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NONEXISTENCE OF UNIVERSAL PARACOMPACT *M*-SPACES

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Abstract

For a cardinal m with $\omega \leqslant m \leqslant 2^{\omega}$, no paracompact M-space of weight m has the universal property that every paracompact M-space of weight $\leqslant m$ is embeddable in it as a closed subspace. We derive this result from the corresponding statement for metric spaces.

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The purpose of this note is to give an answer to the following question raised by J. Nagata (see page 271 in [1]):

(Q1) Does there exist a paracompact M-space X of weight m with the universal property that every paracompact M-space of weight $\leqslant m$ is homeomorphic to some closed subspace of X?

We show that the answer is negative in the case of $m \leqslant \mathfrak{c}$ where $\mathfrak{c}$ denotes the cardinal 2^{ω} of the continuum. The key idea of the proof is to reduce the above question to that of only metric spaces (see Theorem 1). It is, indeed, easily observed that in the special case of $m = \omega$ the question (Q1) is nothing but the problem of separable metric spaces.

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We assume that all spaces in this paper are at least Hausdorff and all maps are continuous. Recall that a *paracompact M-space* is a paracompact space which admits a perfect map onto a metric space, or equivalently, is a closed subspace in the product of a compact space and a metric space (see Nagata [6]). A *perfect map* is a closed map with compact fibers. The class of paracompact M-spaces forms one of most important classes in general topology since it coincides with the class of paracompact p-spaces in the sense of Arhangel'skii [2]. In this section we will establish the next key theorem.

THEOREM 1. *Let m be an infinite cardinal. The following statements are equivalent*:

(1) *There exists a paracompact M-space Z of weight m in which every paracompact M-space of weight $\leqslant m$ is embeddable as a closed subspace.*

(2) *There exists a metric space M of weight m in which every metric space of weight $\leqslant m$ is embeddable as a closed subspace.*

To prove this theorem we need some facts and lemmas. Let us denote by I the unit interval.

FACT 1 (Nagata [6]). *Let X be a paracompact M-space of weight m. Then X admits a perfect map onto a metric space T_X if and only if X is embeddable in $I^m \times T_X$ as a closed subset.*

LEMMA 1. *For a metric space T of weight m, the following are equivalent*:

(a) *Every paracompact M-space X of weight $\leqslant m$ is embeddable as a closed subset in the paracompact M-space $I^m \times T$.*

(b) *Every metric space S of weight $\leqslant m$ admits a perfect map onto some closed subset of T.*

PROOF. (a) $\to$ (b): Let S be a metric space of weight $\leqslant m$. Since every metric space is of course a paracompact M-space, we can assume by (a) that S is a closed subset of $I^m \times T$. Since the projection π: $I^m \times T \to T$ is perfect, its restriction to S is a perfect map onto a closed subspace $\pi[S]$ of T.

(b) $\to$ (a): Let X be a paracompact M-space of weight $\leqslant m$. By Fact 1 we can assume that X is a closed subspace of $I^m \times T_X$ for some metric space T_X of weight $\leqslant m$. It follows from the assumption (b) that T_X admits a perfect map onto some closed subspace F of T. So, again by Fact 1, T_X is assumed to be a closed subspace of $I^m \times F$. Next resulting inclusions show that X is a closed subspace of $I^m \times T$:

$$X \underset{\text{cl.}}{\subset} I^m \times T_X \underset{\text{cl.}}{\subset} I^m \times (I^m \times F) \cong I^m \times F \underset{\text{cl.}}{\subset} I^m \times T.$$

LEMMA 2. *Let X and Y be spaces, that is, Hausdorff spaces, and let f be a perfect map from a (not necessarily closed) subspace A of X onto Y. Then the graph f is a closed subspace of $X \times Y$.*

PROOF. Let (x, y) be a point of $X \times Y$ missing the graph f. Then the compact space $f^{-1}(y)$ misses the point x in the Hausdorff space X, so there exist disjoint open sets U and U' containing x and $f^{-1}(y)$ respectively. Since f is a closed map, we can choose an open neighborhood V of y in Y such that $f^{-1}(V) \subset U' \cap A \subset U'$. Therefore we have $U \cap f^{-1}(V) = \varnothing$, that is, $(U \times V) \cap f = \varnothing$. Since $U \times V$ is a neighborhood of (x, y), the graph f is proved to be closed in $X \times Y$.

Now we can give the proof of Theorem 1.

PROOF OF THEOREM 1. (1) → (2): Let Z be a paracompact M-space as in (1). We can suppose by Fact 1 that Z is a closed subset of $I^m \times T$ for some metric space T of weight m. Notice that this metric space T has the property of Lemma 1 (a), hence also, that of Lemma 1 (b). Consider the metric space H of weight m called the generalized Hilbert space (see [5]), and put $M = H \times T$. We show this metric space M is the desired one. Let S be an arbitrary metric space of weight $\leqslant m$. By the property of T in Lemma 1 (a) there exists a perfect map f of S onto some closed subspace F of T. Since the Hilbert space H possesses the universal property that every metric space of weight $\leqslant m$ is embeddable in it, we can think of S as a subset of H (see page 207 in [5]). Then, by Lemma 2 the graph f is closed in $H \times F$. Therefore we have

$$S \cong f \underset{\text{cl.}}{\subset} H \times F \underset{\text{cl.}}{\subset} H \times T = M.$$

(2) → (1): Let M be a metric space as in (2). Put $Z = I^m \times M$. We show this paracompact M-space Z has the property as in (1). Let X be an arbitrary paracompact M-space of weight $\leqslant m$. Then we may assume by Fact 1 that X is a closed subspace of $I^m \times S_X$ for some metric space S_X of weight $\leqslant m$. By the property of M in (2) we can assume that S_X is a closed subspace of M. Hence X is a closed subspace of $I^m \times M = Z$. This completes the proof of Theorem 1.

Now, in view of Theorem 1, Nagata's question (Q1) is equivalently converted into the following question about metric spaces:

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(Q2) Does there exist a metric space X of weight m with the universal property that every metric space of weight $\leqslant m$ is homeomorphic to some closed subspace of X?

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The above question (Q2) seems to be a very fundamental one about metric spaces, and so we are wondering why this question was not raised before explicitly by general topologists. Here we give a partial, negative answer to (Q2) by computing the cardinality of homeomorphic images of closed subsets. Note that the negative answer to (Q2) in the special case $m = \omega$ is also derivable from Section 35, Theorem 5 in Kuratowski [4]. For the number of closed sets in a metric space, consult the paper of Stone [7].

THEOREM 2. *Let m be an infinite cardinal such that*

$$(*) \qquad \kappa \leqslant m \leqslant 2^{\kappa} \quad \textit{for some cardinal } \kappa \textit{ satisfying } \kappa^{\omega} = 2^{\kappa}.$$

Then, for every metric space X of weight $\leqslant m$ we can find a metric space S_X of weight $\leqslant m$ which is not homeomorphic with any closed subspace of X.

PROOF. Let X be an arbitrary metric space of weight $\leqslant m$, and κ be a cardinal with the property $(*)$ above. Let M be any metric space of weight κ and of cardinal κ^{ω} (for example think of the Baire space D^{ω} where D is a discrete space of cardinal κ). Denote by $P(M)$ the power set of M and define its subfamily $H(M)$ by

$$H(M) = \{S \in P(M) \colon S \text{ is homeomorphic with some closed subspace of } X\}.$$

Clearly it suffices to show that $P(M) \setminus H(M)$ is nonempty; we will prove more, namely, $|P(M)| > |H(M)|$. Since $|P(M)| > |M| = \kappa^{\omega}$, our condition $\kappa^{\omega} = 2^{\kappa}$ implies that $|P(M)| > 2^{\kappa}$. So need only prove that $|H(M)| \leqslant 2^{\kappa}$. Let $F_{\kappa}(X)$ be a family of all closed subspaces of X having weight $\leqslant \kappa$ and for each member E of $F_{\kappa}(X)$ put

$$H(E, M) = \{S \in P(M) \colon S \text{ is homeomorphic with } E\}.$$

Then, since the weight of M does not exceed κ, we have

$$H(M) = \cup \{H(E, M) \colon E \in F_{\kappa}(X)\}.$$

Hence we are done if we show that $F_{\kappa}(X)$ and $H(E, M)$ are both of cardinals $\leqslant 2^{\kappa}$. The cardinal of X is $\leqslant m^{\omega}$ since X is first countable and of weight $\leqslant m$. Notice

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that each member of $F_\kappa(X)$ has density at most κ and a closed subset is completely determined by its dense subset; so we have

$$|F_\kappa(X)| \leqslant |X|^\kappa \leqslant (m^\omega)^\kappa \leqslant 2^\kappa.$$

To compute the cardinal of $H(E, M)$, observe that $|H(E, M)| \leqslant |C(E, M)|$ where $C(E, M)$ denotes the sets of all continuous functions from E to M. Since each $E \in F_\kappa(X)$ contains a dense subset of cardinal $\leqslant \kappa$, we have $|C(E, M)| \leqslant |M|^\kappa = (\kappa^\omega)^\kappa = 2^\kappa$. Hence $|H(E, M)| \leqslant 2^\kappa$, completing the proof.

The condition (*) is quite restricted and a bit nuisance, but it is clearly satisfied if $m \leqslant 2^\omega$ and $\kappa = \omega$. Thus we come up to the next negative answer to (Q2) as well as Nagata's question (Q1).

COROLLARY 1. *The answer to* (Q2) *as well as* (Q1) *is negative in the case* $\omega \leqslant m \leqslant 2^\omega$.

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We believe that the assertion in Theorem 2 remains true for every cardinal m without the condition (*). In other words,

Conjecture: the answer to (Q2) is negative for every cardinal m.

In connection with this conjecture, we note here that each of the next statements is consistent with ZFC:

(1) There are arbitrarily large cardinals m for which the answer to (Q2) is negative.

(2) Let $m = 2^\mathfrak{c}$ where $\mathfrak{c} = 2^\omega$; then for this m the answer to (Q2) is negative.

In fact, let us denote by $\mathfrak{N}$ the class of all cardinals m for which the answer to (Q2) is negative. Then GCH implies that κ and $\kappa^+ = 2^\kappa$ belong to $\mathfrak{N}$ for every cardinal κ with countable cofinality, since such cardinal κ satisfies the equality $\kappa^\omega = 2^\kappa$ in (*) of Theorem 2. Hence the statement (1) is consistent with ZFC. To show the consistency of (2), it will be sufficient to point out that one can build, by Easton's method, a model of ZFC satisfying

$$2^\omega = \omega_1 \quad \text{and} \quad 2^{\omega_1} = 2^{\omega_\omega} = \omega_{\omega+1}$$

(see Theorems 46, 18 in Jech [3]), hence the equality $\kappa^\omega = 2^\kappa$ for $\kappa = \omega_\omega$.

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References

[1] R. A. Alò and H. L. Shapiro, *Normal topological spaces* (Cambridge Univ. Press, 1974).
[2] A. V. Arhangel'skii, 'On a class of spaces containing all metric and all locally bicompact spaces', *Soviet Math.* **4** (1963), 751–754.
[3] T. Jech, *Set theory* (Academic Press, 1978).
[4] K. Kuratowski, *Topology*, vol. I (Academic Press, 1966).
[5] J. Nagata, *Modern general topology* (North-Holland, 1968).
[6] J. Nagata, 'A note on M-spaces and topologically complete spaces', *Proc. Japan Acad.* **45** (1969), 541–543.
[7] A. H. Stone, 'Cardinals of closed sets', *Mathematika* **6** (1959), 99–107.

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