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ON CERTAIN EQUATIONS IN RINGS

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In this paper we prove the following result: Let R be a 2-torsion free semiprime ring. Suppose there exists an additive mapping $T : R \to R$ such that $T(xyx) = T(x)yx - xT(y)x + xyT(x)$ holds for all pairs $x, y \in R$. Then T is of the form $2T(x) = qx + xq$, where q is a fixed element in the symmetric Martindale ring of quotients of R.

1. INTRODUCTION

This research has been motivated by the work of Brešar [**6**] and Zalar [**13**]. Throughout, R will represent an associative ring with centre $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, when $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that R is *prime* if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$holds for all pairs $x, y \in R$ and is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ is satisfied for all $x \in R$. A derivation D is *inner* if there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [**8**] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [**4**]. Cusack [**7**] has generalised Herstein's result to 2-torsion free semiprime rings (see also [**5**] for an alternative proof). We denote by Q_{mr}, Q_r, Q_s and C the maximal right ring of quotients, the right Martindale ring of quotients, the symmetric Martindale ring of quotients and the extended centroid of a semiprime ring R, respectively. For the explanation of Q_{mr}, Q_r, Q_s and C we refer the reader to [**2**]. An additive mapping $T : R \to R$ is called a *left centraliser* if $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. The concept appears naturaly in C^*-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a right R-module R into itself. For a semiprime ring R all such homomorphisms are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Q_r (see [**2**, Chapter 2]). If R has the identity element, $T : R \to R$ is a left

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centraliser if and only if T is of the form $T(x) = ax$ for some fixed element $a \in R$. An additive mapping $T : R \to R$ is called a *left Jordan centraliser* if $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of a right centraliser and a right Jordan centraliser should be self-explanatory. We follow Zalar [**13**] and call an additive mapping $T : R \to R$ a *centraliser* if T is both left and right centraliser. For a semiprime ring R each centraliser T is of the form $T(x) = cx$ for some fixed element $c \in C$ (see [**2**, Theorem 2.3.2]). Following ideas from [**5**] Zalar proved that any left (right) Jordan centraliser on a 2-torsion free semiprime ring is a left (right) centraliser. Molnar [**9**] proved that if we have an additive mapping $T : A \to A$, where A is a semisimple H^*-algebra, satisfying the relation $T(x^3) = T(x)x^2$ (respectively $T(x^3) = x^2T(x)$) for all $x \in A$, then T is a left (respectively right) centraliser. For the definition of an H^*-algebra we refer to [**1**]. Vukman [**10**] proved that if there exists an additive mapping $T : R \to R$, where R is a 2-torsion free semiprime ring, satisfying the relation $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, then T is a centraliser. Recently, Benkovič and Eremita [**3**] obtained the following result: Let $T : R \to R$ be an additive mapping, where R is a prime ring of either $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) \geqslant n$, satisfying the relation $T(x^n) = T(x)x^{n-1}$ for any $x \in R$ and some integer $n > 1$, then T is a left centraliser. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation if

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{1}$$

is satisfied for all pairs $x, y \in R$. One can easily prove that any Jordan derivation on arbitrary ring is a Jordan triple derivation (see for example [**4**]). Brešar [**6**] proved the result below.

Theorem 1.1. [**6**, Theorem 4.3] *Any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation.*

Recently, Vukman [**11**] proved that if there exists an additive mapping $T : R \to R$ where R is a 2-torsion free semiprime ring satisfying the relation $T(xyx) = xT(y)x$ for any pair $x, y \in R$, then T is a centraliser (see also [**12**]). It is easy to see that any centraliser T on arbitrary ring R satisfies the relation

$$T(xyx) = T(x)yx - xT(y)x + xyT(x) \tag{2}$$

for all pairs $x, y \in R$ (compare the relations (1) and (2)). It seems natural to ask whether the above relation characterises centralisers among all additive mappings on 2-torsion free semiprime rings. The answer to this question is negative. Namely, a routine calculation shows that for any fixed element $a \in R$, where R is an arbitrary ring, the mapping $T : R \to R$ defined by $T(x) = ax + xa$ satisfies the relation (2).

2. The result

Theorem 2.1. *Let R be a 2-torsion free semiprime ring. Suppose there exists an additive mapping $T : R \to R$ such that*

$$T(xyx) = T(x)yx - xT(y)x + xyT(x)$$

for all $x, y \in R$. Then there exists $q \in Q_s$ such that $2T(x) = qx + xq$ for all $x \in R$.

For the proof of Theorem 2.1 we need several lemmas.

Lemma 2.2. [5, Lemma 4] *Let R be a 2-torsion free semiprime ring and let $a, b \in R$. If for all $x \in R$ the relation $axb + bxa = 0$ holds, then $axb = bxa = 0$ is satisfied for all $x \in R$.*

Lemma 2.3. [6, Lemma 1.2] *Let $G_1, G_2, \ldots, G_n$ be additive groups and R be a semiprime ring. Suppose that mappings $S : G_1 \times G_2 \times \ldots \times G_n \to R$ and $T : G_1 \times G_2 \times \ldots \times G_n \to R$ are additive in each argument. If $S(a_1, a_2, \ldots, a_n)xT(a_1, a_2, \ldots, a_n) = 0$ for all $x \in R$, $a_i \in G_i$, $i = 1, \ldots, n$, then $S(a_1, a_2, \ldots, a_n)xT(b_1, b_2, \ldots, b_n) = 0$ for all $x \in R$, $a_i, b_i \in G_i$, $i = 1, \ldots, n$.*

Before we write down the next lemma, let us notice that the linearisation of the relation (2) gives

$$T(xyz + zyx) = T(x)yz - xT(y)z + xyT(z) + T(z)yx - zT(y)x + zyT(x) \tag{3}$$

for all $x, y, z \in R$. For the purposes of the next lemma we shall write $A(x, y, z) = T(xyz) - T(x)yz + xT(y)z - xyT(z)$ and $B(x, y, z) = xyz - zyx$. From (3) it follows that $A(x, y, z) = -A(z, y, x)$.

Lemma 2.4. *If R is any ring then*

$$A(x, y, z)uB(x, y, z) + B(x, y, z)uA(x, y, z) = 0$$

holds for all $x, y, z, u \in R$.

Proof: We compute $W = T(xyzuzyx + zyxuxyz)$ in two ways. On the one hand using (2) we have

$$\begin{aligned}
W &= T\big(x(yzuzy)x\big) + T\big(z(yxuxy)z\big) \\
&= T(x)yzuzyx - xT(y(zuz)y)x + xyzuzyT(x) + T(z)yxuxyz \\
&\qquad - zT\big(y(xux)y\big)z + zyxuxyT(z) \\
&= T(x)yzuzyx - xT(y)zuzyx + xyT(zuz)yx - xyzuzT(y)x \\
&\qquad + xyzuzyT(x) + T(z)yxuxyz - zT(y)xuxyz + zyT(xux)yz \\
&\qquad - zyxuxT(y)z + zyxuxyT(z) \\
&= T(x)yzuzyx - xT(y)zuzyx + xyT(z)uzyx - xyzT(u)zyx
\end{aligned}$$

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$$+ xyzuT(z)yx - xyzuzT(y)x + xyzuzyT(x) + T(z)yxuxyz - zT(y)xuxyz + zyT(x)uxyz - zyxT(u)xyz + zyxuT(x)yz - zyxuxT(y)z + zyxuxyT(z)$$

for all $x, y, z, u \in R$. On the other hand using (3) we get

$$\begin{aligned} W &= T\big((xyz)u(zyx) + (zyx)u(xyz)\big) \\ &= T(xyz)uzyx - xyzT(u)zyx + xyzuT(zyx) \\ &\qquad + T(zyx)uxyz - zyxT(u)xyz + zyxuT(xyz) \end{aligned}$$

for all $x, y, z, u \in R$. Comparing two expressions so obtained and using

$$A(x, y, z) = T(xyz) - T(x)yz + xT(y)z - xyT(z)$$

and

$$A(x, y, z) = -A(z, y, x)$$

we obtain the assertion of the lemma. ∎

LEMMA 2.5. *Let R be a semiprime ring and let $f, g : R \to Q_{mr}$ be additive mappings. If*

(4) $$f(x)y + xg(y) = 0$$

for all $x, y \in R$, then there exists a unique $q \in Q_{mr}$ such that $f(x) = -xq$ and $g(x) = qx$ for all $x \in R$.

PROOF: Using (4) we see that

$$xg(yz) = -f(x)yz = xg(y)z$$

and hence $x\big(g(yz) - g(y)z\big) = 0$ for all $x, y, z \in R$. Since R is semiprime we have $g(yz) = g(y)z$ for all $y, z \in R$. This means that g is a right R-module homomorphism. We set $I = RQ_{mr}$ and define the mapping $\widetilde{g} : I \to Q_{mr}$ by

$$\widetilde{g}\Big(\sum x_i q_i\Big) = \sum g(x_i) q_i$$

for all $q_i \in Q_{mr}$ and $x_i \in R$. By [**2**, Lemma 2.1.9], I is a dense right ideal of Q_{mr} and according to [**2**, Lemma 2.1.14] $\widetilde{g}$ is a well-defined homomorphism of right Q_{mr}-modules. Hence by [**2**, Proposition 2.1.7] there exists $q \in Q_{mr}(Q_{mr}) = Q_{mr}$ such that $\widetilde{g}(x) = qx$ for all $x \in I$. In particular, $g(x) = qx$ for all $x \in R$. Now, (4) implies that $f(x) = -xq$ for all $x \in R$. It is also straightforward to see that q is uniquely determined. ∎

Now we are ready to prove Theorem 2.1.

PROOF OF THEOREM 2.1: The proof goes through in several steps.

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FIRST STEP. Let us prove that for any $x, y, z \in R$, we have

$$T(xyz) = T(x)yz - xT(y)z + xyT(z). \tag{5}$$

As an immediate consequence of Lemma 2.2, Lemma 2.3 and Lemma 2.4 we obtain

$$A(x_1, x_2, x_3)uB(y_1, y_2, y_3) = 0 \tag{6}$$

for all u, x_i, $y_i \in R$, $i = 1, 2, 3$. Since $A(x, y, z) = -A(z, y, x)$ we have

$$\begin{aligned} 2A(x,y,z)uA(x,y,z) &= \big(A(x,y,z) - A(z,y,x)\big)uA(x,y,z) \\ &= \Big(T\big(B(x,y,z)\big) + B\big(T(z),y,x\big) - B\big(z,T(y),x\big) \\ &\qquad + B\big(z,y,T(x)\big)\Big)uA(x,y,z) \end{aligned}$$

for all $x, y, z, u \in R$. Using (6) and Lemma 2.2 the relation above reduces to

$$2A(x,y,z)uA(x,y,z) = T\big(B(x,y,z)\big)uA(x,y,z) \tag{7}$$

for all $x, y, z, u \in R$. Similarly we obtain

$$2A(x,y,z)uA(x,y,z) = A(x,y,z)uT\big(B(x,y,z)\big) \tag{8}$$

for all $x, y, z, u \in R$. Next, using Lemma 2.4 and the relation (3) we obtain

$$\begin{aligned} 0 &= T\big(A(x,y,z)uB(x,y,z) + B(x,y,z)uA(x,y,z)\big) \\ &= T\big(A(x,y,z)\big)uB(x,y,z) - A(x,y,z)T(u)B(x,y,z) \\ &\qquad + A(x,y,z)uT\big(B(x,y,z)\big) + T\big(B(x,y,z)\big)uA(x,y,z) \\ &\qquad - B(x,y,z)T(u)A(x,y,z) + B(x,y,z)uT\big(A(x,y,z)\big) \end{aligned}$$

for all $x, y, z, u \in R$, which according to (7) and (8) implies

$$\begin{aligned} 0 = 4A(x,y,z)uA(x,y,z) &+ T\big(A(x,y,z)\big)uB(x,y,z) \\ &- A(x,y,z)T(u)B(x,y,z) - B(x,y,z)T(u)A(x,y,z) \\ &+ B(x,y,z)uT\big(A(x,y,z)\big) \end{aligned}$$

for all $x, y, z, u \in R$. Using (6) the above relation reduces to

$$0 = 4A(x,y,z)uA(x,y,z) + T\big(A(x,y,z)\big)uB(x,y,z) + B(x,y,z)uT\big(A(x,y,z)\big)$$

for all $x, y, z, u \in R$. Left multiplication of the above relation by $A(x,y,z)uA(x,y,z)v$ gives according to (6)

$$4A(x,y,z)uA(x,y,z)vA(x,y,z)uA(x,y,z) = 0$$

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for all $x, y, z, u, v \in R$. Since R is a 2-torsion free semiprime ring it follows immediately that $A(x, y, z) = 0$ for all $x, y, z \in R$, which completes the proof of the relation (5).

SECOND STEP. We intend to prove that

$$(9) \qquad \big(T(xy) - T(x)y\big)z + x\big(T(yz) - yT(z)\big) = 0$$

holds for all $x, y, z \in R$. According to (5) $T(xyzu)$ can be written as

$$(10) \qquad T\big((xy)zu\big) = T(xy)zu - xyT(z)u + xyzT(u)$$

and also as

$$(11) \qquad T\big(x(yz)u\big) = T(x)yzu - xT(yz)u + xyzT(u).$$

Comparing (10) and (11) we arrive at

$$0 = \big(T(xy) - T(x)y\big)zu + x\big(T(yz) - yT(z)\big)u$$

for all $x, y, z, u \in R$ and so

$$\Big(\big(T(xy) - T(x)y\big)z + x\big(T(yz) - yT(z)\big)\Big)R = (0).$$

Since R is semiprime, it follows that (9) holds true.

THIRD STEP. It remains to prove that there exists $q \in Q_s$ such that

$$2T(x) = qx + xq$$

for all $x \in R$. We define mappings $F, G : R \times R \to R$ by $F(x, y) = T(xy) - T(x)y$ and $G(x, y) = T(xy) - xT(y)$ for all $x, y \in R$. Now (9) can be written as

$$F(x, y)z + xG(y, z) = 0$$

for all $x, y, z \in R$. Using Lemma 2.5 we see that for each $y \in R$ there exists a uniquely determined $q_y \in Q_{mr}$ such that $F(x, y) = -xq_y$ and $G(y, z) = q_y z$ for all $x, z \in R$. Thus, the mapping $H : R \to Q_{mr}$ defined by $H : y \mapsto q_y$ is well-defined. Since F is biadditive, it follows easily that H is additive. We have

$$T(xy) - T(x)y = F(x, y) = -xH(y),$$
$$T(xy) - xT(y) = G(x, y) = H(x)y$$

and so $\big(H(x) - T(x)\big)y + x\big(H(y) + T(y)\big) = 0$ for all $x, y \in R$. Again, applying Lemma 2.5 we get $q \in Q_{mr}$ such that $H(x) - T(x) = -xq$ and $H(x) + T(x) = qx$, which in turn implies that

$$(12) \qquad 2T(x) = qx + xq$$

for each $x \in R$. Finally, let us prove that $q \in Q_s$. Since $q \in Q_{mr}$, there exists a dense right ideal J of R such that $qJ \subseteq R$ (see [**2**, Proposition 2.1.7 (ii)]). According to (12) we have $qx+xq \in R$ for all $x \in R$ and so we see that also $Jq \subseteq R$. Let $I = RJ$. Then I is an essential two-sided ideal (see [**2**, Proposition 2.1.1] and [**2**, Remark 2.1.4]). Obviously, $Iq = RJq \subseteq R^2 \subseteq R$. Since $qx + xq \in R$ for all $x \in I$, it follows that $qI \subseteq R$. Thus, $qI \cup Iq \subseteq R$ and hence $q \in Q_s$ (see [**2**, p. 66]). ◻

COROLLARY 2.6. *Let R be a 2-torsion free semiprime ring. If $S, T : R \to R$ are additive mappings such that*

$$S(xyx) = S(x)yx - xT(y)x + xyS(x), \tag{13}$$

$$T(xyx) = T(x)yx - xS(y)x + xyT(x) \tag{14}$$

for all $x \in R$, then there exist a derivation $D : R \to R$ and $q \in Q_s$ such that

$$4S(x) = qx + xq + D(x) \quad \text{and} \quad 4T(x) = qx + xq - D(x)$$

for all $x \in R$.

PROOF: Comparing (13) and (14) we see that $S - T$ is a Jordan triple derivation and

$$(S+T)(xyx) = (S+T)(x)yx - x(S+T)(y)x + xy(S+T)(x)$$

for all $x, y \in R$. Hence by Theorem 2.1 there is $q \in Q_s$ such that $2(S+T)(x) = qx + xq$ for all $x \in R$. On the other hand, Theorem 1.1 implies that $S - T$ is a derivation. By D we denote the derivation $2(S - T)$. Consequently, $4S(x) = qx + xq + D(x)$ and $4T(x) = qx + xq - D(x)$ for all $x \in R$. ◻

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