lambda - 1)L(E_{ii}) + (\lambda^{-1} - 1)L(E_{jj}) \\ &= \begin{bmatrix} 0 & \cdots & 0 \\ a_{i1}(1) + a_{j1}(1) + (\lambda - 1)q_{i1} & \cdots & a_{it}(1) + a_{jt}(1) + (\lambda - 1)q_{it} \\ 0 & \cdots & 0 \\ a_{i1}(1) + a_{j1}(1) + (\lambda^{-1} - 1)q_{j1} & \cdots & a_{it}(1) + a_{jt}(1) + (\lambda^{-1} - 1)q_{jt} \\ 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

Comparing the above equalities,

$$a_{ik}(1) + a_{jk}(1) + (\lambda - 1)q_{ik} = \lambda(a_{ik}(1) + a_{jk}(1) + (\lambda^{-1} - 1)q_{jk})$$

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for all integers $1 \le k \le t$. This yields that for all nonzero scalars $\lambda \in \mathbb{R}$,

$$(1-\lambda)(a_{ik}(1)+a_{jk}(1)-q_{ik}-q_{jk})=0$$

and, hence,

$$a_{ik}(1)+a_{jk}(1)=q_{ik}+q_{jk}, \quad 1 \le k \le t.$$

Thus,

$$\begin{aligned} L(E_{ij}^{(m)}+E_{ji}^{(m)}) &= L(E_{ii}^{(m)}+E_{ij}^{(m)}+E_{ji}^{(m)}+E_{jj}^{(m)})-L(E_{ii}^{(m)})-L(E_{jj}^{(m)}) \\ &= \begin{bmatrix} 0 & \cdots & 0 \\ a_{i1}(1)+a_{j1}(1)-q_{i1} & \cdots & a_{it}(1)+a_{jt}(1)-q_{it} \\ 0 & \cdots & 0 \\ a_{i1}(1)+a_{j1}(1)-q_{j1} & \cdots & a_{it}(1)+a_{jt}(1)-q_{jt} \\ 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \cdots & 0 \\ q_{j1} & \cdots & q_{jt} \\ 0 & \cdots & 0 \\ q_{i1} & \cdots & q_{it} \\ 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

So, we proved that

$$L(E_{ij}^{(m)}+E_{ji}^{(m)})=(E_{ij}^{(m)}+E_{ji}^{(m)})Q, \quad 1 \le i < j \le m.$$

Using similar arguments, we can show that

$$L(\sqrt{-1}E_{ij}^{(m)}-\sqrt{-1}E_{ji}^{(m)})=(\sqrt{-1}E_{ij}^{(m)}-\sqrt{-1}E_{ji}^{(m)})Q, \quad 1 \le i < j \le m.$$

This completes the proof, since the matrices $E_{ij}^{(m)}+E_{ji}^{(m)}$, $\sqrt{-1}E_{ij}^{(m)}-\sqrt{-1}E_{ji}^{(m)}$, $E_{kk}^{(m)}$, $1 \le i < j \le m$, $1 \le k \le m$, form the base of H_m. □

Lemma 4.2. *Let k,t be positive integers with $1 \le k \le m_1$ and $t \ge m_1m_2$, let $f : H_{m_2} \to M_{(m_1m_2)\times t}$ be a map, and let τ be the identity map or the transposition map on H_{m_2}. Suppose that a linear map $L_k : H_{m_2} \to M_{(m_1m_2)\times t}$ satisfies*

$$L_k(A)=(E_{kk}^{(m_1)} \otimes \tau(A))f(A)$$

for all $A \in H_{m_2}^1$. Then

$$L_k(A)=(E_{kk}^{(m_1)} \otimes \tau(A))Q_k, \quad A \in H_{m_2},$$

where $Q_k=\sum_{i=1}^{m_2}(E_{kk}^{(m_1)} \otimes E_{ii}^{(m_2)})f(E_{ii}^{(m_2)})$.

Proof. We observe two cases.

Case 1. First we consider the case $k=1$. Let us write

$$f(A)=\begin{bmatrix} g(A) \\ h(A) \end{bmatrix}, \quad A \in H_{m_2},$$

where $g(A) \in M_{m_2 \times t}$. Then

$$L_1(A) = (E_{11}^{(m_1)} \otimes A) f(A) = (E_{11}^{(m_1)} \otimes A) \begin{bmatrix} g(A) \\ h(A) \end{bmatrix} = \begin{bmatrix} \tau(A) g(A) \\ 0 \end{bmatrix}$$

for all $A \in H_{m_2}^1$. Now let $\tilde{L}_1 : H_{m_2}^1 \to M_{m_2 \times t}$ be a linear map such that $\tilde{L}_1(A) = \tau(A) g(A)$ for all $A \in H_{m_2}^1$. Then, by Lemma 4.1,

$$\tilde{L}_1(A) = \tau(A) \sum_{i=1}^{m_2} E_{ii}^{(m_2)} g(E_{ii}^{(m_2)}), \quad A \in H_{m_2}.$$

Hence,

$$\begin{aligned} L_1(A) &= \begin{bmatrix} \tau(A) \sum_{i=1}^{m_2} E_{ii}^{(m_2)} g(E_{ii}^{(m_2)}) \\ 0 \end{bmatrix} \\ &= (E_{11}^{(m_1)} \otimes \tau(A)) \sum_{i=1}^{m_2} (E_{11}^{(m_1)} \otimes E_{ii}^{(m_2)}) \begin{bmatrix} g(E_{ii}^{(m_2)}) \\ h(E_{ii}^{(m_2)}) \end{bmatrix} \\ &= (E_{11}^{(m_1)} \otimes \tau(A)) \sum_{i=1}^{m_2} (E_{11}^{(m_1)} \otimes E_{ii}^{(m_2)}) f(E_{ii}^{(m_2)}) \\ &= (E_{11}^{(m_1)} \otimes \tau(A)) Q_1. \end{aligned}$$

Case 2. Suppose that $k > 1$. Let $P_{1k} := I_{m_1} - E_{11}^{(m_1)} - E_{kk}^{(m_1)} + E_{1k}^{(m_1)} + E_{k1}^{(m_1)}$. Then $E_{kk}^{(m_1)} = P_{1k} E_{11}^{(m_1)} P_{1k}$ and

$$\begin{aligned} L_k(A) &= ((P_{1k} E_{11}^{(m_1)} P_{1k}) \otimes \tau(A)) f(A) \\ &= (P_{1k} \otimes I_{m_2})(E_{11}^{(m_1)} \otimes \tau(A))(P_{1k} \otimes I_{m_2}) f(A). \end{aligned}$$

Let us define a map $\tilde{f} : H_{m_2} \to M_{(m_1 m_2) \times t}$ by $\tilde{f}(A) = (P_{1k} \otimes I_{m_2}) f(A)$ and a linear map $\tilde{L}_k : H_{m_2} \to M_{(m_1 m_2) \times t}$ by $\tilde{L}_k(A) = (P_{1k} \otimes I_{m_2}) L_k(A)$. Then, for any $A \in H_{m_2}^1$,

$$\tilde{L}_k(A) = (E_{11}^{(m_1)} \otimes \tau(A)) \tilde{f}(A).$$

Applying Case 1,

$$\tilde{L}_k(A) = (E_{11}^{(m_1)} \otimes \tau(A)) \sum_{i=1}^{m_2} (E_{11}^{(m_1)} \otimes E_{ii}^{m_2}) \tilde{f}(E_{ii}^{(m_2)}), \quad A \in H_{m_2}.$$

Thus, for any $A \in H_{m_2}$,

$$\begin{aligned} L_k(A) &= (P_{1k} \otimes I_{m_2}) \tilde{L}_k(A) \\ &= (P_{1k} \otimes I_{m_2})(E_{11}^{(m_1)} \otimes \tau(A)) \sum_{i=1}^{m_2} (E_{11}^{(m_1)} \otimes E_{ii}^{m_2})(P_{1k} \otimes I_{m_2}) f(E_{ii}^{(m_2)}) \end{aligned}$$

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$$= (P_{1k} \otimes I_{m_2})(E_{11}^{(m_1)} \otimes \tau(A))(P_{1k} \otimes I_{m_2}) \cdot \sum_{i=1}^{m_2} (P_{1k} \otimes I_{m_2})(E_{11}^{(m_1)} \otimes E_{ii}^{(m_2)})(P_{1k} \otimes I_{m_2}) f(E_{ii}^{(m_2)})$$
$$= (E_{kk}^{(m_1)} \otimes \tau(A)) \sum_{i=1}^{m_2} (E_{kk}^{(m_1)} \otimes E_{ii}^{(m_2)}) f(E_{ii}^{(m_2)})$$
$$= (E_{kk}^{(m_1)} \otimes \tau(A)) Q_k. \qquad \square$$

Lemma 4.3. *Let k, t be positive integers with $1 \le k \le m_2$ and $t \ge m_1 m_2$, let $f : H_{m_1} \to M_{(m_1 m_2) \times t}$ be an arbitrary map, and let τ be the identity map or the transposition map on H_{m_1}. Suppose that a linear map $L_k : H_{m_1} \to M_{(m_1 m_2) \times t}$ satisfies*

$$L_k(A) = (\tau(A) \otimes E_{kk}^{(m_2)}) f(A)$$

for all $A \in H_{m_1}^1$. Then

$$L_k(A) = (\tau(A) \otimes E_{kk}^{(m_2)}) Q_k, \quad A \in H_{m_1},$$

where $Q_k = \sum_{i=1}^{m_1} (E_{ii}^{(m_1)} \otimes E_{kk}^{(m_2)}) f(E_{ii}^{(m_1)})$.

Proof. Let $P \in M_{m_1 m_2}$ be a permutation matrix such that $P(A \otimes B)P^{-1} = B \otimes A$ for all $A \in H_{m_1}$, $B \in H_{m_2}$. Define a map $\tilde{f} : H_{m_1} \to M_{(m_1 m_2) \times t}$ by $\tilde{f}(A) = P f(A)$ and a linear map $\tilde{L}_k : H_{m_1} \to M_{(m_1 m_2) \times t}$ by $\tilde{L}_k(A) = P L_k(A)$. Then, for any $A \in H_{m_1}^1$,

$$\tilde{L}_k(A) = P(\tau(X) \otimes E_{kk}^{(m_2)}) P^{-1} \tilde{f}(X) = (E_{kk}^{(m_2)} \otimes \tau(A)) \tilde{f}(A).$$

Thus, by Lemma 4.2,

$$\tilde{L}_k(A) = (E_{kk}^{(m_2)} \otimes \tau(A)) \sum_{i=1}^{m_1} (E_{kk}^{(m_2)} \otimes E_{ii}^{(m_1)}) \tilde{f}(E_{ii}^{(m_1)}), \quad A \in H_{m_1}.$$

Hence, for all $A \in H_{m_1}$,

$$L_k(A) = P^{-1}(E_{kk}^{(m_2)} \otimes \tau(A)) \sum_{i=1}^{m_1} (E_{kk}^{(m_2)} \otimes E_{ii}^{(m_1)}) P f(E_{ii}^{(m_1)})$$
$$= P^{-1}(E_{kk}^{(m_2)} \otimes \tau(A)) P \sum_{i=1}^{m_1} P^{-1}(E_{kk}^{(m_2)} \otimes E_{ii}^{(m_1)}) P f(E_{ii}^{(m_1)})$$
$$= (\tau(A) \otimes E_{kk}^{(m_2)}) \sum_{i=1}^{m_1} (E_{ii}^{(m_1)} \otimes E_{kk}^{(m_2)}) f(E_{ii}^{(m_1)})$$
$$= (\tau(A) \otimes E_{kk}^{(m_2)}) Q_k. \qquad \square$$

Lemma 4.4. *Let h, k, t be positive integers with $1 \le h \le m_1$, $1 \le k \le m_3$, and $t \ge m_1 m_2 m_3$, let $f : H_{m_2} \to M_{(m_1 m_2 m_3) \times t}$ be an arbitrary map, and let τ be the identity map or the transposition map on H_{m_2}. Suppose that a linear map $L_{h,k} : H_{m_2} \to M_{(m_1 m_2 m_3) \times t}$*

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satisfies

$$L_{h,k}(A) = (E_{hh}^{(m_1)} \otimes \tau(A) \otimes E_{kk}^{(m_3)}) f(A)$$

for all $A \in H_{m_2}^1$. *Then*

$$L_{h,k}(A) = (E_{hh}^{(m_1)} \otimes \tau(A) \otimes E_{kk}^{(m_3)}) Q_{h,k}, \quad A \in H_{m_2},$$

where $Q_{h,k} = \sum_{i=1}^{m_2} (E_{hh}^{(m_1)} \otimes E_{ii}^{(m_2)} \otimes E_{kk}^{(m_3)}) f(E_{ii}^{(m_2)})$.

Proof. Let $P \in M_{m_1 m_2 m_3}$ be a matrix such that $P(A \otimes B \otimes C)P^{-1} = A \otimes C \otimes B$ for all $A \in H_{m_1}, B \in H_{m_2}, C \in H_{m_3}$. Define a map $\tilde{f} : H_{m_2} \to M_{(m_1 m_2 m_3)\times t}$ by $\tilde{f}(A) = Pf(A)$ and a linear map $\tilde{L}_{h,k} : H_{m_2} \to M_{(m_1 m_2 m_3)\times t}$ by $\tilde{L}_{h,k}(A) = PL_{h,k}(A)$. Then, for any $A \in H_{m_2}^1$,

$$\tilde{L}_{h,k}(A) = (E_{hh}^{(m_1)} \otimes E_{kk}^{(m_3)} \otimes \tau(A)) \tilde{f}(A).$$

Thus, by Lemma 4.2, we conclude that for all $A \in H_{m_2}$,

$$\tilde{L}_{h,k}(A) = (E_{hh}^{(m_1)} \otimes E_{kk}^{(m_3)} \otimes \tau(A)) \sum_{i=1}^{m_1} (E_{hh}^{(m_1)} \otimes E_{kk}^{(m_3)} \otimes E_{ii}^{(m_2)}) \tilde{f}(E_{ii}^{(m_2)}).$$

Hence, for all $A \in H_{m_2}$,

$$\begin{aligned} L_{h,k}(A) &= P^{-1}(E_{hh}^{(m_1)} \otimes E_{kk}^{(m_3)} \otimes \tau(A)) \sum_{i=1}^{m_1} (E_{hh}^{(m_1)} \otimes E_{kk}^{(m_3)} \otimes E_{ii}^{(m_2)}) Pf(E_{ii}^{(m_2)}) \\ &= P^{-1}(E_{hh}^{(m_1)} \otimes \tau(A) \otimes E_{kk}^{(m_3)}) P \\ &\quad \cdot \sum_{i=1}^{m_1} P^{-1}(E_{hh}^{(m_1)} \otimes E_{ii}^{(m_2)} \otimes E_{kk}^{(m_3)}) Pf(E_{ii}^{(m_2)}) \\ &= (E_{hh}^{(m_1)} \otimes \tau(A) \otimes E_{kk}^{(m_3)}) \sum_{i=1}^{m_1} (E_{hh}^{(m_1)} \otimes E_{ii}^{(m_2)} \otimes E_{kk}^{(m_3)}) f(E_{ii}^{(m_2)}) \\ &= (E_{hh}^{(m_1)} \otimes \tau(A) \otimes E_{kk}^{(m_3)}) Q_{h,k}. \end{aligned}$$

□

Before writing the last lemma of this section, let us fix some additional notation. For $H_{m_1 \cdots m_l}$, we define a chain of sets

$$\begin{aligned} \Delta_0 &:= \{E_{i_1 i_1}^{(m_1)} \otimes E_{i_2 i_2}^{(m_2)} \otimes \cdots \otimes E_{i_l i_l}^{(m_l)} : 1 \le i_j \le m_j\}, \\ \Delta_1 &:= \{A_1 \otimes E_{i_2 i_2}^{(m_2)} \otimes \cdots \otimes E_{i_l i_l}^{(m_l)} \ : \ A_1 \in H_{m_1}^1, 1 \le i_j \le m_j\}, \\ &\ \ \vdots \\ \Delta_k &:= \{A_1 \otimes \cdots \otimes A_k \otimes E_{i_{k+1} i_{k+1}}^{(m_{k+1})} \otimes \cdots \otimes E_{i_l i_l}^{(m_l)} \ : \ A_j \in H_{m_j}^1, \ 1 \le i_j \le m_j\}, \\ &\ \ \vdots \\ \Delta_l &:= \{A_1 \otimes \cdots \otimes A_l \ : \ A_j \in H_{m_j}^1\}. \end{aligned}$$

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It is clear that $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_l$, and that the linear span of Δ_l is equal to the whole algebra $H_{m_1\cdots m_l}$.

Lemma 2.7 is a direct consequence of the following result.

Lemma 4.5. *Let t be a positive integer with $t \geq m_1 \cdots m_l$, let $f : H_{m_1\cdots m_l} \to M_{(m_1\cdots m_l)\times t}$ be an arbitrary map, and let π be a canonical map on $H_{m_1\cdots m_l}$. Suppose that a linear map $L : H_{m_1\cdots m_l} \to M_{(m_1\cdots m_l)\times t}$ satisfies*

$$L(A) = \pi(A)f(A)$$

for all $A \in \Delta_l$. Then

$$L(A) = \pi(A)Q, \quad A \in H_{m_1\cdots m_l},$$

where

$$Q = \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_l=1}^{m_l}(E^{(m_1)}_{i_1i_1} \otimes \cdots \otimes E^{(m_l)}_{i_li_l})f(E^{(m_1)}_{i_1i_1} \otimes \cdots \otimes E^{(m_l)}_{i_li_l}).$$

Remark 4.6. If $l = 2$ and $m_1 = m_2 = 2$, then

$$\begin{aligned} Q &= \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}(E^{(m_1)}_{i_1i_1} \otimes E^{(m_2)}_{i_2i_2})f(E^{(m_1)}_{i_1i_1} \otimes E^{(m_2)}_{i_2i_2}) \\ &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}(E^{(2)}_{i_1i_1} \otimes E^{(2)}_{i_2i_2})f(E^{(2)}_{i_1i_1} \otimes E^{(2)}_{i_2i_2}) \\ &= (E^{(2)}_{11} \otimes E^{(2)}_{11})f(E^{(2)}_{11} \otimes E^{(2)}_{11}) + (E^{(2)}_{22} \otimes E^{(2)}_{11})f(E^{(2)}_{22} \otimes E^{(2)}_{11}) \\ &\quad + (E^{(2)}_{11} \otimes E^{(2)}_{22})f(E^{(2)}_{11} \otimes E^{(2)}_{22}) + (E^{(2)}_{22} \otimes E^{(2)}_{22})f(E^{(2)}_{22} \otimes E^{(2)}_{22}). \end{aligned}$$

Proof of Lemma 4.5. It suffices to prove that

$$L(A) = \pi(A)f(A) = \pi(A)Q \tag{4.1}$$

for all $A \in \Delta_k$, $k = 0, \ldots, l$. We use induction on k.

First, let $A \in \Delta_0$. Then we may write $A = E^{(m_1)}_{h_1h_1} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}$ for some $1 \leq h_j \leq m_j$, $j = 1, \ldots, l$, and

$$\begin{aligned} \pi(A)Q &= (E^{(m_1)}_{h_1h_1} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l})Q \\ &= (E^{(m_1)}_{h_1h_1} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l})f(E^{(m_1)}_{h_1h_1} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) \\ &= Af(A) = \pi(A)f(A) = L(A). \end{aligned}$$

So, (4.1) holds for $k = 0$.

Now, assume that (4.1) holds for $0 \leq k < l$. We would like to prove that (4.1) holds for $k + 1$. Let us write

$$\pi = \tau_1 \otimes \cdots \otimes \tau_l,$$

where τ_j is either the identity map or the transposition map on H_{m_j} for each $j = 1, \ldots, l$.

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Case 1. If $k = 0$ and $A = A_1 \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l} \in \Delta_1$ with $A_1 \in H^1_{m_1}$, then

$$L(A) = (\tau_1(A_1) \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) f(A_1 \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}).$$

Let us define a map $\tilde{f} : H_{m_1} \to M_{(m_1\cdots m_l)\times t}$ by

$$\tilde{f}(X_1) = f(X_1 \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}), \qquad X_1 \in H_{m_1}$$

and a map $\tilde{L} : H_{m_1} \to M_{(m_1\cdots m_l)\times t}$ by

$$\tilde{L}(X_1) = L(X_1 \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}), \qquad X_1 \in H_{m_1}.$$

Clearly, $\tilde{L}$ is a linear map. Moreover, $\mathrm{rank}(\tilde{L}(X_1)) = 1$ whenever $\mathrm{rank}(X_1) = 1$. Applying Lemma 4.3,

$$\begin{aligned}\tilde{L}(X_1) &= L(X_1 \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) \\ &= (\tau_1(X_1) \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) \cdot \sum_{i=1}^{m_1} (E^{(m_1)}_{ii} \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) \tilde{f}(E^{(m_1)}_{ii})\end{aligned}$$

for all $X_1 \in H_{m_1}$. On the other hand, we already know that

$$\begin{aligned}&(E^{(m_1)}_{ii} \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) f(E^{(m_1)}_{ii} \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) \\ &\quad = (E^{(m_1)}_{ii} \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) Q\end{aligned}$$

for all $1 \le i \le m_1$. Comparing the above two equations, we arrive at

$$\begin{aligned}L(A) &= L(A_1 \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) \\ &= (\tau_1(A_1) \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) \sum_{i=1}^{m_1} (E^{m_1}_{ii} \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) Q \\ &= (\tau_1(A_1) \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l})(I_{m_1} \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) Q \\ &= (\tau_1(A_1) \otimes E^{(m_2)}_{h_2h_2} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}) Q \\ &= \pi(A) Q.\end{aligned}$$

Case 2. Let $0 < k < l - 1$ and let

$$A = A_1 \otimes \cdots \otimes A_k \otimes A_{k+1} \otimes E^{(m_{k+2})}_{h_{k+2}h_{k+2}} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l} \in \Delta_{k+1}$$

with $A_j \in H^1_{m_j}$ for $j = 1, \ldots, k + 1$. For the sake of readability, let us denote

$$E := E^{(m_{k+2})}_{h_{k+2}h_{k+2}} \otimes \cdots \otimes E^{(m_l)}_{h_lh_l}.$$

So, $A = A_1 \otimes \cdots \otimes A_k \otimes A_{k+1} \otimes E$. Since $\mathrm{rank}(\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k)) = 1$, there exist invertible matrices $U, V \in M_{m_1\cdots m_k}$ such that

$$\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) = U E^{(m_1\cdots m_k)}_{11} V.$$

Let us define a map $\tilde{f} : H_{m_{k+1}} \to M_{(m_1\cdots m_l)\times t}$ by

$$\tilde{f}(X_{k+1}) = (V \otimes I_{m_{k+1}\cdots m_l}) f(\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) \otimes \tau_{k+1}(X_{k+1}) \otimes E)$$

for all $X_{k+1} \in H_{m_{k+1}}$ and a map $\tilde{L} : H_{m_{k+1}} \to M_{(m_1\cdots m_l)\times t}$ by

$$\tilde{L}(X_{k+1}) = (U^{-1} \otimes I_{m_{k+1}\cdots m_l}) L(\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) \otimes \tau_{k+1}(X_{k+1}) \otimes E)$$

for all $X_{k+1} \in H_{m_{k+1}}$. Clearly, $\tilde{L}$ is a linear map and, for any $X_{k+1} \in H^1_{m_{k+1}}$,

$$\begin{aligned}
\tilde{L}(X_{k+1}) &= (U^{-1} \otimes I_{m_{k+1}\cdots m_l})(\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) \otimes \tau_{k+1}(X_{k+1}) \otimes E) \\
&\qquad \cdot (V^{-1} \otimes I_{m_{k+1}\cdots m_l}) \tilde{f}(X_{k+1}) \\
&= (E_{11}^{(m_1\cdots m_k)} \otimes \tau_{k+1}(X_{k+1}) \otimes E) \tilde{f}(X_{k+1}).
\end{aligned}$$

By Lemma 4.4,

$$\begin{aligned}
\tilde{L}(X_{k+1}) &= (E_{11}^{(m_1\cdots m_k)} \otimes \tau_{k+1}(X_{k+1}) \otimes E) \\
&\qquad \cdot \sum_{i=1}^{m_{k+1}} (E_{11}^{(m_1\cdots m_k)} \otimes E_{ii}^{(m_{k+1})} \otimes E) \tilde{f}(E_{ii}^{(m_{k+1})}) \\
&= (E_{11}^{(m_1\cdots m_k)} \otimes \tau_{k+1}(X_{k+1}) \otimes E)(U^{-1} \otimes I_{m_{k+1}\cdots m_l}) \\
&\qquad \cdot \sum_{i=1}^{m_{k+1}} (U \otimes I_{m_{k+1}\cdots m_l})(E_{11}^{(m_1\cdots m_k)} \otimes E_{ii}^{(m_{k+1})} \otimes E) \\
&\qquad \cdot (V \otimes I_{m_{k+1}\cdots m_l}) f(\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) \otimes E_{ii}^{(m_{k+1})} \otimes E) \\
&= (E_{11}^{(m_1\cdots m_k)} \otimes \tau_{k+1}(X_{k+1}) \otimes E)(U^{-1} \otimes I_{m_{k+1}\cdots m_l}) \\
&\qquad \cdot \sum_{i=1}^{m_{k+1}} (\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) \otimes E_{ii}^{(m_{k+1})} \otimes E) \\
&\qquad \cdot f(\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) \otimes E_{ii}^{(m_{k+1})} \otimes E)
\end{aligned}$$

for all $X_{k+1} \in H_{m_{k+1}}$. Using the induction hypothesis,

$$\begin{aligned}
\tilde{L}(X_{k+1}) &= (E_{11}^{(m_1\cdots m_k)} \otimes \tau_{k+1}(X_{k+1}) \otimes E)(U^{-1} \otimes I_{m_{k+1}\cdots m_l}) \\
&\qquad \cdot \sum_{i=1}^{m_{k+1}} (\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) \otimes E_{ii}^{(m_{k+1})} \otimes E) Q \\
&= (E_{11}^{(m_1\cdots m_k)} \otimes \tau_{k+1}(X_{k+1}) \otimes E) \\
&\qquad \cdot \sum_{i=1}^{m_{k+1}} (E_{11}^{(m_1\cdots m_k)} \otimes E_{ii}^{(m_{k+1})} \otimes E)(V \otimes I_{m_{k+1}\cdots m_l}) Q \\
&= (E_{11}^{(m_1\cdots m_k)} \otimes \tau_{k+1}(X_{k+1}) \otimes E) \\
&\qquad \cdot (E_{11}^{(m_1\cdots m_k)} \otimes I_{m_{k+1}} \otimes E)(V \otimes I_{m_{k+1}\cdots m_l}) Q \\
&= (E_{11}^{(m_1\cdots m_k)} \otimes \tau_{k+1}(X_{k+1}) \otimes E)(V \otimes I_{m_{k+1}\cdots m_l}) Q
\end{aligned}$$

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for all $X_{k+1} \in H_{m_{k+1}}$. Hence,

$$\begin{aligned} L(A) &= L(A_1 \otimes \cdots \otimes A_k \otimes A_{k+1} \otimes E) \\ &= (U \otimes I_{m_{k+1} \cdots m_l})\tilde{L}(A_{k+1}) \\ &= (U \otimes I_{m_{k+1} \cdots m_l})(E_{11}^{(m_1 \cdots m_k)} \otimes \tau_{k+1}(A_{k+1}) \otimes E)(V \otimes I_{m_{k+1} \cdots m_l})Q \\ &= (\tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k) \otimes \tau_{k+1}(A_{k+1}) \otimes E)Q \\ &= \pi(A)Q. \end{aligned}$$

Case 3. If $k = l - 1$, then the proof is similar to the proof in Case 1 and the proof in Case 2. We just use Lemma 4.2 instead of Lemma 4.3 or Lemma 4.4. □

REMARK 4.7. If $\operatorname{rank}(Q) < m_1 \cdots m_l$, then we can write

$$Q = U \begin{bmatrix} I_r & 0 \\ 0 & 0_s \end{bmatrix} V$$

for some invertible matrices $U \in M_{m_1 \cdots m_l}$ and $V \in M_t$, where $r < m_1 \cdots m_l$ and $s = m_1 \cdots m_l - r > 0$. Since a canonical map π is bijective on $H_{m_1 \cdots m_l}$, there exists a nonzero matrix $A \in H_{m_1 \cdots m_l}$ such that

$$\pi(A) = (U^{-1})^* \begin{bmatrix} 0_r & 0 \\ 0 & I_s \end{bmatrix} U^{-1}.$$

Consequently,

$$L(A) = \pi(A)Q = (U^{-1})^* \begin{bmatrix} 0_r & 0 \\ 0 & I_s \end{bmatrix} U^{-1} U \begin{bmatrix} I_r & 0 \\ 0 & 0_s \end{bmatrix} V = 0_{(m_1 \cdots m_l) \times t}.$$

So, if $L(A) \neq 0$ whenever $0 \neq A \in H_{m_1 \cdots m_l}$, then $\operatorname{rank}(Q) = m_1 \cdots m_l$.

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